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Some Results on Composition of Analytic Functions in a Unit Polydisc

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Abstract

The manuscript is an attempt to consider all methods which are applicable to investigation a directional index for composition of an analytic function in some domain and an entire function. The approaches are applied to find sufficient conditions of the L-index boundedness in a direction $\mathbf{b}\in\mathbb{C}^n\setminus\{\mathbf{0}\}$, where the continuous function L satisfies some growth condition and the condition of positivity in the unit polydisc. The investigation is based on a counterpart of the Hayman Theorem for the class of analytic functions in the polydisc and a counterpart of logarithmic criterion describing local conduct of logarithmic derivative modulus outside some neighborhoods of zeros. The established results are new advances for the functions analytic in the polydisc and in multidimensional value distribution theory.

1. Main Definitions and Notations

We will use notations from [1,2]. Let $\mathbb{C}^n$ be an n-dimensional complex vector space, $\mathbf{0}=(0,\dots,0)$, and $\mathbf{b}=(b_1,\dots,b_n)\in\mathbb{C}^n\setminus\{\mathbf{0}\}$ be a fixed direction. Other denotations are the following: $\mathbb{R}_+=(0,+\infty)$, the unit polydisc $\mathbb{D}^n$ is the Cartesian products of the discs with radius 1, i.e. $\mathbb{D}^n=\{z\in\mathbb{C}^n : |z_j|<1 \text{ for every } j\in\{1,2,\dots,n\}\}$. A continuous function $L\colon\ \mathbb{D}^n\to\mathbb{R}_+$ is such that for any $z=(z_1,z_2,\dots,z_n)\in\mathbb{D}^n$

$$L(z)>\beta\max_{1\le j\le n}\frac{|b_j|}{1-|z_j|},\quad \beta=\text{const}>1. \tag{1.1}$$

Recently, Salo T. with her co-authors [1] introduced a notion of the directional L-index for functions analytic in the polydisc. They proved many criteria belonging functions to the class. They describe the local behavior of the function and its directional derivative and its value distribution on all slices generated by the vector $\mathbf{b}$ and give estimates of logarithmic derivative modulus in the same vector. Now we justify some application of the results to related topics. In particular, we will examine some compositions of a function analytic in $\mathbb{C}^n$ and a function analytic in the $\mathbb{D}^n$, and will present sufficient conditions of boundedness of the L-index in direction for such a composition. Note there are results [3,4] on the finiteness of the index for analytic functions of single variable for which multidimensional analogs are still unknown.
The notation $\mathscr{A}(\mathbb{D}^n)$ we use for the class of functions which are analytic in $\mathbb{D}^n$. Similarly, $\mathscr{A}(\mathbb{C}^n)$ means the class of entire functions of n complex variables.
Let us remind the main definition from [1]. A function $F\in\mathscr{A}(\mathbb{D}^n)$ is said to be of *bounded L-index in a direction* $\mathbf{b}$, if it is possible to find $m_0\in\mathbb{Z}_+$ such that for every non-negative integer m and for any point z from the polydisc one has

$$\frac{|\partial_{\mathbf{b}}^m F(z)|}{m!L^m(z)}\le\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{k!L^k(z)}\ :\ \text{for every } k\in\{0,1,\dots,m_0\}\right\}, \tag{1.2}$$

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where $\partial_{\mathbf{b}}^0 F(z)$ matches with the function F, $\partial_{\mathbf{b}} F(z)$ is the dot product of the gradient of the function F and the conjugate of the vector $\mathbf{b}$, $\partial_{\mathbf{b}}^k F(z) = \partial_{\mathbf{b}}\left(\partial_{\mathbf{b}}^{k-1} F(z)\right)$, $k \geq 2$. The definition firstly appeared for entire functions of single variable in the paper of B. Lepson [5] if $L \equiv 1$, $\mathbf{b} = 1$ and in paper [6] if L is an arbitrary positive continuous function and $\mathbf{b} = 1$. If such least integer $m_0 = m_0(\mathbf{b})$ exists then it is is called the L-index in the direction $\mathbf{b}$ of F. The value m_0 will be denoted by $N_{\mathbf{b}}(F, L)$.
For a fixed point $z^* = (z_1^*, \dots, z_n^*)$ from the polydisc by D_z we denote an intersection of the $\mathbb{D}^n$ and a complex line crossing the point in a given direction $\mathbf{b}$, i.e. $D_{z^*} = \{t \in \mathbb{C} : (z_1^* + tb_1, \dots, z_n^* + tb_n) \in \mathbb{D}^n\}$. In other words, $D_z = \{t \in \mathbb{C} : |t| < \min_{1\leq j\leq n} \frac{1-|z_j|}{|b_j|}\}$. Here if $b_j = 0$ then we suppose $\frac{1-|z_j|}{|b_j|} = +\infty$. Denote

$$\lambda_{\mathbf{b}}(\zeta) = \sup_{w\in\mathbb{D}^n} \sup_{s_1,s_2\in D_w} \left\{ \frac{L(w+s_2\mathbf{b})}{L(w+s_1\mathbf{b})} : |s_1 - s_2| \leq \frac{\zeta}{\min\{L(z+s_2\mathbf{b}), L(z+s_1\mathbf{b})\}} \right\}.$$

As in [1] the $Q_{\mathbf{b}}(\mathbb{D}^n)$ denotes a class of continuous functions $L : \mathbb{D}^n \to \mathbb{R}_+$, which satisfy (1.1) and for each ζ from the segment $[0, \beta]$ the quantity $\lambda_{\mathbf{b}}(\zeta)$ is finite (the parameter β is defined in condition (1.1)).

2. Boundedness of L-index in Direction for Composition of Analytic Functions in the Polydisc

For simplicity, we suppose that for $\Psi \in \mathscr{A}(\mathbb{D}^n)$ there exist $\kappa > 0$ and natural p such that for all $z \in \mathbb{D}^n$ and for all integer $m \in \{0, 1, \dots, p\}$ next inequality is fulfilled

$$|\partial_{\mathbf{b}}^m \Psi(z)| \leq \kappa |\partial_{\mathbf{b}} \Psi(z)|^m. \tag{2.1}$$

For functions $h : \mathbb{C}^m \to \mathbb{C}$ (or $\mathbb{R}$ instead of $\mathbb{C}$) and $g : \mathbb{D}^n \to \mathbb{C}$ by $h_m^\circ g$ we denote such a composition $h(\underbrace{g(z), \dots, g(z)}_{m \text{ times}})$ The following proposition was early deduced for the unit ball [7] and n-dimensional complex space [8]. Now we formulate it for the class $\mathscr{A}(\mathbb{D}^n)$.

