Çanakkale Onsekiz Mart University
Journal of Advanced Research in Natural and Applied Sciences
dergipark.org.tr/tr/pub/jarnas
Open Access

e-ISSN: 2757-5195 2024, Vol. 10, Issue 3, Pages: 530-541 doi.org/10.28979/jarnas.1451630

Lipschitz Operators Associated with Weakly p-Compact and Unconditionally p-Compact Sets

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Article Info
Received: 12 Mar 2024
Accepted: 04 Jun 2024
Published: 30 Sep 2024
Research Article

Abstract— The study introduces different categories of Lipschitz operators linked with weakly p-compact and unconditionally p-compact sets. It explores some properties of these operator classes derived from linear operators associated with these sets and examines their interconnections. Additionally, it denotes that these classes are extensions of the related linear operators. Moreover, the study evaluates the concept of majorization by scrutinizing both newly obtained and pre-existing results and draws some conclusions based on these findings. The primary method used to obtain the results in the study is the linearization of Lipschitz operators through the Lipschitz-free space constructed over a pointed metric space.

Keywords — *Lipschitz compact operator, Lipschitz p-compact operator, Lipschitz weakly p-compact operator, Lipschitz unconditionally p-compact operator, majorization*

Subject Classification (2020) 47B10, 26A16

1. Introduction

Various authors have recently engaged in the exploration of Lipschitz versions of diverse bounded linear operators. In 2014, Jiménez-Vargas et al. [1] introduced the concepts of Lipschitz finite-rank, Lipschitz compact, Lipschitz weakly compact and Lipschitz approximable operators. They also obtained some outcomes for these concepts. In 1955, Grothendieck [2] effectively demonstrated that compactness could be examined geometrically by obtaining the necessary and sufficient criteria for a set to be compact in a Banach space. In 2002, motivated by Grothendieck's compactness principle, Sinha and Karn [3] defined the notion of p-compactness (respectively, weakly p-compactness) of a set and also defined the concepts of the p-compact (respectively, weakly p-compact) linear operator and p-approximation property in Banach spaces, which arises naturally from this notion. Later, as modifications of these concepts, Kim [4] introduced the concepts of unconditionally p-compact set and unconditionally p-compact linear operator. Many mathematicians studied the linear operators and approximation properties associated with all these sets [4–8]. Various studies have been conducted in recent years to investigate the properties of p-compact sets and p-compact linear operators. These

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studies encompass both nonlinear and linear situations. Inspired by Jiménez-Vargas et al. [1], Achour et al. [9] defined and studied the concept of Lipschitz operators, which are p-compact as a nonlinear extension of p-compact linear operators and considered some features in the linear case for the Lipschitz case. At the same time, they introduced the concepts of Lipschitz free p-compact operator and Lipschitz locally p-compact operator and compared these concepts with each other and showed their different properties. To our knowledge, Lipschitz weakly p-compact and unconditionally p-compact operators have not yet been studied in the literature. In this study, considering the works on weakly p-compact and unconditionally p-compact sets of Kim [4–7] and inspired by the works of Jiménez-Vargas et al. [1] and Achour et al. [9], we consider some classes of Lipschitz operators associated with weakly p-compact and unconditionally p-compact sets. After we provide some notations and basic concepts, we introduce Lipschitz weakly (respectively, unconditionally) p-compact operator and Lipschitz free weakly (respectively, unconditionally) p-compact operator, respectively, and obtain some of their properties. We evaluate the obtained and current results with the concept of majorization and provide some results.

2. Preliminaries

A metric space with a designated base point denoted as 0 is referred to as a pointed metric space. Throughout the study, unless specifically mentioned otherwise, the symbols X and Y represent pointed metric spaces, while E and F represent Banach spaces. B_E represents the closed unit ball of a Banach space E. The symbol $\mathbb{K}$ represents the field space of complex or real numbers. The notation $L(E,F)$ (respectively, $\mathcal{B}(E,F)$) represents the vector space of all the linear operators (respectively, bounded linear operators) from E to F. The dual space of E is $E^* := \mathcal{B}(E,\mathbb{K})$. The notation $\mathrm{Lip}_0(X,Y)$ indicates the set of all the Lipschitz operators $g : X \to Y$ where $g(0) = 0$. The function

$$\mathrm{Lip}(g) = \sup\{\|g(x) - g(y)\|/d(x,y) : x \neq y,\ x,y \in X\}$$

defines a norm on $\mathrm{Lip}_0(X,E)$, and $\mathrm{Lip}_0(X,E)$ is a Banach space with the norm [1]. The Banach space $\mathrm{Lip}_0(X,\mathbb{K})$ is Lipschitz dual of X and is denoted by $X^\sharp$. For a Banach space E, it is clear that $E^* \subset E^\sharp$. For thorough information on Lipschitz operators and their properties, see [10]. For $x \in X$, the function $\delta_x : X^\sharp \to \mathbb{K}$ is defined as $\delta_x(g) = g(x)$ such that $g \in X^\sharp$ [1]. Lipschitz-free Banach space of X is defined as the closed linear span of the set $\{\delta_x : x \in X\}$ and denoted by $\mathcal{F}(X)$ [1]. The map $\delta_X : X \to \mathcal{F}(X)$ is described by $\delta_X(x)(g) = g(x)$, for all $x \in X$ and $g \in X^\sharp$ [1].

We recall certain properties of the Lipschitz-free Banach space in the next lemma.

