International Electronic Journal of Algebra
Published Online: March 2024
DOI: 10.24330/ieja.1449459

SIMPLE-SEPARABLE MODULES

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Received: 19 April 2023; Revised: 20 December 2023; Accepted: 26 December 2023
Communicated by A. Çiğdem Özcan

Abstract. A module M over a ring is called simple-separable if every simple submodule of M is contained in a finitely generated direct summand of M. While a direct sum of any family of simple-separable modules is shown to be always simple-separable, we prove that a direct summand of a simple-separable module does not inherit the property, in general. It is also shown that an injective module M over a right noetherian ring is simple-separable if and only if $M = M_1 \oplus M_2$ such that M_1 is separable and M_2 has zero socle. The structure of simple-separable abelian groups is completely described.

Mathematics Subject Classification (2020): 16D10, 16D40, 16D80
Keywords: Separable module, simple-separable module, V-ring, π-V-ring

1. Introduction

Throughout this article, R is an associative ring with an identity element and all modules considered are unital right R-modules unless stated otherwise. Let M be an R-module. By $E(M)$ we denote the injective hull of M. The notations $N \subseteq M$ and $N \leq M$ mean that N is a subset and N is a submodule of M, respectively. By $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ we denote the ring of rational numbers, the ring of integer numbers and the set of natural numbers, respectively. In 1937 [2], Baer introduced the notion of separable abelian groups to mean torsion-free abelian groups G for which every finite subset of G can be embedded in a completely decomposable direct summand of G. The first example given by Baer of groups satisfying this property was the direct product of countably many copies of $\mathbb{Z}$. In 1973 [9, p. 1], Fuchs called an abelian group G for which every finite subset of G can be embedded in a direct summand A of G such that A is a direct sum of groups of rank 1 a separable group. On the other hand, another version of separability was introduced in 1968 [10] by Griffith who called an abelian group G separable if every finitely generated subgroup of G is contained in a finitely generated direct summand of G. This variation of separable groups was extended by Zöschinger in 1979 [24] to the general module theoretic setting. Following Zöschinger, a module M over an arbitrary ring R is called separable if every finitely generated submodule of M is contained in a finitely generated direct summand of M.

In this paper, we study the "simple" version of separable modules. A module M is called simple-separable if every simple submodule of M is contained in a finitely generated direct summand of M. Note that this notion can also be considered as the dual of the notion of $\mathfrak{m}$-coseparable modules studied in [5]. In Section 2, we present some basic properties of these modules. It is shown that the property of being simple-separable is closed under direct sums, while a direct summand of a simple-separable module may not inherit the property. We investigate the class of rings R for which every injective R-module is simple-separable. We also prove that the class of commutative rings R for which every finitely cogenerated R-module is simple-separable is precisely that of the π-V-rings. Moreover, we determine the structure of simple-separable abelian groups. In Section 3, we shed some light on the modules M for which every direct summand of M is simple-separable. We conclude the paper by a short section on modules M for which every proper simple submodule of M is contained in a proper finitely generated direct summand.

2. Simple-separable modules

Definition 2.1. A module M is called *simple-separable* if every simple submodule of M is contained in a finitely generated direct summand of M.

Clearly, every separable module is simple-separable. However, the converse is not true, in general. To see this, we can consider the $\mathbb{Z}$-module $\mathbb{Q}$ which is simple-separable since $\mathbb{Q}$ has no simple submodules. On the other hand, $\mathbb{Q}$ is not separable since $\mathbb{Q}$ has no nonzero finitely generated direct summands.

Recall that a submodule N of a module M is called *small* in M (denoted by $N \ll M$) if $M \neq N + X$ for any proper submodule X of M.

Remark 2.2. Let M be an R-module. It is well known that a simple submodule of M is either small in M or a direct summand of M. It follows that M is simple-separable if and only if every simple small submodule of M is contained in a finitely generated direct summand of M. For example, if M is a module with $\mathrm{Rad}(M) = 0$, then M is a simple-separable module.

Example 2.3. (i) It is obvious that any module M with $\mathrm{Soc}(M) = 0$ is simple-separable.

(ii) It is clear that finitely generated modules are simple-separable. Also, any module which is a direct sum of finitely generated submodules (e.g., a free module) is simple-separable. On the other hand, note that a module with small radical need not be simple-separable (see Example 2.23).

(iii) Let a module $M = \sum_{i\in I} L_i$ such that $\{L_i \mid i \in I\}$ is a chain of finitely generated direct summands of the module M. It is clear that M is separable and hence M is simple-separable.

(iv) It is easily seen that for any separable module M_1 and any module M_2 with $\mathrm{Soc}(M_2) = 0$, the module $M = M_1 \oplus M_2$ is simple-separable.

The proof of the next result is straightforward and hence is omitted.

Proposition 2.4. *Let M be an indecomposable module. Then the following statements are equivalent:*

(i) *M is simple-separable;*

(ii) *$Soc(M) = 0$ or M is finitely generated.*

The following corollary is an immediate consequence of Proposition 2.4.

Corollary 2.5. Let S be a simple module. Then $E(S)$ is simple-separable if and only if $E(S)$ is a finitely generated module.

In the following example, we present some indecomposable simple-separable modules. Moreover, we provide an example of a simple-separable module which has a factor module which is not simple-separable.

Example 2.6. (i) Let p be a prime number. From Proposition 2.4, it follows that the indecomposable $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is not simple-separable but the indecomposable $\mathbb{Z}$-modules $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{Z}/p^k\mathbb{Z}$ $(k \in \mathbb{N})$ are simple-separable.

(ii) Let p be a prime number. Then the $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is not simple-separable. On the other hand, there exists a free $\mathbb{Z}$-module F such that $\mathbb{Z}(p^\infty) \cong F/L$ for some submodule L of F. It is clear that F is simple-separable since F is a direct sum of cyclic submodules.

Next, we will be concerned with direct summands of simple-separable modules. We begin by providing an example which shows that being simple-separable is not preserved by taking direct summands.