Theorem 2.1. *Let* $\mathbf{b}$ *be non-zero n-dimensional complex vector,* $f \in \mathscr{A}(\mathbb{C}^m)$, $\Psi \in \mathscr{A}(\mathbb{D}^n)$ *and its derivative in the direction* $\mathbf{b}$ *has empty zero set. Suppose that function* l *belongs to the class* $Q_{\mathbf{1}}^m$ *and its values are not lesser than* 1, *and the function* L *is defined as* $L(z) = \left|\partial_{\mathbf{b}} \Psi(z)\right| l_m^\circ \Psi(z)$ *and it belongs to the class* $Q_{\mathbf{b}}(\mathbb{D}^n)$.
If the l*-index in the direction* $\mathbf{1}$ *of the function* $f \in \mathscr{A}(\mathbb{C}^m)$ *is finite and the function* Ψ *satisfies* (2.1) *with* $N_{\mathbf{1}}(f, l)$ *instead of* p *then the* L*-index in the direction* $\mathbf{b}$ *of the function* $F(z) = f_m^\circ \Psi(z)$ *is also finite.*
And if the function $F(z) = f_m^\circ \Psi(z)$ *has finite* $N_{\mathbf{b}}(F, L)$ *and inequality* (2.1) *is fulfilled for the function* Ψ *and* $p = N_{\mathbf{b}}(F, L)$ *then* $N_{\mathbf{1}}(f, l)$ *is finite.*

Let us formulate some auxiliary propositions. They are counterparts the Hayman Theorem for the class $\mathscr{A}(\mathbb{C}^n)$ [9] and the class $\mathscr{A}(\mathbb{D}^n)$ [1], It was firstly proved by W. Hayman [10] for entire functions of one variable having bounded index.

Theorem 2.2 ([9]). *Let* $L \in Q_{\mathbf{b}}^n$. *A function* $F \in \mathscr{A}(\mathbb{C}^n)$ *is of bounded* L*-index in the direction* $\mathbf{b}$ *if and only if there exist numbers* $p \in \mathbb{Z}_+$, $R > 0$ *and* $C > 0$ *such that for every* $z \in \mathbb{C}^n$ *outside the disc of radii* R *one has*

$$\frac{|\partial_{\mathbf{b}}^{p+1} F(z)|}{L^{p+1}(z)} \leq C \max\left\{ \frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} : k \in \{0, \dots, p\} \right\}. \tag{2.2}$$

Theorem 2.3 ([1]). *Let* $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$. *A function* $F \in \mathscr{A}(\mathbb{D}^n)$ *has finite* $N_{\mathbf{b}}(F, L)$ *if and only if for some positive integer* p *and positive real* C, *and for every* z *belonging the polydisc inequality* (2.2) *holds.*

Proof of Theorem 2.1. Denote $\nabla f = \partial_{\mathbf{1}} f = \sum_{j=1}^m \frac{\partial f}{\partial z_j}$, $\nabla^k f \equiv \partial_{\mathbf{1}}^k f$ for $k \geq 2$. Firstly, we present two following formulas from [7, 8, 11]

$$\partial_{\mathbf{b}}^k F(z) = \nabla^k f_m^\circ \Psi(z) \left(\partial_{\mathbf{b}} \Psi(z)\right)^k + \sum_{j=1}^{k-1} \nabla^j f_m^\circ \Psi(z) Q_{j,k}(z), \tag{2.3}$$

where

$$Q_{j,k}(z) = \sum_{\substack{n_1+2n_2+\dots+kn_k=k \\ 0\leq n_1\leq j-1}} c_{j,k,n_1,\dots,n_k} \left(\partial_{\mathbf{b}} \Psi(z)\right)^{n_1} \left(\partial_{\mathbf{b}}^2 \Psi(z)\right)^{n_2} \dots \left(\partial_{\mathbf{b}}^k \Psi(z)\right)^{n_k},$$

$c_{j,k,n_1,\dots,n_k}$ are non-negative integer numbers, and

$$\nabla^k f_m^\circ \Psi(z) = \partial_{\mathbf{b}}^k F(z) \left(\partial_{\mathbf{b}} \Psi(z)\right)^{-k} + \left(\partial_{\mathbf{b}} \Psi(z)\right)^{-2k} \sum_{j=1}^{k-1} \partial_{\mathbf{b}}^j F(z) \left(\partial_{\mathbf{b}} \Psi(z)\right)^j Q^*(z; j, k), \tag{2.4}$$

with

$$Q^*(z; j, k) = \sum_{m_1+2m_2+\dots+km_k=2(k-j)} b_{j,k,m_1,\dots,m_k} \left(\partial_{\mathbf{b}} \Psi(z)\right)^{m_1} \left(\partial_{\mathbf{b}}^2 \Psi(z)\right)^{m_2} \dots \left(\partial_{\mathbf{b}}^k \Psi(z)\right)^{m_k},$$

$b_{j,k,m_1,\dots,m_k}$ are some integer coefficients. Their detailed proofs were presented in [7] for the unit ball and use the mathematical induction method. Obviously, their proofs for the polydisc is the same, so we omit them.