Lemma 2.1. [1,11,12] Let E be a Banach space and X and Y be two pointed metric spaces.

i. The function $\delta_X : X \to \mathcal{F}(X)$ described as $\delta_X(x)(g) = g(x)$, for all $x \in X$ and $g \in X^\sharp$, is a nonlinear isometry.

ii. The map $Q_X : X^\sharp \to \mathcal{F}(X)^*$ given by $Q_X(g)(v) = v(g)$, for all $g \in X^\sharp, v \in \mathcal{F}(X)$, is an isometric isomorphism.

iii. The set $B_{\mathcal{F}(X)}$ is convex closed and balanced span of the set $\{(\delta_x - \delta_y)/d(x,y) : x \neq y,\ x,y \in X\}$.

iv. The Lipschitz adjoint of $g \in \mathrm{Lip}_0(X,E)$ is $g^\sharp : E^\sharp \to X^\sharp$ defined by $g^\sharp(h) = h \circ g$, for all $h \in E^\sharp$. Moreover, $g^\sharp \in \mathcal{B}(E^\sharp, X^\sharp)$ and $\|g^\sharp\| = \mathrm{Lip}(g)$.

v. For every operator $g \in \mathrm{Lip}_0(X,E)$, a unique operator $T_g \in \mathcal{B}(\mathcal{F}(X),E)$ exists so that $g = T_g \circ \delta_X$, in other words, the following diagram commutes and $\|T_g\| = \mathrm{Lip}(g)$:

$$\begin{tikzcd} X \arrow[r, "\delta_X"] \arrow[dr, "g"'] & \mathcal{F}(X) \arrow[d, "T_g"] \\ & E \end{tikzcd}$$

vi. For every operator $g \in \mathrm{Lip}_0(X,Y)$, a unique operator $\widehat{g} \in \mathcal{B}(\mathcal{F}(X), \mathcal{F}(Y))$ exists so that $\widehat{g} \circ \delta_X = \delta_Y \circ g$, in other words, the following diagram commutes and $\|\widehat{g}\| = \mathrm{Lip}(g)$:

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\delta_X"] & Y \arrow[d, "\delta_Y"] \\ \mathcal{F}(X) \arrow[r, "\widehat{g}"] & \mathcal{F}(Y) \end{tikzcd}$$

The restriction E^* of $g^\sharp$ in Lemma 2.1 *iv* is called Lipschitz transpose of g and denoted by g^t. It is clear that $Q_X \circ g^t = T_g^*$ and $\widehat{g}^* \circ Q_Y = Q_X \circ g^\sharp$, where T_g^* and $\widehat{g}^*$ are adjoint operators of T_g and $\widehat{g}$, respectively [9].

Unless otherwise stated, throughout the study, $p \geq 1$ and $1/p + 1/p^* = 1$. The Banach space denoted as $l_p(E)$ is defined as the space of all the p-summable sequences in the space E. Here, the norm is given by the formula [3]:

$$\|(x_n)_n\|_p = \left(\sum_{n=1}^{\infty} \|x_n\|^p\right)^{1/p}$$

Besides, $l_p^w(E)$ is a Banach space representing all weakly p-summable sequences in E. The norm of an element $(x_n)_n$ in this space is defined as [3]:

$$\|(x_n)_n\|_p^w = \sup\{\|(x^*(x_n))_n\|_p : \ x^* \in B_{E^*}\}$$

The space

$$l_p^u(E) := \{(x_n)_n \in l_p^w(E) : \|(0, ..., 0, x_n, x_{n+1}, ...)\|_p^w \to 0 \text{ as } n \to \infty\}$$

is referred to as the space of all the sequences in E that are unconditionally p-summable. The space is a closed subspace of $l_p^w(E)$ [4]. If a sequence $(x_n)_n \in l_p(E)$ satisfying

$$C \subset \text{p–co}(\{x_n\}) = \{\sum_{n=1}^{\infty} \alpha_n x_n : (\alpha_n)_n \in B_{l_{p^*}}\}$$

exists, then the subset C of E is relatively p-compact [3]. If there is $(x_n)_n \in l_p^w(X)$ (respectively, $(x_n)_n \in l_p^u(X)$) satisfying $C \subset \text{p–co}(\{x_n\})$, then C is called relatively weakly p-compact (respectively, relatively unconditionally p-compact) [3] (respectively, [4]). We note that the sets that are relatively p-compact are also relatively unconditionally p-compact. Further, the relatively unconditionally p-compact sets are relatively compact and relatively weakly p-compact [4,13]. An operator $T \in L(E,F)$ is said to be p-compact (respectively, weakly p-compact, unconditionally p-compact) if $T(B_E)$ is relatively p-compact set (respectively, relatively weakly p-compact set, relatively unconditionally p-compact set) in F [3,4]. The space of all the p-compact linear operators (respectively, weakly p-compact linear operators, unconditionally p-compact linear operators) from E to F is denoted by the symbol $\mathcal{K}_p(E,F)$ (respectively, $\mathcal{W}_p(E,F)$, $\mathcal{K}_{up}(E,F)$). If $S \in \mathcal{K}_p(E,F)$, then

$$\|S\|_{K_p} := \inf\{\|(x_n)_n\|_p : \ S(B_E) \subset \text{p–co}(\{x_n\}), \ (x_n)_n \in l_p(F)\}$$

is a norm function. Moreover, $(\mathcal{K}_p, \|.\|_{\mathcal{K}_p})$ is a Banach operator ideal [14]. If $S \in \mathcal{W}_p(E,F)$, then

$$\|S\|_{\mathcal{W}_p} := \inf\{\|(x_n)_n\|_p^w : S(B_E) \subset \text{p–co}(\{x_n\}), \ (x_n)_n \in l_p^w(F)\}$$

is a norm function and also $(\mathcal{W}_p, \|.\|_{\mathcal{W}_p})$ is a Banach operator ideal [7]. If $S \in \mathcal{K}_{up}(E,F)$, then

$$\|S\|_{\mathcal{K}_{up}} := \inf\{\|(x_n)_n\|_p^w : S(B_E) \subset \text{p–co}(\{x_n\}),\ (x_n)_n \in l_p^u(F)\}$$