Example 2.7. It was shown in [16, Proposition 3.3] that there is a cyclic artinian module M over a ring R and a direct summand N of $M^{(\mathbb{N})}$ such that N has no nonzero finitely generated direct summands. Since M is artinian, its socle is essential. Therefore $M^{(\mathbb{N})}$ has an essential socle by [1, Propositions 6.17 and 9.19]. This implies that $\mathrm{Soc}(N) \neq 0$. It follows that N is not simple-separable.

A module M is called a *D*3*-module* if for every pair (M_1, M_2) of direct summands of M with $M_1 + M_2 = M$, $M_1 \cap M_2$ is also a direct summand of M. It is well known that quasi-projective modules are D3-modules (see [17, Proposition 4.38]).

In contrast to Example 2.7, we next exhibit some sufficient conditions under which some special direct summands of a simple-separable module inherit the property.

Proposition 2.8. *Let M be a simple-separable R-module such that $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2 where M_2 is finitely generated and semisimple. Assume that one of the following conditions is satisfied:*

(i) *M is a $D3$-module, or*

(ii) *M_2 is projective.*

Then M_1 is simple-separable.

Proof. Note that M_2 is a finite direct sum of simple submodules. Then by induction, we can assume that M_2 is a simple module. Now to prove that M_1 is simple-separable, assume that $Soc(M_1) \neq 0$ and let S_1 be a simple submodule of M_1. Since M is simple-sparable, there exists a finitely generated direct summand K of M such that $S_1 \subseteq K$. If $K \subseteq M_1$, we are done. Suppose now that K is not contained in M_1. Then $K + M_1 = M$ as M_1 is a maximal submodule of M.

(i) Since M is a $D3$-module, $K \cap M_1$ is a direct summand of M. Therefore $K \cap M_1$ is a direct summand of K.

(ii) Note that $M_2 \cong M/M_1 = (K + M_1)/M_1 \cong K/(K \cap M_1)$ is projective. Then $K \cap M_1$ is a direct summand of K (see [1, Proposition 17.2]).

Hence $K \cap M_1$ is a finitely generated direct summand of M_1 such that $S_1 \subseteq K \cap M_1$. It follows that M_1 is a simple-separable module. □

Recall that a submodule N of a module M is called *fully invariant* if $f(N) \subseteq N$ for every endomorphism f of M. A module M is said to have the *SIP* (Summand Intersection Property) if the intersection of any two direct summands of M is again a direct summand of M.

Proposition 2.9. *Let N be a submodule of a simple-separable R-module M. Assume that one of the following conditions is satisfied:*

(i) *N is a direct summand of M and M has the SIP, or*

(ii) *N is a direct summand of M and $K \cap N$ is a direct summand of M for every finitely generated direct summand K of M, or*

(iii) *N is a fully invariant direct summand of M, or*

(iv) *R is a right noetherian ring and N is a fully invariant submodule of M.*

Then N is a simple-separable module.

Proof. Let N be a submodule of M and let S be a simple submodule of N. Since M is simple-separable, there exists a finitely generated direct summand K of M such that $S \subseteq K$ and $M = K \oplus K'$ for some submodule K' of M. Moreover, $S \subseteq K \cap N$. The proof is completed by showing that $K \cap N$ is a direct summand of N which is finitely generated.

(i)-(ii) Suppose that N is a direct summand of M. By hypothesis, $K \cap N$ is a direct summand of M and hence of K. Therefore $K \cap N$ is a finitely generated direct summand of N.

To prove (iii)-(iv), note that $N = (K \cap N) \oplus (K' \cap N)$ since N is fully invariant in M.

(iii) As N is a direct summand of M, $K \cap N$ is a direct summand of K and so $K \cap N$ is finitely generated.

(iv) Since R is right noetherian, K is a noetherian module and so $K \cap N$ is finitely generated. This proves the proposition. □

Next, we will show that being simple-separable is preserved under direct sums.

Theorem 2.10. *Every direct sum of simple-separable modules is simple-separable.*

Proof. First note that without loss of generality, we can only prove the result for a finite direct sum of simple-separable modules. Let a module $M = M_1 \oplus M_2$ be a direct sum of simple-separable submodules M_1 and M_2. Let S be a simple submodule of M. If $S \subseteq M_i$ for some $i \in \{1, 2\}$, then clearly S is contained in a finitely generated direct summand of M. Now suppose that $S \cap M_1 = S \cap M_2 = 0$. Then $S \oplus M_1 = M_1 \oplus [(S \oplus M_1) \cap M_2]$. Hence $(S \oplus M_1) \cap M_2$ is a simple submodule of M_2. Since M_2 is simple-separable, there exists a finitely generated direct summand K_2 of M_2 such that $(S \oplus M_1) \cap M_2 \subseteq K_2$. Thus $S \oplus M_1 \subseteq M_1 \oplus K_2$. On the other hand, $S \oplus M_2 = [(S \oplus M_2) \cap M_1] \oplus M_2$. Hence $(S \oplus M_2) \cap M_1$ is a simple submodule of M_1. Since M_1 is simple-separable, there exists a finitely generated direct summand K_1 of M_1 such that $(S \oplus M_2) \cap M_1 \subseteq K_1$. Thus $S \oplus M_2 \subseteq K_1 \oplus M_2$. Therefore

$$S \subseteq (S \oplus M_1) \cap (S \oplus M_2) \subseteq (M_1 \oplus K_2) \cap (K_1 \oplus M_2) = K_1 \oplus K_2.$$

Note that $K_1 \oplus K_2$ is a finitely generated direct summand of M. The result follows. □

Corollary 2.11. *Let N be a fully invariant direct summand of a module M. Then the following conditions are equivalent:*

(i) *M is simple-separable;*

(ii) *N and M/N are both simple-separable modules.*

Proof. (i) $\Rightarrow$ (ii) First note that N is simple-separable by Proposition 2.9(iii). To prove that M/N is simple-separable, let U be a submodule of M such that $N \subseteq U$ and U/N is simple. By hypothesis, there exists a submodule K of M such that $M = N \oplus K$. Then $U = N \oplus (U \cap K)$ and $S = U \cap K$ is simple. Since M is simple-separable, there exist submodules A and B of M such that $M = A \oplus B$, A is finitely generated and $S \subseteq A$. As N is fully invariant in M, we have

$$M/N = [(A + N)/N] \oplus [(B + N)/N].$$

Moreover, $U/N \subseteq (A + N)/N$ and $(A + N)/N \cong A/(A \cap N)$ is finitely generated.