Suppose that $N_{\mathbf{1}}(f,l)$ is finite and f belongs to the class $\mathscr{A}(\mathbb{C}^m)$. By Theorem 2.2 inequality (2.2) holds for $n=m$, $F=f$, $L=l$, $\mathbf{b}=\mathbf{1}$. Taking into account (2.1) and (2.3), for $k=p+1$ we obtain

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)} \le \frac{|\nabla^{p+1} f_m^{\circ}\Psi(z)|}{L^{p+1}(z)}|\partial_{\mathbf{b}}\Psi(z)|^{p+1} + \sum_{j=1}^{p}\frac{|\nabla^j f_m^{\circ}\Psi(z)||Q_{j,p+1}(z)|}{L^{p+1}(z)} \le$$

$$\le \max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} : k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\frac{|Q_{j,p+1}(z)|}{(l_m^{\circ}\Psi(z))^{p+1-j}|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right) \le \max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} : k\in\{0,\dots,p\}\right\}\times$$

$$\times\left(C+\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} c_{j,p+1,n_1,\dots,n_{p+1}}\frac{|(\partial_{\mathbf{b}}\Psi(z))^{n_1}(\partial_{\mathbf{b}}^2\Psi(z))^{n_2}\dots\left(\partial_{\mathbf{b}}^{p+1}\Psi(z)\right)^{n_{p+1}}|}{(l_m^{\circ}\Psi(z))^{p+1-j}|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right) \le$$

$$\le \max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} : k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\ldots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}}\frac{c_{j,p+1,n_1,\dots,n_{p+1}}\kappa^{p+1}}{(l_m^{\circ}\Psi(z))^{p+1-j}}\right) \le C_1\max_{k\in\{0,\dots,p\}}\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}.$$

Now we substitute the right-hand side of (2.4) instead of $\nabla^k f_m^{\circ}\Psi(z)$ and perform some algebraic transformations:

$$\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} \le \frac{|\partial_{\mathbf{b}}^k F(z)|}{(l_m^{\circ}\Psi(z))^k|\partial_{\mathbf{b}}\Psi(z)|^k} + \sum_{j=1}^{k-1}\frac{|\partial_{\mathbf{b}}^j F(z)||Q^*(z;j,k)|}{(l_m^{\circ}\Psi(z))^k|\partial_{\mathbf{b}}\Psi(z)|^{2k-j}} \le \max_{1\le j\le k}\frac{|\partial_{\mathbf{b}}^j F(z)|}{L^j(z)}\left(1+\sum_{j=1}^{k-1}\frac{|Q^*(z;j,k)|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\right) \le$$

$$\le \max_{j\in\{1,2,\dots,k\}}\left\{L^{-j}(z)\left|\partial_{\mathbf{b}}^j F(z)\right|\right\}\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\dots,m_k}|\frac{|(\partial_{\mathbf{b}}\Psi(z))^{m_1}(\partial_{\mathbf{b}}^2\Psi(z))^{m_2}\dots(\partial_{\mathbf{b}}^k\Psi(z))^{m_k}|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\right) \le$$

$$\le \max\left\{\frac{|\partial_{\mathbf{b}}^j F(z)|}{L^j(z)} : 1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}\frac{|b_{j,k,m_1,\dots,m_k}|\kappa^k}{(l_m^{\circ}\Psi(z))^{k-j}}\right) \le C_2\max_{1\le j\le k}\frac{|\partial_{\mathbf{b}}^j F(z)|}{L^j(z)}.$$

Hence, it follows that

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L^{p+1}(z)} \le C_1C_2\max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} : k\in\{0,\dots,p\}\right\}.$$

The last inequality is the same as (2.2) in Theorem 2.3. It means that the theorem is applicable. Hence, we conclude that the directional L-index of the function F is bounded. The first part is proved.
Now we will start coinsiderations vice versa. Assume that the L-index in the direction $\mathbf{b}$ of the function F is bounded. In view of Hayman's Theorem the function must satisfies (2.2). Using (2.1) and (2.4), we will estimate

$$\frac{|\nabla^{p+1} f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^{p+1}} \le \frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{(l_m^{\circ}\Psi(z))^{p+1}|\partial_{\mathbf{b}}\Psi(z)|^{p+1}} + \sum_{j=1}^{p}\frac{|\partial_{\mathbf{b}}^j F(z)||Q^*(z;j,p+1)|}{(l_m^{\circ}\Psi(z))^{p+1}|\partial_{\mathbf{b}}\Psi(z)|^{2p+2-j}} \le$$

$$\le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} : k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\frac{|Q^*(z;j,p+1)|}{(l_m^{\circ}\Psi(z))^{p+1-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(p+1-j)}}\right) \le$$

$$\le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} : k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\ =2(p+1-j)}}|b_{j,p+1,m_1,\dots,m_{p+1}}|\frac{|(\partial_{\mathbf{b}}\Psi(z))^{m_1}(\partial_{\mathbf{b}}^2\Psi(z))^{m_2}\dots\left(\partial_{\mathbf{b}}^{p+1}\Psi(z)\right)^{m_{p+1}}|}{(l_m^{\circ}\Psi(z))^{p+1-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(p+1-j)}}\right) \le$$

$$\le \max\left\{\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} : k\in\{0,\dots,p\}\right\}\left(C+\sum_{j=1}^{p}\sum_{\substack{m_1+\ldots+(p+1)m_{p+1}=\\ =2(p+1-j)}}\frac{|b_{j,p+1,m_1,\dots,m_{p+1}}|\kappa^{2p+2-2j}}{l^{p+1-j}(\Psi(z))}\right) \le C_3\max_{k\in\{0,\dots,p\}}\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)}.$$

Instead $\partial_{\mathbf{b}}^k F(z)$ in previous expression we substitute (2.3) and again deduce

$$\frac{|\partial_{\mathbf{b}}^k F(z)|}{L^k(z)} \le \frac{|\nabla^k f_m^{\circ}\Psi(z)||\partial_{\mathbf{b}}\Psi(z)|^k}{L^k(z)} + \sum_{j=1}^{k-1}\frac{|\nabla^j f_m^{\circ}\Psi(z)||Q_{j,k}(z)|}{L^k(z)} \le$$

$$\le \max\left\{\frac{|\nabla^j f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j} : 1\le j\le k\right\}\left(1+\sum_{j=1}^{k-1}\frac{|Q_{j,k}(z)|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^k}\right) \le C_4\max\left\{\frac{|\nabla^j f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j} : j\in\{1,2,\dots,k\}\right\}.$$

It implies that

$$\frac{|\nabla^{p+1} f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^{p+1}} \le C_3C_4\max\left\{\frac{|\nabla^j f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j} : j\in\{0,\dots,p\}\right\}.$$