is a norm function, and $(\mathcal{K}_{up}, \|.\|_{\mathcal{K}_{up}})$ is a Banach operator ideal [4]. The operator $S \in L(E,F)$ is quasi unconditionally p-nuclear (respectively, quasi weakly p-nuclear) if there is $(x_n^*)_n \in l_p^u(E^*)$ (respectively, $(x_n^*)_n \in l_p^w(E^*)$) such that $\|Sx\| \leq \|(x_n^*(x))_n\|_p$, for all $x \in E$ [4,6]. The space of all the quasi unconditionally p-nuclear operators (respectively, quasi weakly p-nuclear operators) from E to F is represented by $\mathcal{N}_{up}^Q(E,F)$ (respectively, $\mathcal{N}_{wp}^Q(E,F)$). For $S \in \mathcal{N}_{up}^Q(E,F)$ (respectively, $S \in \mathcal{N}_{wp}^Q(E,F)$), let $\|S\|_{\mathcal{N}_{up}^Q} := \inf \|(x_n^*)_n\|_p^w$, where infimum is taken over all the sequences satisfying the condition quasi unconditionally p-nuclear (respectively, quasi weakly p-nuclear). Then, $(\mathcal{N}_{up}^Q, \|.\|_{\mathcal{N}_{up}^Q})$ (respectively, $(\mathcal{N}_{wp}^Q, \|.\|_{\mathcal{N}_{wp}^Q})$) is a Banach operator ideal [4,6]. Let G be a Banach space, $S \in \mathcal{B}(E,F)$, and $T \in \mathcal{B}(E,G)$. If there is a $K > 0$ satisfying $\|Tx\| \leq K\|Sx\|$, for all $x \in E$, then it is said that S majorizes T [15]. Assume that $f \in \text{Lip}_0(X,E)$ and $g \in \text{Lip}(X,F)$. If there is an $M > 0$ satisfying $\|g(x_1) - g(x_2)\| \leq M\|f(x_1) - f(x_2)\|$, for all $x_1, x_2 \in X$, then it is said that f majorizes g [16].

Jiménez-Vargas et al. [1] defined the compactness (respectively, weakly compactness) of a Lipschitz operator as follows:

Definition 2.2. [1] Let $g \in \text{Lip}_0(X,E)$. The operator g is called Lipschitz compact (respectively, Lipschitz weakly compact) if

$$\text{Im}_{\text{Lip}}(g) := \{\frac{g(x) - g(y)}{d(x,y)} : x \neq y,\ x, y \in X\} \subset E$$

is a relatively compact set (respectively, relatively weakly compact set).

Inspired by this definition, Achour et al. [9] defined the p-compactness of a Lipschitz operator as follows.

Definition 2.3. [9] Let $g \in \text{Lip}_0(X,E)$. The operator g is called Lipschitz $p-$compact if the set $\text{Im}_{\text{Lip}}(g)$ is a relatively p-compact in E.

3. The Main Results

In this section, we introduce the various classes of Lipschitz operators associated with weakly p-compact sets and unconditionally p-compact sets and investigate some of their properties, respectively.

3.1. Lipschitz Weakly p-Compact Operators and Some of Their Properties

Taking inspiration from the concepts of the Lipschitz compact operator defined by [1] and the Lipschitz p-compact operator defined by [9], we define the concept of Lipschitz weakly p-compact operator.

Definition 3.1. Let $g \in \text{Lip}_0(X,E)$. Then, g is called a Lipschitz weakly p-compact operator if the set $\text{Im}_{\text{Lip}}(g)$ is a relatively weakly p-compact subset of E.

The set of all the Lipschitz weakly p-compact operators defined from X to E is represented by the symbol $\text{Lip}_0^{\mathcal{W}_p}(X,E)$.

Remark 3.2. We note that $\text{Im}_{\text{Lip}}(g) \subset g(B_X)$ when X is a Banach space and $g \in L(X,E)$. Thus, $g \in \text{Lip}_0^{\mathcal{W}_p}(X,E)$ as $g \in \mathcal{W}_p(X,E)$. Therefore, Lipschitz operators, which are weakly p-compact, can be thought of as a generalization of linear operators that are weakly p-compact.

Remark 3.3. By [14], it is well known that the set $\text{p–co}(\{x_n\})$ is absolutely convex and norm closed if $p > 1$ and $(x_n)_n \in l_p^w(E)$.

Since the smallest absolutely convex set containing set A is $\operatorname{abco}(A)$, the following lemma is obtained by Remark 3.3.

Lemma 3.4. Let E be a Banach space and $p > 1$. The set $\overline{\operatorname{abco}}(A)$ is a weakly p-compact subset of E if A is a relatively weakly p-compact subset of E.

The following result has been obtained as a modification of Proposition 2.1 in [1] (see also Theorem 3.4 in [9]) for weakly p-compact sets.

Theorem 3.5. Let $g \in \operatorname{Lip}_0(X, E)$ and $p > 1$. Then, $g \in \operatorname{Lip}_0^{\mathcal{W}_p}(X, E)$ if and only if $T_g \in \mathcal{W}_p(\mathcal{F}(X), E)$.

The proof of the theorem is omitted as it can be obtained similarly to steps in Proposition 2.1 [1] using Lemma 3.4.

Kim [6] proved Theorem 3.7 (c) that the adjoint of a linear operator that is weakly p-compact is quasi-weakly p-nuclear. The following proposition is an extension of this result to the Lipschitz case.

Proposition 3.6. Let $p > 1$. If $g \in \operatorname{Lip}_0^{\mathcal{W}_p}(X, E)$, then $g^t \in \mathcal{N}_{wp}^{Q}(E^*, X^\sharp)$ and $\|g^t\|_{\mathcal{N}_{wp}^{Q}} \leq \|T_g\|_{\mathcal{W}_p}$.