(ii) $\Rightarrow$ (i) This follows from Theorem 2.10. □

Recall that a module M is called *separable* if every finitely generated submodule of M is contained in a finitely generated direct summand of M. Next, we investigate simple-separable injective modules.

Proposition 2.12. *The following are equivalent for an injective R-module M:*

(i) *M is a simple-separable module;*

(ii) *Either $Soc(M) = 0$ or $E(S)$ is finitely generated for any simple submodule S of M.*

If, moreover, R is right noetherian, then (i)-(ii) are equivalent to:

(iii) *$M = (\oplus_{i \in I} M_i) \oplus N$ such that each M_i is an indecomposable finitely generated submodule of M and $Soc(N) = 0$;*

(iv) *$M = (\oplus_{i \in I} M_i) \oplus N$ such that each M_i is a finitely generated submodule of M and $Soc(N) = 0$;*

(v) *$M = L \oplus N$ such that L is a separable submodule of M and $Soc(N) = 0$.*

Proof. (i) $\Rightarrow$ (ii) Assume that $\mathrm{Soc}(M) \neq 0$ and let S be a simple submodule of M. By (i), there exist submodules K and K' of M such that $M = K \oplus K'$, $S \subseteq K$ and K is finitely generated. Since K is injective, $E(S)$ is a direct summand of K. Hence $E(S)$ is finitely generated.

(ii) $\Rightarrow$ (i) This is immediate.

(ii) $\Rightarrow$ (iii) Since M is injective, there exists a submodule $N \leq M$ such that $M = E(\mathrm{Soc}(M)) \oplus N$. Set $\mathrm{Soc}(M) = \oplus_{i \in I} S_i$ where S_i ($i \in I$) are simple submodules of M. Then $M = (\oplus_{i \in I} E(S_i)) \oplus N$ since R is right noetherian (see [1, Proposition 18.13]). Moreover, note that each $E(S_i)$ $(i \in I)$ is a finitely generated indecomposable submodule of M and $\mathrm{Soc}(N) = 0$.

(iii) $\Rightarrow$ (iv) This is obvious.

(iv) $\Rightarrow$ (v) This follows from the fact that any module which is a direct sum of finitely generated submodules is separable.

(v) $\Rightarrow$ (i) This follows from Theorem 2.10. □

Following Caldwell's terminology in [3], a ring R is called *hypercyclic* if each cyclic right R-module has a cyclic injective hull. It was shown in [7, Theorems 4.1 and 4.2] that any artinian principal ideal ring is hypercyclic (see also [3, Theorem 1.5]). Commutative hypercyclic rings are characterized in [3]. From Proposition 2.12, we infer that every injective module over a hypercyclic ring is simple-separable.

In the next two corollaries, we describe simple-separable injective modules over commutative domains and over right artinian rings.

Corollary 2.13. *Let R be a commutative domain which is not a field. Then the following are equivalent for an injective R-module M:*

(i) *M is a simple-separable R-module;*
(ii) *$Soc(M) = 0$.*

Proof. Let E be an injective R-module. It is clear that E is divisible and hence $\mathrm{Rad}(E) = E$.

(i) $\Rightarrow$ (ii) Suppose that $\mathrm{Soc}(M) \neq 0$. Then M contains a simple submodule S. By Proposition 2.12, $E(S)$ is finitely generated. This contradicts the fact that $E(S)$ has no maximal submodules (see also [11, Corollary 2]).

(ii) $\Rightarrow$ (i) This is clear. □

Corollary 2.14. *Let M be an injective module over a right artinian ring R. Then the following are equivalent:*

(i) *M is a simple-separable R-module;*
(ii) *M is a direct sum of finitely generated submodules.*

Proof. This follows from Proposition 2.12 and [22, Theorem 4.5]. □

As exhibited in Example 2.6, for any prime number p, $\mathbb{Z}(p^\infty) \cong E(\mathbb{Z}/p\mathbb{Z})$ is not simple-separable. Next, we will be concerned with the class of rings R for which every injective R-module is simple-separable.

Proposition 2.15. *The following are equivalent for a ring R:*

(i) *Every injective R-module is simple-separable;*

(ii) *$E(S)$ is simple-separable for any simple R-module S;*

(iii) *$E(S)$ is finitely generated for any simple R-module S.*

Proof. (i) $\Rightarrow$ (ii) This is immediate.

(ii) $\Rightarrow$ (iii) By Corollary 2.5.

(iii) $\Rightarrow$ (i) Let M be an injective R-module and let S be a simple submodule of M. By (iii), $E(S)$ is a finitely generated direct summand of M which contains S. Therefore M is simple-separable. This completes the proof. □

Recall that a module M is said to be *finitely cogenerated* (or *finitely embedded*) if for any family of submodules $\{N_i : i \in I\}$ in M, if $\cap_{i\in I} N_i = 0$, then $\cap_{i\in J} N_i = 0$ for some finite subset $J \subseteq I$. This is equivalent to the fact that $E(M) \cong E(S_1) \oplus E(S_2) \oplus \cdots \oplus E(S_k)$ for some finitely many simple modules S_1, S_2, ..., S_k (see [14, Proposition 19.1] and [22, p. 70]).

Corollary 2.16. *Let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$. Then the following are equivalent:*

(i) *Every injective R-module is simple-separable;*

(ii) *$E(R/\mathfrak{m})$ is finitely generated;*

(iii) *R is an artinian ring.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from Proposition 2.15.

(ii) $\Rightarrow$ (iii) Using Proposition 2.15, we see that $E(R/\mathfrak{m})$ is finitely generated. Since R is noetherian, it follows that every finitely cogenerated R-module is finitely generated. Thus R is an artinian ring by [23, Theorem 3].