Application of Theorem 2.2 for such values $n=m$, $F=f$, $L=l$, $\mathbf{b}=\mathbf{1}$ give us finiteness of the l-index in the direction $\mathbf{b}$. □

Theorem 2.4. *Let* $\mathbf{b}$ *be a fixed n-dimensional non-zero complex direction, the functions* l, f, Ψ *belong to the classes* $Q_{\mathbf{1}}^m$, $\mathscr{A}(\mathbb{C}^m)$, $\mathscr{A}(\mathbb{D}^n)$, *respectively. For each* $w \in \mathbb{C}^m$ *the values of* $l(w)$ *are not lesser than* 1, *and the l-index in the direction* $\mathbf{1}$ *of the function* f *is bounded. Suppose that the function* $L(z) = \max\{1, |\partial_{\mathbf{b}}\Psi(z)|\} l_m^{\circ}\Psi(z)$ *belongs to the class* $Q_{\mathbf{b}}(\mathbb{D}^n)$ *and for every point z from the polydisc* $\mathbb{D}^n$ *and for each* $k \in \{1,2,\dots,N_{\mathbf{1}}(f,l)+1\}$ *the function* Ψ *satisfies*

$$|\partial_{\mathbf{b}}^k\Psi(z)| \le \kappa(l_m^{\circ}\Psi(z))^{1/(N_{\mathbf{1}}(f,l)+1)}|\partial_{\mathbf{b}}\Psi(z)|^k, \ \ (1 \le kappa \equiv const). \tag{2.5}$$

Then the function $F(z) = f_m^{\circ}\Psi(z)$ *belongs to the function class having bounded L-index in the direction* $\mathbf{b}$.

Proof of Theorem 2.4. As above, we will merge methods from appropriate statements in [7,8].
Denote $L_0(z) = l_m^{\circ}\Psi(z)|\partial_{\mathbf{b}}\Psi(z)|$. We estimate Equation (2.3) with L_0 instead of L by modulus and substitute $l_m^{\circ}\Psi(z)|\partial_{\mathbf{b}}\Psi(z)|$ instead of the function L_0, for $k = p+1$ we conclude

$$|\partial_{\mathbf{b}}^{p+1}F(z)|L_0^{-p-1}(z) \le |\nabla^{p+1}f_m^{\circ}\Psi(z)|L_0^{-p-1}(z)\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1} + \sum_{j=1}^{p}|\nabla^j f_m^{\circ}\Psi(z)||Q_{j,p+1}(z)|L_0^{-p-1}(z) \le$$
$$\le \frac{|\nabla^{p+1}f_m^{\circ}\Psi(z)|\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}{(l_m^{\circ}\Psi(z))^{p+1}\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1}} + \sum_{j=1}^{p}\frac{|\nabla^j f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j}\cdot\frac{|Q_{j,p+1}(z)|(l_m^{\circ}\Psi(z))^j}{|\partial_{\mathbf{b}}\Psi(z)|^{p+1}\,(l_m^{\circ}\Psi(z))^{p+1}}. \tag{2.6}$$

Let us remind that $f \in \mathscr{A}(\mathbb{C}^m)$ has finite $N_{\mathbf{b}}(f,l)$ (by hypothesis of the assertion). Theorem 2.2 yields validity of inequality (2.2) in this form

$$(\forall \tau \in \mathbb{C}^m): \quad \frac{|\nabla^{p+1}f(\tau)|}{l^{p+1}(\tau)} \le C\max\left\{\frac{|\nabla^k f(\tau)|}{l^k(\tau)}: \ k\in\{0,\dots,p\}\right\}$$

for such values of parameters $n = m$, $F = f$, $L = l$, $\mathbf{b} = \mathbf{1}$ and $p = N_{\mathbf{1}}(f,l)$. We enhance (2.6), if we substitute previous inequality with $\tau = (\underbrace{\Psi(z),\dots,\Psi(z)}_{m \text{ times}})$

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le \max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}: \ k\in\{0,\dots,p\}\right\}\left(C + \sum_{j=1}^{p}\frac{|Q_{j,p+1}(z)|(l_m^{\circ}\Psi(z))^{j-p-1}}{|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right) \le$$
$$\le \max_{k\in\{0,\dots,p\}}\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}\left(C + \sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\dots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} c_{j,p+1,n_1,\dots,n_{p+1}}\frac{|(\partial_{\mathbf{b}}\Psi(z))^{n_1}(\partial_{\mathbf{b}}^2\Psi(z))^{n_2}\dots\left(\partial_{\mathbf{b}}^{p+1}\Psi(z)\right)^{n_{p+1}}|}{(l_m^{\circ}\Psi(z))^{p+1-j}\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right). \tag{2.7}$$

Now we use condition (2.5) for the function Ψ. Then inequality (2.7) transforms in the following

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le \max\left\{\frac{|\nabla^k f(\Psi(z))|}{(l_m^{\circ}\Psi(z))^k}: k\in\{0,1,\dots,p\}\right\}\times$$
$$\times\left(C + \sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\dots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}}\frac{c_{j,p+1,n_1,\dots,n_{p+1}}\kappa^{p+1}l(\Psi(z),\dots,\Psi(z))|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}{(l_m^{\circ}\Psi(z))^{p+1-j}\,|\partial_{\mathbf{b}}\Psi(z)|^{p+1}}\right) \le$$
$$\le \max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}: k\in\{0,1,2,\dots,p\}\right\}\left(C + \sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\dots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}}\frac{c_{j,p+1,n_1,\dots,n_{p+1}}\kappa^{p+1}}{(l_m^{\circ}\Psi(z))^{p-j}}\right). \tag{2.8}$$

Since the values of the function l are not lesser than 1, the composition $l_m^{\circ}\Psi(z)$ is also not lesser than 1. We substitute it in (2.8)