Proof. Let $g \in \operatorname{Lip}_0^{\mathcal{W}_p}(X, E)$. By Theorem 3.5, T_g is a linear operator that is weakly p-compact. Thus, by the result Theorem 3.7 (c) in [6], $T_g^* \in \mathcal{N}_{wp}^{Q}(E^*, \mathcal{F}(X)^*)$ and $\|T_g^*\|_{\mathcal{N}_{wp}^{Q}} \leq \|T_g\|_{\mathcal{W}_p}$. Since $Q_X^{-1} \circ T_g^* = g^t$, by the ideal property of quasi weakly p-nuclear operators, $g^t \in \mathcal{N}_{wp}^{Q}(E^*, X^\sharp)$, and also $\|T_g^*\|_{\mathcal{N}_{wp}^{Q}} \leq \|Q_X\|\|g^t\|_{\mathcal{N}_{wp}^{Q}}$ and $\|g^t\|_{\mathcal{N}_{wp}^{Q}} \leq \|Q_X^{-1}\|\|T_g^*\|_{\mathcal{N}_{wp}^{Q}}$. By Lemma 2.1 *ii*, since Q_X is an isometric isomorphism, it is obtained $\|g^t\|_{\mathcal{N}_{wp}^{Q}} = \|T_g^*\|_{\mathcal{N}_{wp}^{Q}}$. Using $\|T_g^*\|_{\mathcal{N}_{wp}^{Q}} \leq \|T_g\|_{\mathcal{W}_p}$, it is obtained $\|g^t\|_{\mathcal{N}_{wp}^{Q}} \leq \|T_g\|_{\mathcal{W}_p}$. □

By [3], any sets that are relatively weakly p-compact are also relatively weakly q-compact as $1 \leq p \leq q < \infty$, and by [14], any sets that are relatively weakly p-compact (with $p > 1$) are also relatively weakly compact. Therefore, we directly obtain the following proposition.

Proposition 3.7. Let $g \in \operatorname{Lip}_0(X, E)$ and $1 \leq p \leq q < \infty$. Then, g is Lipschitz weakly q-compact whenever g is Lipschitz weakly p-compact. In particular, if $p > 1$ and g is Lipschitz weakly p-compact, then g is Lipschitz weakly compact.

The following proposition demonstrates that if the hypotheses of Proposition 3.13 in [9] are replaced p-summability of the operator T_f with p-summability of the operator T_f^*, and the Lipschitz compactness of Lipschitz operator g with the Lipschitz weakly p-compactness, then the same result can be obtained.

Proposition 3.8. Let $p > 1$, $f \in \operatorname{Lip}_0(X, E)$, and $g \in \operatorname{Lip}_0(Z, E^*)$, where Z is a pointed metric space. If $g \in \operatorname{Lip}_0^{\mathcal{W}_p}(Z, E^*)$ and the operator T_f^* is p-summing, then $f^t \circ g : Z \to X^\sharp$ is Lipschitz p-compact.

Proof. Assume that g is Lipschitz weakly p-compact. By Theorem 3.5, T_g is weakly p-compact. Since T_f^* is p-summing and T_g is weakly p-compact, by Proposition 5.4 [3], the operator $T_f^* \circ T_g$ is p-compact. Moreover, $f^t \circ T_g = Q_X^{-1} \circ T_f^* \circ T_g$ and thus, by the ideal property of $\mathcal{K}_p$, $f^t \circ T_g$ is a p-compact operator. By Lemma 2.1 *v*, since $T_g \circ \delta_Z = g$, the linerization of $f^t \circ g$ is $f^t \circ T_g$. Thus, by Theorem 3.4 [9], $f^t \circ g$ is Lipschitz p-compact. □

Kim [7] has obtained a result in Proposition 2.4, which characterizes the factorization of linear operators that are weakly p-compact. Through Kim's result, we get the following result, which characterizes the factorization of a Lipschitz operator that is weakly p-compact.

Theorem 3.9. Let $p > 1$ and $f \in \operatorname{Lip}_0(X, E)$. Then, $f \in \operatorname{Lip}_0^{\mathcal{W}_p}(X, E)$ if and only if there are a quotient space G of l_{p^*}, $g \in \operatorname{Lip}_0^{\mathcal{W}_p}(X, G)$, and $S \in \mathcal{W}_p(G, E)$ such that $f = S \circ g$. Further, $\operatorname{Lip}(f) \leq \|S\|_{\mathcal{W}_p} \operatorname{Lip}(g)$.

Proof. ($\Leftarrow$): Assume that $f \in \mathrm{Lip}_0(X,E)$ has the factorization in theorem. Then, $\mathrm{Im}_{\mathrm{Lip}}(f) = S(\mathrm{Im}_{\mathrm{Lip}}(g))$. Since $g \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,G)$, $S(\mathrm{Im}_{\mathrm{Lip}}(g))$ is relatively weakly p-compact. Thus, $f \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,E)$.

($\Rightarrow$): Let $f \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,E)$. Thus, by Theorem 3.5, $T_f \in \mathcal{W}_p(\mathcal{F}(X),E)$. By Kim's result Proposition 2.4 [7], there are a quotient space G of l_{p^*}, $S \in \mathcal{W}_p(G,E)$ and $R \in \mathcal{W}_p(\mathcal{F}(X),G)$ such that $T_f = S \circ R$. By Lemma 2.1 v, we obtain $f = S \circ R \circ \delta_X$. Let $g := R \circ \delta_X$. It is clear that $g \in \mathrm{Lip}_0(X,G)$. Hence, by unique of the linearization of g in Lemma 2.1 v is $T_g = R$. Thus, the desired factorization is obtained. Further, by [3], since $\|S\| \leq \|S\|_{\mathcal{W}_p}$, using the norm property in Lemma 2.1 v,

$$\mathrm{Lip}(f) = \|T_f\| \leq \|S\|\|R\| = \|S\|\|T_g\| \leq \|S\|_{\mathcal{W}_p}\,\mathrm{Lip}(g).$$

□

It is well known that Davis-Figiel-Johnson-Pełczynski [17] obtained a significant theorem that states that any linear operator that is weakly compact can be factored via a reflexive Banach space. Since every set that is weakly p-compact is also weakly compact while $p > 1$ [14], combining Davis-Figiel-Johnson-Pełczynski theorem [17] and Theorem 3.9, we get the following proposition.

Proposition 3.10. Let $p > 1$ and $f \in \mathrm{Lip}_0(X,E)$. Then, $f \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,E)$ if and only if there are a quotient space G of l_{p^*}, a reflexive Banach space W, $T \in \mathcal{B}(G,W)$, $Q \in \mathcal{B}(W,E)$, and $g \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,G)$ such that $f = Q \circ T \circ g$. Further, $\mathrm{Lip}(f) \leq \|Q\|\|T\|\,\mathrm{Lip}(g)$.