(iii) $\Rightarrow$ (ii) This follows by using again [23, Theorem 3]. □

Remark 2.17. Not every two-sided artinian ring satisfies the conditions in Proposition 2.15. In fact, even a two-sided artinian ring R may have a simple right R-module S such that $E(S)$ is not finitely generated as illustrated in an example constructed in [15, Ex. 3.34] (see also [21, Theorem 2]).

A ring R is called a left (right) π-V-ring if, for every simple left (right) R-module S, the injective hull $E(S)$ is of finite length (see [12]). Note that left and right artinian PI-rings and quasi-Frobenius rings are left and right π-V-rings by [21, p. 372] (see also [20, Lemma 6 and Proposition 10]).

Example 2.18. Using Proposition 2.15, it follows that over any right π-V-ring R (e.g., we can take R to be any commutative artinian ring), every injective R-module is simple-separable.

Next, we characterize the class of rings R for which every finitely cogenerated R-module is simple-separable. First we need the following lemma.

Lemma 2.19. *Let S be a simple module. Then the following are equivalent for $M = E(S)$:*

(i) *Every submodule of M is simple-separable;*

(ii) *M is a noetherian module.*

If, moreover, R is commutative, then (i)-(ii) *are equivalent to:*

(iii) *M has finite length.*

Proof. (i) $\Rightarrow$ (ii) Let U be a nonzero submodule of M. Note that M is a uniform module with essential socle. Then U is indecomposable and $\mathrm{Soc}(U) \neq 0$. Since U is simple-separable, U is finitely generated by Proposition 2.4. Therefore M is a noetherian module.

(ii) $\Rightarrow$ (i) This is clear.

(ii) $\Leftrightarrow$ (iii) Clearly, M is finitely cogenerated. The equivalence follows from [23, Proposition 4]. □

Proposition 2.20. *The following statements are equivalent for a commutative ring R:*

(i) *Every finitely cogenerated R-module is simple-separable;*

(ii) *R is a π-V-ring;*

(iii) *$R_{\mathfrak{m}}$ is an artinian ring for every maximal ideal $\mathfrak{m}$ of R.*

Proof. (i) $\Rightarrow$ (ii) This follows by using Lemma 2.19.

(ii) $\Rightarrow$ (iii) This follows from [21, Theorem 5].

(iii) $\Rightarrow$ (i) Let M be a finitely cogenerated R-module. From [23, Theorem 3], we infer that M is finitely generated. Hence M is simple-separable. This completes the proof. □

The next result is presumably well known but is included for completeness.

Lemma 2.21. *Let R be a commutative semilocal ring such that $R_{\mathfrak{m}}$ is an artinian ring for every maximal ideal $\mathfrak{m}$ of R. Then R is an artinian ring.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of R and put $S_{\mathfrak{m}} = R \setminus \mathfrak{m}$. Let $I_1 \supseteq I_2 \supseteq \dots$ be a descending chain of ideals of R. Since $R_{\mathfrak{m}}$ is an artinian ring, there exists an integer $n_{\mathfrak{m}} \geq 1$ such that $S_{\mathfrak{m}}^{-1} I_{n_{\mathfrak{m}}} = S_{\mathfrak{m}}^{-1} I_{n_{\mathfrak{m}}+i}$ for each $i \geq 1$. But R has only finitely many maximal ideals. So let n be the maximum of all the integers $n_{\mathfrak{m}}$'s (where $\mathfrak{m}$ ranges over all of the maximal ideals of R). It follows that $S_{\mathfrak{m}}^{-1} I_n = S_{\mathfrak{m}}^{-1} I_{n+i}$ for every maximal ideal $\mathfrak{m}$ of R and all $i \geq 1$. This implies that $S_{\mathfrak{m}}^{-1}(I_n/I_{n+i}) = 0$ for every maximal ideal $\mathfrak{m}$ of R and all $i \geq 1$. Consequently, $I_n = I_{n+i}$ for all $i \geq 1$. This shows that R is artinian. □

Corollary 2.22. *The following are equivalent for a commutative semilocal ring R:*

(i) *Every finitely cogenerated R-module is simple-separable;*

(ii) *R is an artinian ring.*

Proof. (i) ⇒ (ii) This is obvious by Proposition 2.20 and Lemma 2.21.

(ii) ⇒ (i) This is clear by Proposition 2.20. □

In the next example we provide a module with small radical which is not simple-separable. This example also shows that both Corollaries 2.16 and 2.22 are not true, in general, if R is not a commutative ring.

Example 2.23. Let $K = F(x_1, x_2, \ldots)$ with F a field. Consider the field monomorphism $\sigma : K \to K$ defined by $\sigma(x_i) = x_{i+1}$ for all i and σ is equal to the identity on F. Then $R = K \times K$ with coordinate-wise addition and multiplication $(x, y)(x', y') = (xx', xy' + \sigma(x')y)$ is a ring with identity. It is shown in [21, p. 375] that R is a local left artinian ring with maximal left ideal $L = \{0\} \times K$ such that the left R-module $E(R/L)$ is not of finite length. This implies that $E(R/L)$ is not finitely generated since R is left artinian. Now Proposition 2.12 shows that $E(R/L)$ is not simple-separable. On the other hand, note that $\mathrm{Rad}(E(R/L)) \ll E(R/L)$ by [1, Corollary 15.21].

Remark 2.24. There exist some commutative rings which satisfy the conditions in Proposition 2.15 but do not satisfy the statements in Proposition 2.20. For example, consider the ring R constructed in [3, Example p. 42]. In fact, R is a commutative local nonartinian hypercyclic ring. So every injective R-module is simple-separable by Proposition 2.12. On the other hand, it follows from Corollary 2.22 that not every finitely cogenerated R-module is simple-separable.

The following result shows that a simple-separable module M with $\mathrm{Rad}(M) = M$ contains no simple submodules.