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)} \le C_1\max\left\{\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}: k\in\{0,\dots,p\}\right\}, \tag{2.9}$$

with $C_1 = C + \kappa^{p+1}\sum_{j=1}^{p}\sum_{\substack{n_1+2n_2+\dots+(p+1)n_{p+1}=p+1\\ 0\le n_1\le j-1}} c_{j,p+1,n_1,\dots,n_{p+1}}$. To estimate the fraction $\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}$, we find the modulus of equality (2.4)

$$\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k} \le \frac{|\partial_{\mathbf{b}}^k F(z)|}{(l_m^{\circ}\Psi(z))^k|\partial_{\mathbf{b}}\Psi(z)|^k} + \sum_{j=1}^{k-1}\frac{|\partial_{\mathbf{b}}^j F(z)||Q^*(z;j,k)|}{(l_m^{\circ}\Psi(z))^k|\partial_{\mathbf{b}}\Psi(z)|^{2k-j}} \le$$
$$\le \max_{1\le j\le k}\left\{\frac{|\partial_{\mathbf{b}}^j\Psi(z)|}{(l_m^{\circ}\Psi(z))^j|\partial_{\mathbf{b}}\Psi(z)|^j}\right\}\left(1 + \sum_{j=1}^{k-1}\frac{|Q^*(z;j,k)|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\right) \le$$
$$\le \max_{1\le j\le k}\frac{|\partial_{\mathbf{b}}^j\Psi(z)|}{(l_m^{\circ}\Psi(z))^j|\partial_{\mathbf{b}}\Psi(z)|^j}\left(1 + \sum_{j=1}^{k-1}\sum_{m_1+2m_2+\dots+km_k=2(k-j)}|b_{j,k,m_1,\dots,m_k}|\frac{|(\partial_{\mathbf{b}}\Psi(z))^{m_1}(\partial_{\mathbf{b}}^2\Psi(z))^{m_2}\dots(\partial_{\mathbf{b}}^k\Psi(z))^{m_k}|}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\right). \tag{2.10}$$

Since $l(w)\ge 1$ and for $s\in\{1,2,\dots,N_{\mathbf{1}}(f,l)+1\}$ and $N_{\mathbf{1}}(f,l)\ge 1$ one has $s/2\ge 1/(N_{\mathbf{1}}(f,l)+1)$, inequality (2.5) can be reinforced $|\partial_{\mathbf{b}}^{s}\Psi(z)|\le \kappa l^{s/2}(\Psi(z))|\partial_{\mathbf{b}}\Psi(z)|^{s}$. Applying this inequality to (2.10), we deduce

$$\frac{|\nabla^k f_m^{\circ}\Psi(z)|}{(l_m^{\circ}\Psi(z))^k}\le \max_{1\le j\le k}\frac{|\partial_{\mathbf{b}}^{j}F(z)|}{(l_m^{\circ}\Psi(z))^j|\partial_{\mathbf{b}}\Psi(z)|^j}\Bigg(1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|\kappa^{m_1+m_2+\ldots+m_k}\times$$
$$\times\frac{(l_m^{\circ}\Psi(z))^{(m_1+2m_2+\ldots+km_k)/2}|\partial_{\mathbf{b}}\Psi(z)|^{m_1+2m_2+\ldots+km_k}}{(l_m^{\circ}\Psi(z))^{k-j}|\partial_{\mathbf{b}}\Psi(z)|^{2(k-j)}}\Bigg)\le C_2\max\left\{\frac{|\partial_{\mathbf{b}}^{j}\Psi(z)|}{(l_m^{\circ}\Psi(z))^j|\partial_{\mathbf{b}}\Psi(z)|^j}:\ j\in\{1,2,\ldots,k\}\right\},$$

with

$$C_2=1+\sum_{j=1}^{k-1}\sum_{m_1+2m_2+\ldots+km_k=2(k-j)}|b_{j,k,m_1,\ldots,m_k}|\kappa^{m_1+m_2+\ldots+m_k}.$$

Then from inequality (2.9) we get

$$\frac{|\partial_{\mathbf{b}}^{p+1}F(z)|}{L_0^{p+1}(z)}\le C_1\max_{k\in\{0,\ldots,p\}}\frac{|f^{(k)}(\Psi(z),\ldots,\Psi(z))|}{(l_m^{\circ}\Psi(z))^k}\le C_1C_2\max\left\{\frac{|\partial_{\mathbf{b}}^{j}F(z)|}{L_0^{j}(z)}:\ j\in\{0,\ldots,p\}\right\},\tag{2.11}$$

$p=N_{\mathbf{1}}(f,l)$. Remind that inequality (2.11) is proved for all z outside zero set of the function $\partial_{\mathbf{b}}\Phi$ and with usage the condtion $N_{\mathbf{1}}(f,l)\ge 1$. If $N_{\mathbf{1}}(f,l)=0$ then the parameter p also equals zero and estimate (2.9) yields

$$\frac{|\partial_{\mathbf{b}}F(z)|}{L_0(z)}\le C_1|f_m^{\circ}\Psi(z)|=C_1|F(z)|.$$

Thus, (2.11) is proved for all possible finite values of the directional l-index for the function f.
Since $L(z)=(l_m^{\circ}\Psi(z)\max\{1,|\partial_{\mathbf{b}}\Psi(z)|\}$, we can rewrite inequality (2.11):

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\cdot\frac{L^{p+1}(z)}{L_0^{p+1}(z)}\le C_1C_2\max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}\frac{L^{k}(z)}{L_0^{k}(z)}:\ k\in\{0,\ldots,p\}\right\}.$$

Then

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\le C_1C_2\frac{L_0^{p+1}(z)}{L^{p+1}(z)}\max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}\frac{L^{k}(z)}{L_0^{k}(z)}:\ k\in\{0,\ldots,p\}\right\}\le$$
$$\le C_1C_2\frac{L_0^{p+1}(z)}{L^{p+1}(z)}\max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}:k\in\{0,\ldots,p\}\right\}\max\left\{\frac{L^{k}(z)}{L_0^{k}(z)}:k\in\{0,\ldots,p\}\right\}=\frac{C_1C_2(L_0(z)/L(z))^{p+1}}{\min\limits_{k\in\{0,\ldots,p\}}(L_0(z)/L(z))^k}\max_{k\in\{0,\ldots,p\}}\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}.\tag{2.12}$$