The following proposition is a modification of the ideal property of Jiménez-Vargas et al., Proposition 2.3 [1], for Lipschitz operators that are weakly p-compact. The proof has been omitted as it can be easily done following their methods and using Theorem 3.5.

Proposition 3.11. Let $p > 1$, $S \in \mathcal{B}(E,F)$, and $f \in \mathrm{Lip}_0(Y,X)$. If $g \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,E)$, then $S \circ g \circ f \in \mathrm{Lip}_0^{\mathcal{W}_p}(Y,F)$.

3.2. Lipschitz Unconditionally p-Compact Operators and Some of Their Properties

In this section, taking inspiration from the concepts of the Lipschitz compact operator defined by [1], and the Lipschitz p-compact operator defined by [9], we define the concept of Lipschitz unconditionally p-compact operator.

Definition 3.12. Let $g \in \mathrm{Lip}_0(X,E)$. Then, g is called Lipschitz unconditionally p-compact operator if the set $\mathrm{Im}_{\mathrm{Lip}}(g)$ is a relatively unconditionally p-compact subset of E.

The set of all the Lipschitz unconditionally p-compact operators defined from X to E is represented by the notation $\mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$.

Similar to Remark 3.2, Lipschitz unconditionally p-compact operators can be thought of as a generalization of linear operators that are unconditionally p-compact.

Considering relationships among relatively p-compact, weakly p-compact, and unconditionally p-compact sets and compact sets, we get the following proposition.

Proposition 3.13. Let $g \in \mathrm{Lip}_0(X,E)$. Then, $g \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$ if g is Lipschitz p-compact. Moreover, if $g \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$, then $g \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,E)$ and also g is Lipschitz compact.

Since $(x_n)_n \in l_p^w(E)$ while $(x_n)_n \in l_p^u(E)$, a result similar to Lemma 3.4 is provided as follows.

Lemma 3.14. Let $p > 1$. The set $\overline{\mathrm{abco}}(A)$ is an unconditionally p-compact set in E if A is a relatively unconditionally p-compact subset of a Banach space E.

The following result has been obtained as a modification of Proposition 2.1 in [1] (see also Theorem 3.4 [9]) for unconditionally p-compact sets.

Theorem 3.15. Let $g \in \mathrm{Lip}_0(X,E)$ and $p>1$. Then, $g \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$ if and only if $T_g \in \mathcal{K}_{up}(\mathcal{F}(X),E)$.

The proof of the theorem has been omitted since it can be done using Lemma 3.14 and following the steps in Proposition 2.1 [1].

Kim [4, 5] demonstrated Theorem 2.4 and Theorem 5.6, respectively, that the unconditionally p-compactness of a linear operator and the quasi unconditionally p-nuclearity of the adjoint of this operator are equivalent. The following proposition is an extension of this result to the Lipschitz case. This proposition also extends Proposition 3.12 [9] to unconditionally p-compact sets.

Proposition 3.16. Let $f \in \mathrm{Lip}_0(X,E)$ and $p>1$. Then, $f \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$ if and only if $f^t \in \mathcal{N}_{up}^{Q}(E^*,X^\sharp)$. Moreover, $\|f^t\|_{\mathcal{N}_{up}^{Q}} = \|T_f\|_{\mathcal{K}_{up}}$.

Proof. Let $f \in \mathrm{Lip}_0(X,E)$ and $p>1$. By Theorem 3.15, $f \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$ if and only if T_f is an unconditionally p-compact operator. By Kim's Theorem 2.4 [4], $T_f^* \in \mathcal{N}_{up}^{Q}(E^*,\mathcal{F}(X)^*)$ if and only if $T_f \in \mathcal{K}_{up}(\mathcal{F}(X),E)$. Thus, using $Q_X^{-1} \circ T_f^* = f^t$, the desired equivalence is achieved. Further, by Theorem 5.6 [5], since $\|T_f^*\|_{\mathcal{N}_{up}^{Q}} = \|T_f\|_{\mathcal{K}_{up}}$, $\|f^t\|_{\mathcal{N}_{up}^{Q}} = \|T_f\|_{\mathcal{K}_{up}}$ is obtained. □

We obtain the following proposition by combining Kim's Theorem 2.4 [4] and Theorem 3.5 herein.

Proposition 3.17. Let $f \in \mathrm{Lip}_0(X,E)$ and $p>1$. Assume that the operator T_f maps weakly p-summable sequences into unconditionally p-summable sequences and $S \in \mathrm{Lip}_0^{\mathcal{W}_p}(Z,\mathcal{F}(X))$. Then, $S^t \circ T_f^* \in \mathcal{N}_{up}^{Q}(E^*,Z^\sharp)$.

Proof. Since $S \in \mathrm{Lip}_0^{\mathcal{W}_p}(Z,\mathcal{F}(X))$, by Theorem 3.5, $T_S \in \mathcal{W}_p(\mathcal{F}(Z),\mathcal{F}(X))$. Since the operator T_f maps weakly p-summable sequences into unconditionally p-summable sequences, it is obtained $T_f \circ T_S \in \mathcal{K}_{up}(\mathcal{F}(Z),E)$. Thus, by Theorem 2.4 [4], $(T_f \circ T_S)^* \in \mathcal{N}_{up}^{Q}(E^*,\mathcal{F}(Z)^*)$. Since $T_S^* = Q_Z \circ S^t$, it is obtained $Q_Z \circ S^t \circ T_f^* \in \mathcal{N}_{up}^{Q}(E^*,\mathcal{F}(Z)^*)$. If it is used that Q_Z is an isometric isomorphism and $\mathcal{N}_{up}^{Q}$ has the ideal property, then it is obtained $S^t \circ T_f^* \in \mathcal{N}_{up}^{Q}(E^*,Z^\sharp)$. □

Combining Theorem 3.15 and some results in [4], we obtained the following theorem.