Proposition 2.25. *Let M be a nonzero module with $Rad(M) = M$. Then M is simple-separable if and only if $Soc(M) = 0$.*

Proof. (⇒) Assume that $\mathrm{Soc}(M) \neq 0$ and let S be a simple submodule of M. Since M is simple-separable, there exists a finitely generated direct summand K of M such that $S \subseteq K$ and $M = K \oplus K'$ for some submodule K' of M. Hence K

contains a maximal submodule U with $S \subseteq U$. It is easily seen that $U \oplus K'$ is a maximal submodule of M, a contradiction.

($\Leftarrow$) This implication is immediate. □

Let G be an abelian group. We denote the torsion subgroup of G by $T(G)$. For any prime number p, let $T_p(G) = \{x \in G \mid p^n x = 0$ for some non-negative integer $n\}$ which is a subgroup of G called the p-primary component of G. Note that if G is a torsion abelian group, then G is a direct sum of its p-primary components. An abelian group G is said to be a primary group (or p-group) if $G = T_p(G)$ for some prime p.

Let G be an abelian p-group (for some prime p), $x \in G$, and n be a non-negative integer. Then x is said to have *height* n if x is divisible by p^n but not by p^{n+1} (i.e. $x \in p^n G$ but $x \notin p^{n+1}G$). In this case, we write $h(x) = n$. If x is divisible by p^k for every non-negative integer k (i.e. $x \in \cap_{k\geq 1} p^k G$), then x is called an *element of infinite height* and we write $h(x) = \infty$. If x is an element of a subgroup U of G, then we can define two heights for x. When it is necessary, we will write $h_U(x)$ and $h_G(x)$ for the height of x in U and G, respectively. We always have $h_U(x) \leq h_G(x)$.

Recall that a subgroup U of an abelian group G is called *pure* if $nU = U \cap nG$ for every non-negative integer n. An abelian group G is said to be of *bounded* if $nG = 0$ for some positive integer n.

In the next theorem, we determine the structure of simple-separable abelian groups. First, we give the following four lemmas. The proof of the second one is adapted from that of [13, Theorem 9] (see also [8, Corollary 27.2]).

Lemma 2.26. *Let K be a finitely generated subgroup of an abelian group G with $K \subseteq T(G)$. Then K is a direct summand of $T(G)$ if and only if K is a direct summand of G.*

Proof. The sufficiency follows by modularity. Conversely, suppose that K is a direct summand of $T(G)$. Then K is a pure subgroup of $T(G)$ which is itself a pure subgroup of G. Thus K is pure in G. Moreover, note that K is a direct sum of a finite number of finite cyclic abelian groups since K is finitely generated and $K \subseteq T(G)$ (see [8, Theorem 15.5]). Hence K is bounded. Now using [13, Theorem 7], we conclude that K is a direct summand of G. □

Lemma 2.27. *Let G be an abelian group such that $\cap_{n\geq 1} p^n T_p(G) = 0$ for every prime number p. Then every simple subgroup of G is contained in a finite cyclic primary direct summand of G.*

Proof. Suppose that $\mathrm{Soc}(G) \neq 0$ and let S be a simple subgroup of G. Then there exist a prime number p and $0 \neq x \in G$ such that $S = \mathbb{Z}x \cong \mathbb{Z}/p\mathbb{Z}$. Since $\cap_{n\geq 1} p^n T_p(G) = 0$, it follows that the subgroup $U = T_p(G)$ has no elements of infinite height. Therefore x has finite height in U. Let $h_U(x) = m$ for some non-negative integer m. Then there exists $y \in U$ such that $x = p^m y$. Put $H = \mathbb{Z}y$. Clearly, $S \subseteq H$ and $H \cong \mathbb{Z}/p^{m+1}\mathbb{Z}$ is primary. It is easily seen that the only elements of order p in H are the multiples of x by integers prime to p. So these elements have the same height in H as in U. Thus H is a pure subgroup of U by [13, Lemma 7]. Note that $p^{m+1}H = 0$. It follows from [13, Theorem 7] that H is a direct summand of U. But U is a direct summand of $T(G)$, so H is a direct summand of $T(G)$. From Lemma 2.26, it follows that H is a direct summand of G. This completes the proof. □

Lemma 2.28. *Let G be a simple-separable abelian group. Then $\cap_{n\geq 1} p^n T_p(G) = 0$ for every prime number p.*

Proof. Assume that $\cap_{n\geq 1} p^n(T_p(G)) \neq 0$ for some prime number p. Then there exists in $\cap_{n\geq 1} p^n(T_p(G))$ a nonzero element x of order p. Clearly, $\mathbb{Z}x$ is a simple subgroup of G. Since G is simple-separable, there exists a decomposition $G = K \oplus L$ such that K is finitely generated and $\mathbb{Z}x \subseteq K$. Note that $\mathbb{Z}x \subseteq T_p(K)$. Moreover, since $T(K)$ is finitely generated, there exists an integer $n \geq 1$ such that $nT(K) = 0$. But $T(K)$ is a pure subgroup of K, so $T(K)$ is a direct summand of K by [13, Theorem 7]. Note that $T_p(K)$ is a direct summand of $T(K)$. Then $T_p(K)$ is a direct summand of K which is finitely generated. Therefore there exists an integer $s \geq 1$ such that $p^s T_p(K) = 0$. Since $x \in \cap_{n\geq 1} p^n(T_p(G))$, we have $x = p^s y$ for some $y \in T_p(G)$. Now, since $G = K \oplus L$, $y = a + b$ for some $a \in K$ and $b \in L$. Clearly, $a \in T_p(K)$. Therefore $p^s a \in p^s T_p(K) = 0$ and hence $x = p^s b \in L$. But $x \in K$, so $x \in K \cap L = 0$, a contradiction. □

Lemma 2.29. *Let G be a torsion abelian group. Then G is separable if and only if G is simple-separable.*