Let $t_0=t(z)=L_0(z)/L(z)$ and $k_0\le p$ ($k_0\in\mathbb{Z}_+$) be such that $(t_0)^{k_0}=\min_{k\in\{0,\ldots,p\}}t_0^{k}$. One should observe that $t_0\in(0,1]$ and $p+1-k_0\ge 1$. Hence, $\frac{t_0^{p+1}}{t_0^{k_0}}=t_0^{p+1-k_0}\le t_0\le 1$. Therefore, $\frac{(L_0(z)/L(z))^{p+1}}{\min_{k\in\{0,\ldots,p\}}(L_0(z)/L(z))^k}=t_0^{p+1-k_0}\le t_0\le 1$. Thus, from inequality (2.12) we get

$$\frac{\left|\partial_{\mathbf{b}}^{p+1}F(z)\right|}{L^{p+1}(z)}\le C_1C_2\max\left\{\frac{\left|\partial_{\mathbf{b}}^{k}F(z)\right|}{L^{k}(z)}:\ k\in\{0,\ldots,p\}\right\}\tag{2.13}$$

for all z outside zero set of the $\mathbf{b}$-directional derivative of the function Ψ.
If for some point z from the polydisc $\mathbb{D}^n$ the $\mathbf{b}$-directional derivative of the function Ψ vanishes then for any natural value of k does not exceeding $N(f,l)+1$ condition (2.5) means that k-th order $\mathbf{b}$-directional derivative of the function Ψ also vanishes at this same point. Substituting this point in (2.3) we conclude that k-th order $\mathbf{b}$-directional derivative of the function Ψ also vanishes at this same point for each natural $1\le k\le N(f,l)+1$. Hence, for all points z belonging zero set of the $\mathbf{b}$-directional derivative of the function Ψ inequality (2.13) is true.
Applying Theorem 2.3 we establish that the function F belong to the class of functions with bounded L-index in the direction $\mathbf{b}$. □

3. Application of Logarithmic Criterion to Composition

In this section, we consider an application of the logarithmic criterion to investigation of the index boundedness for a composition of functions from the classes $\mathscr{A}(\mathbb{D}^n)$ and $\mathscr{A}(\mathbb{C}^m)$. Another applications of the statement in function theory of bounded index are decribed in [12–15].
Let us introduce the slice function as $g_z(t):=F(z+t\mathbf{b})$ ($z\in\mathbb{D}^n$). If one has for some z from the unit polydisc the slice function $g_z(t)$ has empty zero set, then we put $G_r^{\mathbf{b}}(F,z):=\varnothing$; otherwise if $g_z(t)$ identically equals zero then we put $G_r^{\mathbf{b}}(F,z):=\{z+t\mathbf{b}\colon\ |t|\le\min\limits_{j\in\{1,\ldots,n\}}\frac{1-|z_j|}{|b_j|}\}$.
And last possible case is if $g_z(t)\not\equiv 0$ and $a_{k,z}$ are zeros of $g_z(t)$, then we denote $G_r^{\mathbf{b}}(F,z):=\bigcup_k\left\{z+t\mathbf{b}\colon\ |t-a_{k,z}|\le\frac{r}{L(z+a_{k,z}\mathbf{b})}\right\}$, $r>0$.
Let $G_r^{\mathbf{b}}(F)=\bigcup_{z^0\in\mathbb{D}^n}G_r^{\mathbf{b}}(F,z^0)$, $n\big(r,z^0,1/F\big)=\sum_{|a_k^0|\le r}1$ is the counting function of zeros (a_k^0) of the function $F(z^0+t\mathbf{b})$ in the disk $\{t\in\mathbb{C}\colon\ |t|\le r\}$. Below we formulate two auxiliary propositions proved in [1]. The first of them is the logarithmic criterion analog, and the second of them is weaker sufficient conditions for functions belonging to the class $\mathscr{A}(\mathbb{D}^n)$.

Theorem 3.1. *[1] Let $F :\in \mathscr{A}(\mathbb{D}^n)$, $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and $\mathbb{D}^n \setminus G_\beta^{\mathbf{b}}(F) \neq \varnothing$. The function F has finite $N_{\mathbf{b}}(F,L)$ if and only if*

1) *for every radius r belonging to the half-closed interval $(0,\beta]$ there exists a positive real $P = P(r)$ such that for every point $z \in \mathbb{D}^n$ outside the set $G_r^{\mathbf{b}}(F)$ the following directional logarithmic derivative estimate is true*

$$|\partial_{\mathbf{b}} F(z)| \leq PL(z)|F(z)|; \tag{3.1}$$

2) *for every radius r belonging to the segment $[0,\beta]$ and some $\widetilde{n}(r) \in \mathbb{Z}_+$ amount of zeros for the slice function in some circles within the unit polydisc is uniformly bounded, i.e.*

$$n\left(r/L(z^0), z^0, 1/F\right) \leq \widetilde{n}(r). \tag{3.2}$$

for each $z^0 \in \mathbb{D}^n$ with $F(z^0 + t\mathbf{b}) \not\equiv 0$.

Theorem 3.2. *[1] Let $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, $\mathbb{D}^n \setminus G_\beta^{\mathbf{b}}(F) \neq \emptyset$, $F : \mathbb{D}^n \to \mathbb{C}$ be an analytic function. If the following conditions are satisfied*

1) *there exists $r_1 \in (0, \beta/2)$ (or there exists $r_1 \in [\beta/2, \beta)$ and $(\forall z \in \mathbb{D}^n) : L(z) > \frac{2\beta|b|}{1-|z|}$) such that $n(r_1) \in [-1;\infty)$;*
2) *there exist $r_2 \in (0,\beta)$, $P > 0$ such that $2r_2 \cdot n(r_1) < r_1/\lambda_{\mathbf{b}}(r_1)$ and for all $z \in \mathbb{D}^n \setminus G_{r_2}(F)$ inequality (3.1) holds,*

then the function F has bounded L-index in the direction $\mathbf{b}$.