Theorem 3.18. Let X be a pointed metric space such that $\mathcal{F}(X)^*$ is an injective Banach space, E is an injective Banach space, and $p>1$. If $f \in \mathrm{Lip}_0^{\mathcal{K}_{up^*}}(X,E)$, then T_f is unconditionally p-nuclear operator.

Proof. If Theorem 3.15 and Theorem 2.4 [4] are used in order, then $T_f^* \in \mathcal{N}_{up^*}^{Q}(E^*,\mathcal{F}(X)^*)$ is obtained. By injectivity of $\mathcal{F}(X)^*$ and the result Lemma 2.6 [4], $T_f^* \in \mathcal{N}_{up^*}(E^*,\mathcal{F}(X)^*)$. From the proof of Proposition 2.2 [4], $\mathcal{N}_{up^*} \subset \mathcal{K}_{up}$. Thus, $T_f^* \in \mathcal{K}_{up}(E^*,\mathcal{F}(X)^*)$. By Theorem 2.3 [4], $T_f \in \mathcal{N}_{up}^{Q}(\mathcal{F}(X),E)$. Since E is an injective Banach space, by Lemma 2.6 [4], T_f is an unconditionally p-nuclear operator. □

The following remark shows a pointed metric space X such that $\mathcal{F}(X)^*$ is an injective Banach space.

Remark 3.19. By [18], it is well known that if $X=\mathbb{R}$, then $\mathcal{F}(X)=\mathcal{F}(\mathbb{R})=L_1$. Moreover, $L_1^*=L_\infty$ and L_∞ is an injective Banach space (see Proposition 4.3.8 (ii) [19]). Thus, $\mathcal{F}(X)^*$ is an injective Banach space.

Using Theorem 3.15 and the factorization result of linear operators that are unconditionally p-compact in Theorem 2.2 [5], we obtain the following result concerning the factorization of Lipschitz unconditionally p-compact operators.

Proposition 3.20. Let $f \in \mathrm{Lip}_0(X,E)$ and $p>1$. Then, $f \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$ if and only if a quotient space G of l_{p^*}, an operator $S \in \mathcal{K}_{up}(G,E)$ and an operator $g \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,G)$ satisfying $f = S \circ g$ exist. Further, $\mathrm{Lip}(f) \le \|S\|_{\mathcal{K}_{up}} \mathrm{Lip}(g)$.

Proof. Since steps in Theorem obtain the proof 3.9 using Theorem 2.2 [5] and Theorem 3.15, it is omitted. For the norm inequality, if steps in Theorem 3.9 are used with the definitions of $\|.\|_{\mathcal{W}_p}$ and $\|.\|_{\mathcal{K}_{up}}$, then the following inequality is obtained:

$$\mathrm{Lip}(f) = \|T_f\| \leq \|S\|\|R\| \leq \|S\|_{\mathcal{W}_p} \mathrm{Lip}(g) \leq \|S\|_{\mathcal{K}_{up}} \mathrm{Lip}(g).$$

□

Using Theorem 3.4 [9] and the factorization of linear operators that are p-compact in Theorem 2.3 [5], we obtain the following factorization result for Lipschitz p-compact operators.

Theorem 3.21. Let $f \in \mathrm{Lip}_0(X, E)$ and $p > 1$. Then, $f \in \mathrm{Lip}_0^{\mathcal{K}_p}(X, E)$ if and only if a quotient space G of l_{p^*}, an operator $S \in \mathcal{K}_p(G, E)$ and an operator $g \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X, G)$ satisfying $f = S \circ g$ exist. Further, $\mathrm{Lip}(f) \leq \|S\|_{\mathcal{K}_p} \mathrm{Lip}(g)$.

The proof of the theorem has been omitted since it can be done using Theorem 3.4 [9] and Theorem 2.3 [5] and following the steps in Theorem 3.9.

We modify the ideal property of Jiménez-Vargas et al. [1] for Lipschitz operators that are unconditionally p-compact. The proof has been omitted as it can be done following their methods and using Theorem 3.15.

Proposition 3.22. Let $p > 1$, $S \in \mathcal{B}(E, F)$, and $f \in \mathrm{Lip}_0(Y, X)$. If $g \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X, E)$, then $S \circ g \circ f \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(Y, F)$.

3.3. Lipschitz Free Weakly and Unconditionally p-Compact Operators

In this section, inspired by the concepts of Lipschitz free (weakly) compact operators defined by [12] and Lipschitz free p-compact operators defined by [9], we define Lipschitz free weakly and unconditionally p-compact operators, investigate some of their properties, and provide relationships among them.

Definition 3.23. Let $p \geq 1$ and $g \in \mathrm{Lip}_0(X, Y)$. If the mapping $\delta_Y \circ g : X \to \mathcal{F}(Y)$ is Lipschitz weakly p-compact (respectively, Lipschitz unconditionally p-compact), then g is said to be Lipschitz free weakly p-compact (respectively, Lipschitz free unconditionally p-compact).

We denote the set of all the Lipschitz free weakly p-compact (Lipschitz free unconditionally p-compact) operators defined from X to Y with the notation $\mathrm{FLip}_0^{\mathcal{W}_p}(X, Y)$ (respectively, $\mathrm{FLip}_0^{\mathcal{K}_{up}}(X, Y)$).

Some characterizations have been obtained for Lipschitz-free (weakly) compact operators in Theorem 2.3 and Theorem 2.4 [12]. Similar characterizations have been obtained for the p-compactness case by Theorem 4.2 [9]. In the following theorems, we get identical characterizations for unconditionally p-compactness and weakly p-compactness cases. The proofs of these theorems will be made by following their proof steps.