Proof. The necessity is obvious. Conversely, suppose that G is simple-separable and let A be a finitely generated subgroup of G. Clearly $A = \oplus_{i=1}^n T_{p_i}(A)$ for some positive integer n and distinct prime numbers p_i $(1 \leq i \leq n)$. Note that for every $1 \leq i \leq n$, $T_{p_i}(A)$ is a finitely generated subgroup of $T_{p_i}(G)$. Since each $T_{p_i}(G)$ is a fully invariant direct summand of G, it follows from Proposition 2.9 that each $T_{p_i}(G)$ is a simple-separable abelian group. Moreover, $\oplus_{i=1}^n T_{p_i}(G)$ is a direct summand of G. The proof is completed by showing that each $T_{p_i}(A)$ is contained in a finitely generated direct summand of $T_{p_i}(G)$. So there is no loss of generality in assuming that G is a p-group for some prime number p. Since A is finitely generated, A is a finite direct sum of finite cyclic subgroups. This implies that A itself is a finite group. Since G has no nonzero elements of infinite height by Lemma 2.28, it follows that the heights of the nonzero elements of A (relative to G) are bounded. Applying [8, Corollary 27.8], we see that $A \subseteq B$ for some bounded direct summand B of G. Note that B is a direct sum of finite cyclic subgroups by [8, Theorem 17.2]. Since A is finitely generated, there exist subgroups B_1 and B_2 of B such that $B = B_1 \oplus B_2$, B_1 is finitely generated and $A \subseteq B_1$. It is clear that B_1 is a direct summand of G. This finishes the proof. □

The next result should be compared with [9, Proposition 65.1] which characterized reduced abelian p-groups satisfying another variation of separability.

Theorem 2.30. *The following are equivalent for an abelian group G:*

(i) *G is simple-separable;*

(ii) *For every prime number p, $\cap_{n \geq 1} p^n(T_p(G)) = 0$ (i.e. $T_p(G)$ has no nonzero elements of infinite height);*

(iii) *Every simple subgroup of G is contained in a finite cyclic primary direct summand of G;*

(iv) *$T(G)$ is simple-separable;*

(v) *$T(G)$ is separable.*

Proof. (i) ⇒ (ii) This implication is proved in Lemma 2.28.

(ii) ⇒ (iii) This is clear by Lemma 2.27.

(iii) ⇒ (iv) This follows immediately from Lemma 2.26 and the fact that a cyclic abelian group is either torsion or torsion-free.

(iv) ⇒ (v) This follows from Lemma 2.29.

(v) ⇒ (i) This is an immediate consequence of Lemma 2.26. □

3. Completely simple-separable modules

Motivated by Example 2.7, we introduce the following notion.

Definition 3.1. A module M is called *completely simple-separable* if every direct summand of M is simple-separable.

Recall that a module M is called a *duo module* (resp., *weak duo module*) if every submodule (resp., every direct summand) of M is fully invariant (see for example [19]).

Example 3.2. (i) It is clear that every module M with $Soc(M) \cap Rad(M) = 0$ is completely simple-separable.

(ii) Every finitely generated module is completely simple-separable.

(iii) Using Proposition 2.9(iii), we see that every simple-separable weak duo module is completely simple-separable.

(iv) Let R be a semiperfect ring or a simple right noetherian ring or a one-sided semihereditary ring or a one-sided principal ideal ring. Then every projective R-module is a direct sum of finitely generated submodules by [18, Theorem 3] and [16, Fact 3.4, Corollary 5.5 and Proposition 6.3]. It follows that every projective R-module is completely simple-separable.

Proposition 3.3. *Let R be a ring and let M be a completely simple-separable R-module. Assume that M has the ascending chain condition (ACC) on finitely generated direct summands (e.g., M is noetherian). Then $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2 such that $Soc(M_1) = 0$ and M_2 is finitely generated.*

Proof. Suppose, to the contrary, that the module M does not have such a decomposition. Then $\mathrm{Soc}(M) \neq 0$. Let S_1 be a simple submodule of M. Since M is simple-separable, there exists a finitely generated direct summand K_1 of M such that $S_1 \subseteq K_1$. Let N_1 be a submodule of M such that $M = K_1 \oplus N_1$. Note that N_1 is simple-separable and $\mathrm{Soc}(N_1) \neq 0$. By similar arguments as before, it follows that $N_1 = K_2 \oplus N_2$ such that K_2 is finitely generated and N_2 is a simple-separable submodule with $\mathrm{Soc}(N_2) \neq 0$. By continuing this process, we get a strictly ascending chain of finitely generated direct summands $K_1 \subsetneq K_1 \oplus K_2 \subsetneq \cdots$ of M. This contradicts our assumption. □

Recall that a module M is said to have finite uniform dimension if M does not contain an infinite independent set of submodules. Dually, a module M is said to have finite hollow dimension if M does not contain an infinite coindependent family of submodules; that is, for some $n \in \mathbb{N}$, there exists an epimorphism from M to a direct sum of n nonzero modules but no epimorphism from M to a direct sum of more than n nonzero modules (see, for example [4, p. 47]).

It is well known that a module M has ACC on direct summands if and only if $S = End_R(M)$ has ACC on right direct summands if and only if S contains no infinite set of nonzero orthogonal idempotents (see e.g., [5, Lemma 3.12]). Next, we present some sufficient conditions for a module to satisfy ACC on direct summands.

Remark 3.4. Let R be a ring and let M be an R-module. Then M has the ACC on direct summands when one of the following conditions holds.

(i) M is artinian or noetherian (see [1, Proposition 10.14]);
(ii) M has either finite uniform dimension or finite hollow dimension (see [4, 5.3] and [14, Proposition (6.30)′]);
(iii) $End_R(M)$ is a semilocal ring (see [4, 5.3 and Corollary 18.7]).

In the following two results, we provide more examples of completely simple-separable modules.