Within the notion of bounded index the local properties of analytic solutions of ordinary [5, 16, 17], directional [13] and partial differential equations [18] and their systems [19] are considered in many papers. Moreover, application of the Hayman theorem and its analogs is main method to justify sufficent conditions for boundedness of L-index in direction, if they are applied to composition of entire [4, 8, 15] and analytic functions [2, 7].
Below there are presented other results on functions' composition from the classes $\mathscr{A}(\mathbb{D}^n)$ and $\mathscr{A}(\mathbb{C}^m)$. They are proved with usage of logarithmic criterion analog for the unit polydisc (similar results for the unit ball see in [2]). In this section we suppose that $\mathbf{1} = (1,\ldots,1) \in \mathbb{R}^m$.

Proposition 3.3. *Let $\Psi \in \mathscr{A}(\mathbb{D}^n)$, $f \in \mathscr{A}(\mathbb{C}^m)$ with an empty zero set.*
1) Suppose that $l \in Q_{\mathbf{1}}(\mathbb{C}^m)$, $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and for every point z from the unit polydisc the value $L(z)$ is not lesser than $\left|\partial_{\mathbf{b}}\Psi(z)\right| l_m^\circ \Psi(z)$. If the $\mathbf{1}$-directional l-index of the function f is finite, then the function $F(z) = f_m^\circ \Psi(z)$ has finite $N_{\mathbf{b}}(F,L)$.
2) Suppose that $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$, the $\mathbf{b}$-directional derivative of the function Ψ has empty zero set and $l \in Q_{\mathbf{1}}(\mathbb{C}^m)$ and such a function $l_m^\circ \Psi(z)$ is not lesser than $L(z)/\left|\partial_{\mathbf{b}}\Psi(z)\right|$ for every point z from the polydisc $\mathbb{D}^n$. And if the function $F(z) = f_m^\circ \Psi(z)$ is of bounded L-index in the direction $\mathbf{b}$, then the $\mathbf{1}$-directional l-index of the function f is also finite.

Proof. It is not difficult to verify that

$$\partial_{\mathbf{b}} F(z) = \partial_{\mathbf{1}} f_m^\circ \Psi(z) \cdot \partial_{\mathbf{b}} \Psi(z). \tag{3.3}$$

Remind that zero set of f is empty. So such a function $f_m^\circ \Psi(z)$ has also empty zero set. Then $G_r^{\mathbf{b}}(F) = \varnothing$. Thus, it leaves to validate condition 2) in Theorem 3.2. Indeed, we need to justify inequality (3.1) for every point z belonging the polydisc $\mathbb{D}^n$. Using (3.3) for the directional logarithmic derivative estimate we obtain

$$\left|\partial_{\mathbf{b}} F(z)/F(z)\right| = \left|\partial_{\mathbf{1}} f_m^\circ \Psi(z)\right| \cdot |\partial_{\mathbf{b}} \Psi(z)|/|f_m^\circ \Psi(z)| \tag{3.4}$$

Let f be of bounded l-index in the direction $\mathbf{1}$. By Theorem 3.1 (see also [20]) for ther multivariate entire functions inequality (3.1) is valid for the function f and for all $w \in \mathbb{C}^m$:

$$|\partial_{\mathbf{1}} f(w)| \leq Pl(w) \cdot |f(w)| \tag{3.5}$$

After substitution $w = (\underbrace{\Psi(z),\ldots,\Psi(z)}_{m \text{ times}})$ in (3.5) and usage (3.4) the following estimate become valid

$$|\partial_{\mathbf{b}} F(z)|/|F(z)| = Pl_m^\circ \Psi(z) \cdot |\partial_{\mathbf{b}} \Psi(z)| \leq PL(z). \tag{3.6}$$

The function F also does not vanish. Thus, we have proved validity of condition 2) in Theorem 3.1. It means that the function F belongs to the class of functions with bounded L-index in the direction $\mathbf{b}$.
By analogy to the first part of the proof we can justify the second part of the assertion. □

By $\mathbf{1}_j$ we denote m-dimensional complex vector, in which j-th component equals one, other components are zeros.

Proposition 3.4. *Let $\Psi_j \in \mathscr{A}(\mathbb{D}^n)$ and $l \in Q_{\mathbf{1}_j}^m$ for $j \in \{1,\ldots,m\}$, $f \in \mathscr{A}(\mathbb{C}^m)$ with empty zero set. Suppose that $L \in Q_{\mathbf{b}}(\mathbb{D}^n)$ and $L(z) \geq \sum_{j=1}^m \left|\partial_{\mathbf{b}}\Psi_j(z)\right| l(\Psi_1(z),\Psi_2(z),\ldots,\Psi_m(z))$ for every point z within the polydisc $\mathbb{D}^n$. If for every $j \in \{1,\ldots,m\}$ the function f is of bounded l-index in the direction $\mathbf{1}_j$, then the composite function $F(z) = f(\Psi_1(z),\Psi_2(z),\ldots,\Psi_m(z))$ is of bounded L-index in the direction $\mathbf{b}$.*

Proof. Using direct calculations it can be substantiated

$$\partial_{\mathbf{b}} F(z) = \sum_{j=1}^m f'_{\Psi_j}(\Psi_1(z),\Psi_2(z),\ldots,\Psi_m(z)) \partial_{\mathbf{b}} \Psi_j(z). \tag{3.7}$$

Since f has empty zero set, the composite function $f(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))$ does not vanish for all z from the polydisc $\mathbb{D}^n$ that is $G_r^{\mathbf{b}}(F)=\varnothing$. It leaves to validate inequality (3.1) within the polydisc $\mathbb{D}^n$ because it is equivalent condition 2) in Theorem 3.2. From (3.7) it follows that

$$|\partial_{\mathbf{b}}F(z)|/|F(z)|\le\sum_{j=1}^{m}\left|\frac{f'_{\Psi_j}(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))}{f(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))}\right|\cdot|\partial_{\mathbf{b}}\Psi_j(z)| \tag{3.8}$$