Theorem 3.24. Let $g \in \mathrm{Lip}_0(X, Y)$ and $p > 1$. Then, the following are equivalent.

i. $g \in \mathrm{FLip}_0^{\mathcal{K}_{up}}(X, Y)$

ii. $\widehat{g} \in \mathcal{K}_{up}(\mathcal{F}(X), \mathcal{F}(Y))$

iii. $g^\sharp \in \mathcal{N}_{up}^{Q}(Y^\sharp, X^\sharp)$

Proof. Let $g \in \mathrm{Lip}_0(X, Y)$. Then, $\delta_Y \circ g \in \mathrm{Lip}_0(X, \mathcal{F}(Y))$. By the uniqueness of linearizations in Lemma 2.1, it is obtained $\widehat{g} = T_{\delta_Y \circ g}$. Using Theorem 3.15, we obtain that $g \in \mathrm{FLip}_0^{\mathcal{K}_{up}}(X, Y)$ if and only if $T_{\delta_Y \circ g} \in \mathcal{K}_{up}(X, Y)$. Thus, the equivalence of i and ii is obtained. The equivalence of ii and iii follows from the equality $(Q_X)^{-1} \circ \widehat{g}^* \circ Q_Y = g^\sharp$ and the result Theorem 2.4 [4]. □

Remark 3.25. Let $p \geq 1$ and E and F be Banach spaces. By Theorem 3.7 [6], we know that $T^* \in \mathcal{N}^Q_{wp}(F^*, E^*)$ while $T \in \mathcal{W}_p(E, F)$. Since we do not know whether the converse is true, we obtain the following theorem for the weakly p-compactness case of Theorem 3.24. Since the proof of theorem can be analogous to Theorem 3.24 using Theorem 3.5, we provide the following theorem.

Theorem 3.26. Let $g \in \mathrm{Lip}_0(X, Y)$ and $p > 1$. Then, the following are equivalent.

i. $g \in \mathrm{FLip}_0^{\mathcal{W}_p}(X, Y)$

ii. $\widehat{g} \in \mathcal{W}_p(\mathcal{F}(X), \mathcal{F}(Y))$

The notations $\mathrm{FLip}_0^{\mathcal{W}}$, $\mathrm{FLip}_0^{\mathcal{K}}$, and $\mathrm{FLip}_0^{\mathcal{K}_p}$ denote the sets of all the Lipschitz free operators that are weakly compact, compact, and p-compact, respectively.

If the relations among relatively p-compact, unconditionally p-compact, weakly p-compact, compact, and weakly compact sets are considered, the following proposition can be obtained.

Proposition 3.27. Let $1 < p \leq q$. Then, we have the following inclusions.:

i. $\mathrm{FLip}_0^{\mathcal{K}_p} \subset \mathrm{FLip}_0^{\mathcal{K}_{up}} \subset \mathrm{FLip}_0^{\mathcal{W}_p} \subset \mathrm{FLip}_0^{\mathcal{W}}$

ii. $\mathrm{FLip}_0^{\mathcal{K}_p} \subset \mathrm{FLip}_0^{\mathcal{K}_{up}} \subset \mathrm{FLip}_0^{\mathcal{K}} \subset \mathrm{FLip}_0^{\mathcal{W}}$

iii. $\mathrm{FLip}_0^{\mathcal{K}_p} \subset \mathrm{FLip}_0^{\mathcal{K}_{up}} \subset \mathrm{FLip}_0^{\mathcal{W}_p} \subset \mathrm{FLip}_0^{\mathcal{W}_q} \subset \mathrm{FLip}_0^{\mathcal{W}}$

The following proposition demonstrates the relationships among the classes of Lipschitz operators defined about weakly p-compact and unconditionally p-compact sets. The proof of the proposition is done by following the steps in Proposition 4.6 [9].

Proposition 3.28. Let $p > 1$. Then, $\mathrm{FLip}_0^{\mathcal{W}_p} \subset \mathrm{Lip}_0^{\mathcal{W}_p}$ and $\mathrm{FLip}_0^{\mathcal{K}_{up}} \subset \mathrm{Lip}_0^{\mathcal{K}_{up}}$.

Proof. Let $f \in \mathrm{FLip}_0^{\mathcal{W}_p}(X, E)$. Thus, the set $\mathrm{Im}_{\mathrm{Lip}}(\delta_E \circ f)$ is relatively weakly p-compact. Moreover, by the proof of Proposition 2.2 in [12], $\beta_E(\mathrm{Im}_{\mathrm{Lip}}(\delta_E \circ f) = \mathrm{Im}_{\mathrm{Lip}}(f))$, where $\beta_E : \mathcal{F}(E) \to E$ is a bounded linear operator, the barycentric map. Then, since the set $\mathrm{Im}_{\mathrm{Lip}}(f)$ is relatively weakly p-compact, $f \in \mathrm{Lip}_0^{\mathcal{W}_p}(X, E)$ is obtained. □

The following results with ideal properties provide modifications of Proposition 2.6 [12] and Theorem 4.8 [9] for unconditionally p-compact and weakly p-compact sets, respectively. The proof will be done similarly to the proof of Proposition 2.6 [12].

Proposition 3.29. Let W and Z be pointed metric spaces and $p > 1$. Then, $S \circ f \circ T \in \mathrm{FLip}_0^{\mathcal{K}_{up}}(X, W)$ (respectively, $S \circ f \circ T \in \mathrm{FLip}_0^{\mathcal{W}_p}(X, W)$) if $T \in \mathrm{Lip}_0(X, Y), f \in \mathrm{FLip}_0^{\mathcal{K}_{up}}(Y, Z)$ (respectively, $f \in \mathrm{FLip}_0^{\mathcal{W}_p}(Y, Z)$), and $S \in \mathrm{Lip}_0(Z, W)$.