Proposition 3.5. *Every injective simple-separable R-module is completely simple-separable.*

Proof. This follows directly from Proposition 2.12. □

Proposition 3.6. *If G is a simple-separable abelian group, then so is every subgroup of G. In particular, G is completely simple-separable.*

Proof. Let G be a simple-separable abelian group. From Theorem 2.30, we see that $\cap_{n\geq 1}p^n(T_p(G)) = 0$ for all primes p. This implies that $\cap_{n\geq 1}p^n(T_p(N)) = 0$ for any subgroup N of G and for all primes p. Now the result follows by using again Theorem 2.30. □

Proposition 3.7. *Let M be an artinian module. Then M is completely simple-separable if and only if M is finitely generated.*

Proof. The sufficiency is clear. Conversely, assume that M is completely simple-separable. From Proposition 3.3, we conclude that $M = M_1 \oplus M_2$ such that $Soc(M_1) = 0$ and M_2 is finitely generated. As M is artinian, $Soc(M) = Soc(M_2)$ is an essential submodule of M. This yields $M_1 = 0$. The result follows. □

Proposition 3.8. *Let M be a completely simple-separable module. Then any finitely generated semisimple submodule of M is contained in a finitely generated direct summand of M.*

Proof. Let n be a positive integer. We will prove that every semisimple submodule of M having uniform dimension n is contained in a finitely generated direct summand of M. This is clearly true for $n = 1$. Now assume that $n \geq 2$ and every semisimple submodule of M having uniform dimension $n-1$ is contained in a finitely generated direct summand of M. Let $U = S_1 \oplus S_2 \oplus \cdots \oplus S_n$ be a submodule of M which is a direct sum of n simple submodules S_i $(1 \leq i \leq n)$. By hypothesis, $S_1 \oplus S_2 \oplus \cdots \oplus S_{n-1}$ is contained in a finitely generated direct summand K of M. Hence $M = K \oplus N$ for some submodule N of M. If $S_n \subseteq K$, then

$U \subseteq K$. Suppose now that $S_n \nsubseteq K$. In this case we have $S_n \cap K = 0$ and hence $S_n \oplus K = K \oplus [(S_n \oplus K) \cap N]$. Therefore $(S_n \oplus K) \cap N$ is a simple submodule of N. Since N is simple-separable, there exists a finitely generated direct summand L of N such that $(S_n \oplus K) \cap N \subseteq L$. Hence $U \subseteq K \oplus [(S_n \oplus K) \cap N] \subseteq K \oplus L$ and $K \oplus L$ is a finitely generated direct summand of M. This completes the proof. □

The following corollary is an immediate consequence of Proposition 3.8.

Corollary 3.9. *If M is a completely simple-separable module such that $Soc(M)$ is finitely generated, then $M = N \oplus K$ is a direct sum of submodules N and K such that $Soc(N) = 0$ and K is finitely generated.*

Remark 3.10. The module M of Example 2.7 shows also that an infinite direct sum of completely simple-separable modules need not be completely simple-separable.

The next result deals with a special case of direct sums of two completely simple-separable modules.

Proposition 3.11. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2 such that M_1 is completely simple-separable and M_2 is semisimple. Assume that one of the following conditions is satisfied:*

(i) *M_2 is projective, or*
(ii) *M_2 is finitely generated and M is a D3-module.*

Then M is completely simple-separable.

Proof. Note first that every direct summand of a D3-module is also a D3-module by [17, Lemma 4.7]. Thus by induction it is sufficient to prove (ii) when M_2 is a simple module. To prove the result, let N be a direct summand of M and let S be a simple submodule of N. We need only consider two cases:
Case 1: Assume that S is not contained in M_1. Then $S \oplus M_1 = M_1 \oplus [(S \oplus M_1) \cap M_2]$ is a direct summand of M since M_2 is semisimple. Hence S is a direct summand of N.
Case 2: Assume that $S \subseteq M_1$. Then $S \subseteq N \cap M_1$. If we prove that $N \cap M_1$ is a direct summand of M, the assertion follows. Indeed, in this case $N \cap M_1$ is a direct summand of M_1. This implies that $N \cap M_1$ is simple-separable since M_1 is completely simple-separable. Therefore there exists a finitely generated direct summand K of $N \cap M_1$ such that $S \subseteq K$. Clearly, K is a direct summand of N.

(i) Note that $N + M_1 = M_1 \oplus [(N + M_1) \cap M_2]$ and hence $N/(N \cap M_1) \cong (N+M_1)/M_1 \cong (N+M_1) \cap M_2$. Since $(N+M_1) \cap M_2$ is a direct summand of M_2, $(N + M_1) \cap M_2$ is projective. Therefore $N \cap M_1$ is a direct summand of M.

(ii) Suppose that M_2 is a simple module. If $N \subseteq M_1$, then $N \cap M_1 = N$ is a direct summand of M. Now assume that N is not contained in M_1. Then $N + M_1 = M$ since M_1 is a maximal submodule of M. As M is a D3-module, it follows that $N \cap M_1$ is a direct summand of M. This completes the proof. □

4. Strongly simple-separable modules

In this section, we introduce the following stronger form of simple-separability.

Definition 4.1. A module M is called *strongly simple-separable* if every proper simple submodule of M is contained in a proper finitely generated direct summand of M.

Note that the above notion can be considered as the "simple" version of the concept of $\mathcal{A}$-separable modules (see [6]).

Example 4.2.
- (i) It is easily seen that for any finitely generated module M_1 and any nonzero module M_2 with $\mathrm{Soc}(M_2) = 0$, the module $M = M_1 \oplus M_2$ is strongly simple-separable.
- (ii) Every regular module M (i.e., every cyclic submodule of M is a direct summand) is strongly simple-separable. In particular, every semisimple module is strongly simple-separable.
- (iii) If R is a right V-ring, then every R-module is strongly simple-separable since every simple R-module is injective.

Remark 4.3. *If a module M is not finitely generated, then M is strongly simple-separable if and only if M is simple-separable.*

The proof of the following proposition is straightforward.

Proposition 4.4. *Let M be an indecomposable module. Then the following conditions are equivalent:*

- (i) *M is strongly simple-separable;*
- (ii) *$Soc(M) = 0$ or M is a simple module.*

Remark 4.5. Let S be a simple module. From the preceding proposition, it follows that $E(S)$ is strongly simple-separable if and only if S is an injective module.