Since f is of bounded l-index in each direction $\mathbf{1}_j$, by analog of Theorem 3.1 for entire functions of m complex variables (see [20]) inequality (3.1) holds for the function f and for all $w\in\mathbb{C}^m$:

$$\frac{|\partial_{\mathbf{1}_j}f(w)|}{|f(w)|}\le Pl(w) \tag{3.9}$$

Replacing w by $(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))$ in (3.9) and using it in (3.8) we establish such a directional logarithmic derivative estimate

$$|\partial_{\mathbf{b}}F(z)|/|F(z)|=Pl(\Psi_1(z),\Psi_2(z),\dots,\Psi_m(z))\cdot\sum_{j=1}^{m}|\partial_{\mathbf{b}}\Psi_j(z)|\le PL(z). \tag{3.10}$$

Since function F has not zero points as the function f, from (3.10) it follows that by Theorem 3.1 $\mathbf{b}$-directional L-index of the function F is finite. Proposition 3.4 is proved. □

The condition of absence zero points in the function f can be replaced by another condition on the function Ψ generated of the notion of multidimensional directional multivalence.
Let us remind the definition of function having bounded value L-distribution in a direction.
Function $F\in\mathscr{A}(\mathbb{D}^n)$ is called [1] a function of *bounded value L-distribution in the direction* $\mathbf{b}$, if for some natural p and for any complex w and for every point z_0 within the polydisc $\mathbb{D}^n$ such that the slice function $F(z^0+t\mathbf{b})$ does not equal identically w, the inequality holds $n\big(1/L(z^0),z^0,1/(F-w)\big)\le p$, i.e. the equation $F(z^0+t\mathbf{b})=w$ has at most p solutions in the disc $\{t:\ |t|\le 1/L(z^0)\}$. Using the one-dimensional notion of multivalence, we can claim that the slice function $F(z^0+t\mathbf{b})$ is p-valent in every disc $\{t:\ |t|\le 1/L(z^0)\}$ for every point $z^0\in\mathbb{D}^n$. For another classes of multivariate analytic and slice holomorphic functions the notion is considered in [21]. If $n=1$, $\mathbf{b}=1$ and $L\equiv 1$ then the notion matches with a definition of function of bounded value distribution [22–25], and if $n=1$, $\mathbf{b}=1$, $L=l\not\equiv 1$ then it is a definition of function of bounded value l-distribution [6, 26]. Another approach to multivalence of bivariate function is considered in [27]. Our main result on this topic is the following

Proposition 3.5. *Let $\Psi\in\mathscr{A}(\mathbb{D}^n)$, $f\in\mathscr{A}(\mathbb{S})$, $F(z)=f\circ\Psi(z)$. $l\in Q$, $L\in Q_{\mathbf{b}}(\mathbb{D}^n)$ be such that $L(z)\ge\big|\partial_{\mathbf{b}}\Psi(z)\big|l\circ\Psi(z)$ for any z with $\mathbb{D}^n$. If these functions satisfy such hypotheses*
1) $N(f,l)$ is finite;
2) the function Ψ has bounded value L-distribution in the direction $\mathbf{b}$*,*
3) for any $r_1\in(0;\beta]$ there exist $r_2>0$ and $r_3>0$ for which the following inclusion $G_{r_2}(f;l)\subset\Psi(G^{\mathbf{b}}_{r_1}(F;L))\subset G_{r_3}(f;l)$ is true, then F is of bounded L-index in the direction $\mathbf{b}$.

Proof. The condition 3) allows us to prove inequality (3.6) by similarity to Proposition 3.3.
Inequality (3.2) is valid for F because equality $F(z^0+t\mathbf{b})=0$ yields the equation $\Psi(z^0+t\mathbf{b})=c_k$, where c_k span whole zero set of the function f, $k\in\mathbb{N}$. Since Ψ has bounded value L-distribution in the direction $\mathbf{b}$, the last equation $\Psi(z^0+t\mathbf{b})=c_k$ has at most $p(r_1)$ solutions for given k at the disc $\{t:\ |t|\le\frac{r_1}{L(z^0)}\}$, if $r_1\in(0;\beta)$. Condition 3) means that the set $\{\Psi(z^0+t\mathbf{b}):|t|\le\frac{r_1}{L(z^0)}\}$ includes at most $n(r_3)$ zeros of f. Thus, such a set $\{z^0+t\mathbf{b}:\ |t|\le\frac{r_1}{L(z^0)}\}$ holds at most $p(r_1)\cdot n(r_3)$ zeros of F. In other words, zeros of the F are uniformly distributed in the sense of validity (3.2). Then by the logarithnmic criterion analog (Theorem 3.2) the function F is of bounded L-index in the direction $\mathbf{b}$. □

It is worth recognizing that Theorems 2.1 and 2.4, Propositions 3.3 and 3.5 are varied assumptions by the outer and inner function of the composition. But their consequence is the similar: a composite function is of bounded L-index in the direction $\mathbf{b}$ with alike functions

$$L(z)=\big|\partial_{\mathbf{b}}\Psi(z)\big|\cdot l_m^{\circ}\Psi(z)\text{ or }L(z)=\max\{1,|\partial_{\mathbf{b}}\Psi(z)|\}\cdot l_m^{\circ}\Psi(z).$$

But there were presented examples of analytic functions in the unit ball which dissatisfy concurrently assumptions of these statements (see examples in [2]).

4. Conclusion

Proposition 3.5 has not an analog for another multidimensional approach — so-called index in joint variables. Recent results for composite entire functions with bounded index in joint variables were deduced in [28]. They are similar to Theorem 2.1 and Theorem 2.4. Proposition 3.5 uses the notion of bounded value distribution in a direction. For multivariate complexvalued entire functions F. Nuray [27] introduced a notion of multivalence and indicated some connection between multivalued functions and functions with finite index in joint variables. The multivalence means bounded value distribution in some sense. But we do not know whether is it possible to deduce analogs of Propositon 3.5 for this class of functions which is intensively examined in papers of F. Nuray and R. Patterson [19, 29–31].
Let us present a brief description of possible investigations. Other important meanings of the obtained results is their application to composite differential equations. Changing variables we can reduce such a equation to simpler form and investigate the form by index boundedness of its solution. Further, we perform the inverse changing variables and obtain composition of analytic solutions of simpler equations and a mapping given by the changing variables. Therefore, we can apply the obtained results to such compositions and conclude about L-index boundedness in direction of primary equation for some function L and direction $\mathbf{b}$.

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