Proof. If we show $\widehat{S \circ f \circ T} \in \mathcal{K}_{up}(\mathcal{F}(X), \mathcal{F}(W))$, then, by Theorem 3.24, we obtain that $S \circ f \circ T \in \mathrm{FLip}_0^{\mathcal{K}_{up}}(X, W)$. Since $f \in \mathrm{FLip}_0^{\mathcal{K}_{up}}(Y, Z)$, by Theorem 3.24, we get $\widehat{f} \in \mathcal{K}_{up}(\mathcal{F}(Y), \mathcal{F}(Z))$. By the ideal property of $\mathcal{K}_{up}$, $\widehat{S} \circ \hat{f} \circ \widehat{T} \in \mathcal{K}_{up}(\mathcal{F}(X), \mathcal{F}(W))$. By Lemma 2.1 *vi*, it is obtained that $\widehat{S \circ f \circ T} = \widehat{S} \circ \hat{f} \circ \widehat{T}$. Thus, $\widehat{S \circ f \circ T} \in \mathcal{K}_{up}(\mathcal{F}(X), \mathcal{F}(W))$, and by Theorem 3.24, $S \circ f \circ T \in \mathrm{FLip}_0^{\mathcal{K}_{up}}(X, W)$. For weakly p-compact sets, the proof is similar. □

3.4. Some Results Obtained for Lipschitz Operators Using Majorizations

Inspired by the result Proposition 3.1.3 of Sahraoui [16], using the result Proposition 3 of Barnes [15], the results obtained from this study, and current findings, we obtain the following proposition.

Proposition 3.30. Let $f \in \mathrm{Lip}_0(X, E)$ and $g \in \mathrm{Lip}_0(X, F)$ so that T_f majorizes T_g and $p > 1$. Then, we have the following:

i. g is Lipschitz p-compact operator if f is Lipschitz p-compact operator.

ii. $g \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,F)$ if $f \in \mathrm{Lip}_0^{\mathcal{W}_p}(X,E)$

iii. $g \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,F)$ if $f \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$

Proof. If T_f majorizes T_g, then, by Proposition 3 [15], we know that there is an operator $V \in \mathcal{B}(\overline{T_f(\mathcal{F}(X))}, F)$ such that $T_g = V \circ T_f$.

i. If f is a Lipschitz p-compact operator, then, by Theorem 3.4 [9], T_f is a p-compact operator. By the ideal property of $\mathcal{K}_p$, T_g is a p-compact operator. By Theorem 3.4 [9], g is a Lipschitz operator which is p-compact.

The proofs of *ii* and *iii* can be compared to *i*, using Theorem 3.5 and Theorem 3.15, respectively. □

Proposition 3.31. Let $f \in \mathrm{Lip}_0(X,E)$ and $g \in \mathrm{Lip}_0(Y,E)$ so that f^t majorizes g^t and $p > 1$. Then, we have the following:

i. g is a Lipschitz p-compact operator if f is a Lipschitz p-compact operator.

ii. $g \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(Y,E)$ if $f \in \mathrm{Lip}_0^{\mathcal{K}_{up}}(X,E)$

Proof. *i.* If f is Lipschitz p-compact, then, by Proposition 3.12 [9], $f^t : E^* \to X^\sharp$ is a linear operator that is quasi p-nuclear. Since f^t majorizes g^t, by Proposition 3 [15], there is an operator $V \in \mathcal{B}(\overline{f^t(E^*)}, X^\sharp)$ such that $g^t = V \circ f^t$. Thus, by the ideal property of quasi p-nuclear operators [9], $g^t : E^* \to X^\sharp$ is a linear operator that is quasi p-nuclear. By Proposition 3.12 [9], g is Lipschitz p-compact operator.

ii. The proof can be compared to *i* using Proposition 3.16. □

Proposition 3.32. Let $f \in \mathrm{Lip}_0(X,E)$ and $g \in \mathrm{Lip}_0(Y,E)$ such that $\mathrm{Im}_{\mathrm{Lip}}(g) \subset \mathrm{Im}_{\mathrm{Lip}}(f)$ and $p > 1$. If f^t is quasi weakly p-nuclear, then g^t is quasi weakly p-nuclear.

Proof. If $\mathrm{Im}_{\mathrm{Lip}}(g) \subset \mathrm{Im}_{\mathrm{Lip}}(f)$, then, by Theorem 3.1.1 [16], f^t majorizes g^t. Thus, the proof can be easily obtained by Proposition 3 [15] and the ideal property of quasi-weakly p-nuclear operators. □

By Proposition 3.1.2 [16], we know that if $f \in \mathrm{Lip}_0(X,E)$, $g \in \mathrm{Lip}_0(X,F)$, and T_f majorizes T_g, then f majorizes g. Moreover, for $f \in \mathrm{Lip}_0(X,E)$, $g \in \mathrm{Lip}_0(X,F)$ which T_f majorizes T_g, by Proposition 3.1.3 [16], we know that f is Lipschitz (weakly) compact, then g is Lipschitz (weakly) compact. The following proposition demonstrates that the same result is obtained if the condition T_f majorizes T_g (see Proposition 3.1.3 [16]) replaced by f majorizes g.

Proposition 3.33. Let $f \in \mathrm{Lip}_0(X,E)$ and $g \in \mathrm{Lip}_0(X,F)$ so that f majorizes g. If f is Lipschitz compact (respectively, Lipschitz weakly compact), then g is Lipschitz compact (respectively, Lipschitz weakly compact).

Proof. If f majorizes g, then, by Theorem 3.1.1 [16], $R(g^t) \subset R(f^t)$. Moreover, by Proposition 3.5 [1] (respectively, Proposition 3.4 [1]), f is Lipschitz compact (respectively, Lipschitz weakly compact) if and only if f^t is compact (respectively, weakly compact). Since $R(g^t) \subset R(f^t)$, by Proposition 8 [15], g^t is compact (respectively, weakly compact). Then, by Proposition 3.5 [1] (respectively, Proposition 3.4 [1]), g is Lipschitz compact (respectively, Lipschitz weakly compact). □

4. Conclusion

This study has extensively explored the various classes of Lipschitz operators linked to weakly p-compact and unconditionally p-compact sets. By introducing these operator classes and scrutinizing the properties of associated linear operators, various results have been obtained regarding the properties of these operators. Notably, the study has highlighted that some of these classes are generalizations of related linear operators. Additionally, through evaluating majorization and comparing newly acquired and existing results, the study has drawn meaningful conclusions. Overall, this study contributes to understanding Lipschitz operators related to various sets.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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