Next, we provide an example to show that the class of simple-separable modules and the class of strongly simple-separable modules are different.

Example 4.6. Let R be a commutative local artinian ring which is not a field. Let $\mathfrak{m}$ be the maximal ideal of R. Clearly, R is not a V-ring and hence the R-module $R/\mathfrak{m}$ is not injective. Note that $E(R/\mathfrak{m})$ is a finitely generated R-module by [23, Theorem 3]. Then $E(R/\mathfrak{m})$ is simple-separable. On the other hand, $E(R/\mathfrak{m})$ is not strongly simple-separable by Remark 4.5. For example, we can take the ring $R = \mathbb{Z}/p^n\mathbb{Z}$ for some prime number p and some integer $n \geq 2$. Note that in this case $S = p^{n-1}\mathbb{Z}/p^n\mathbb{Z}$ is the unique simple R-module (up to isomorphism). Moreover, $E(S) = R$ (see [22, Theorem 6.7]).

Proposition 4.7. *The following are equivalent for a ring R:*

(i) *Every R-module is strongly simple-separable;*
(ii) *Every injective R-module is strongly simple-separable;*
(iii) *Every finitely cogenerated R-module is strongly simple-separable;*
(iv) *R is a right V-ring.*

Proof. This follows from Example 4.2(iii) and Remark 4.5. □

In the next example, we show that the strongly simple-separable property does not always transfer from a module to each of its direct summands.

Example 4.8. (i) Let $M = \oplus_{i\geq 1} M_i$ be a direct sum of nonzero nonsimple indecomposable finitely generated submodules M_i $(i \geq 1)$ such that $\mathrm{Soc}(M_{i_0}) \neq 0$ for some $i_0 \geq 1$ (for example, for each $i \geq 1$, we can take M_i to be the $\mathbb{Z}$-module $\mathbb{Z}/p_i^{n_i}\mathbb{Z}$ where p_i is a prime number and $n_i \geq 2$ is an integer). It is clear that M is strongly simple-separable. On the other hand, using Proposition 4.4, it follows that M_{i_0} is not strongly simple-separable.

(ii) We can also consider the module $M^{(\mathbb{N})}$ given in Example 2.7. In fact, it is easily seen that $M^{(\mathbb{N})}$ is strongly simple-separable. But $M^{(\mathbb{N})}$ has a direct summand which is not simple-separable.

Proposition 4.9. *Every direct sum of strongly simple-separable modules is strongly simple-separable.*

Proof. The proof can be adapted from that of Theorem 2.10 by taking into account the fact that any semisimple module is strongly simple-separable. □

The following corollary is a direct consequence of Proposition 4.9.

Corollary 4.10. *The following conditions are equivalent for a ring R:*

(i) *The R-module R_R is strongly simple-separable;*
(ii) *Every free R-module is strongly simple-separable.*

In the next result, we characterize finitely generated duo strongly simple-separable modules.

Proposition 4.11. *Let M be a finitely generated duo R-module which is not simple. Then M is strongly simple-separable if and only if $Soc(M) = 0$ or M is not indecomposable.*

Proof. To prove the necessity, assume that $\mathrm{Soc}(M) \neq 0$ and let S be a simple submodule of M. Since M is strongly simple-separable and $S \neq M$, there exists a finitely generated proper direct summand K of M such that $S \subseteq K$. Hence M is not indecomposable as $K \neq 0$. Conversely, suppose that $M = A \oplus B$ for some proper nonzero submodules A and B of M. Let T be a simple submodule of M. Since M is duo, T is fully invariant in M. This implies that $T = (T \cap A) \oplus (T \cap B)$. Since T is simple, we have $T \subseteq A$ or $T \subseteq B$. This proves that M is strongly simple-separable. □

Recall that a ring R is called *right duo* if the right R-module R_R is duo. The next corollaries are direct consequences of Proposition 4.11.

Corollary 4.12. *Let R be a right duo ring which is not a division ring. Then the R-module R_R is strongly simple-separable if and only if $Soc(R_R) = 0$ or R has at least one non-trivial idempotent element.*

A prime ideal $\mathfrak{p}$ of a commutative ring R is said to be an associated prime ideal of an R-module M provided $\mathfrak{p} = Ann_R(x)$ for some nonzero element x of M. The set of associated prime ideals of M is denoted by $Ass(M)$.

Corollary 4.13. *Let R be a commutative ring which is not a field and let Ω be the set of all maximal ideals of R. Then the R-module R is strongly simple-separable if and only if $Ass(R) \cap \Omega = \emptyset$ or R has at least one non-trivial idempotent element.*

We finally give the structure of strongly simple-separable abelian groups.

Proposition 4.14. *Let G be a simple-separable abelian group. Then the following conditions are equivalent:*

(i) *G is strongly simple-separable;*
(ii) *G is not isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ for every prime number p and any integer $n \geq 2$.*

Proof. (i) ⇒ (ii) Given a prime number p and an integer $n \geq 2$, it is clear that the indecomposable nonsimple $\mathbb{Z}$-module $\mathbb{Z}/p^n\mathbb{Z}$ is not strongly simple-separable since $\mathrm{Soc}(\mathbb{Z}/p^n\mathbb{Z}) \neq 0$ (see Proposition 4.4).

(ii) ⇒ (i) Let G be a simple-separable abelian group which is not isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$ for every prime number p and any integer $n \geq 2$. If G contains no simple proper subgroups, then clearly G is strongly simple-separable. Now assume that G contains a simple proper subgroup S. Then S is isomorphic to $\mathbb{Z}/p_0\mathbb{Z}$ for some prime number p_0. By Theorem 2.30, S is contained in a direct summand H of G with $H \cong \mathbb{Z}/p_0^k\mathbb{Z}$ for some positive integer k. If $k = 1$, then $H = S$ and hence $H \neq G$. Moreover, if $k \geq 2$, then $H \neq G$ by (ii). This completes the proof. □

Acknowledgement. The authors would like to thank the referee for the valuable suggestions and comments which improved this paper.

Declarations

Disclosure statement: The authors report that there are no competing interests to declare.

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