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On New Pell Spinor Sequences

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Abstract

Our motivation for this study is to define two new and particular sequences. The most essential feature of these sequences is that they are spinor sequences. In this study, these new spinor sequences obtained using spinor representations of Pell and Pell-Lucas quaternions are expressed. Moreover, some formulas such that Binet formulas, Cassini formulas and generating functions of these spinor sequences, which are called as Pell and Pell-Lucas spinor sequences, are given. Then, some relationships between Pell and Pell-Lucas spinor sequences are obtained. Therefore, an easier and more interesting representations of Pell and Pell-Lucas quaternions, which are a generalization of Pell and Pell-Lucas number sequences, are obtained. We believe that these new spinor sequences will be useful and advantageable in many branches of science, such as geometry, algebra and physics.

Keywords: *Pell, Pell-Lucas, Spinor*

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1. Introduction and Preliminaries

The number sequences are a subject that is frequently used in mathematics and attracts the attention of readers. The first number sequences that come to mind are the Fibonacci number sequences expressed by Fibonacci (1170-1250), which are frequently encountered in nature [1–3]. The Lucas number sequence, which is obtained by writing the next term as the sum of the previous two terms but with different initial conditions, is another example of a number sequence. In addition, there are many number sequences in the literature, such as the Fibonacci number sequence, whose characteristic equation is different. Moreover, considering different characteristic equations and initial values, different number sequences can be obtained, such as Pell, Pell-Lucas, Modified Pell, Jacobsthal and Jacobsthal-Lucas number sequences etc. [4–6]. Moreover, another studies of this subject are [7, 8, 10, 11, 27]. Horadam discussed Pell numbers and their properties [5]. Patel and Shrivastava obtained some of these with their proofs using Binet forms of some Pell and Pell-Lucas identities [12]. These properties are used to derive generator functions, polynomials, divisibility properties, matrices, determinants of Pell and Pell-Lucas sequences, and many other applications. Koshy mentioned that Pell numbers and Pell-Lucas numbers are special values of Pell and Pell-Lucas polynomials, respectively [13]. Halıcı and Daşdemir studied some relationships between Pell, Pell-Lucas,

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and Modified Pell sequences [14]. Szynal and Wloch studied Pell, Pell-Lucas numbers, quaternions, octonions and recurrence relations [15]. Catarino discussed k-Pell quaternions and octanions and offered some features, including the Binet formula and a generating function [16]. Moreover, Çimen and İpek gave a new quaternion sequence such that Pell and Pell-Lucas quaternion sequence [17].

Spinors can be defined in a simple way as vectors of a space whose transformations are related to spins in physical space. The person who first introduced spinors in a geometric sense was Cartan [18]. Cartan's study [18] is an admirable study in spinor geometry because in this study, spinor representations of the some basic geometric definitions are expressed by Cartan in an easy and understandable way. Another inspiring study on the spinors in geometry was done by Vivarelli [19]. In Vivarelli's study [19], the relationships between quaternions and spinors and spinor representations of 3D rotations were obtained. In the study of Torres del Castillo and Barrales, the spinor representations of the Frenet frame and curvatures of any curve in Euclidean 3-space were given [20]. The spinor representation of the Darboux frame in Euclidean 3-space was obtained [21]. Moreover, in [22], the spinor representation of the Bishop frame in Euclidean 3-space was expressed. On the other hand, the spinor equations for some special curves such as Bertrand, involute-evolute, successor, and Mannheim curves and for Lie groups were obtained [23–27]. Then, for any Minkowski space, hyperbolic spinor equations were given [28–31]. In addition to that, Fibonacci and Lucas spinors were expressed in [32].

Now, the spinors, real quaternions, relationships between them spinors, and Pell, Pell-Lucas quaternions are given.

Assume that any isotropic vector is $\boldsymbol{v} = (v_1,\ v_2,\ v_3) \in \mathbb{C}^3$ where ${v_1}^2 + {v_2}^2 + {v_3}^2 = 0$ and the complex vector space with 3-dimensional is $\mathbb{C}^3$. We can express the set of isotropic vectors in $\mathbb{C}^3$ with the aid of a two-dimensional surface in $\mathbb{C}^2$. Suppose that this two-dimensional surface has coordinates ϖ_1 and ϖ_2. So, we can write $v_1 = {\varpi_1}^2 - {\varpi_2}^2$, $v_2 = \mathbf{i}({\varpi_1}^2 + {\varpi_2}^2)$, $v_3 = -2\varpi_1\varpi_2$ and $\varpi_1 = \pm\sqrt{\frac{v_1 - \mathbf{i}v_2}{2}}$, $\varpi_2 = \pm\sqrt{\frac{-v_1 - \mathbf{i}v_2}{2}}$. Two-dimensional complex vector mentioned above is called as ***spinor*** by Cartan such that

$$\varpi = (\varpi_1,\ \varpi_2) = \begin{bmatrix} \varpi_1 \\ \varpi_2 \end{bmatrix}$$

in spinor space $\mathbb{S}$ [18].

Suppose that any real quaternion is $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ where $q_0, q_1, q_2, q_3 \in \mathbb{R}$. $\{\boldsymbol{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is called the quaternion basis such that

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1, \quad \boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}$$

[33]. We can write $q = S_q + \mathbf{V}_q$ where $q_0 = S_q$ and $\mathbf{V}_q = \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ is called scalar and vector parts of q, respectively [33]. Assume that two any real quaternions $p = S_p + \mathbf{V}_p,\ q = S_q + \mathbf{V}_q$. So, the quaternion product of these quaternions is

$$p \times q = S_pS_q - \langle \mathbf{V}_p, \mathbf{V}_q \rangle + S_p\mathbf{V}_q + S_q\mathbf{V}_p + \mathbf{V}_p \wedge \mathbf{V}_q,$$

where $\langle , \rangle$ is inner product and $\wedge$ is vector product in $\mathbb{R}^3$ [33]. We know that the product of two real quaternions is non-commutative. In addition to that, the quaternion conjugate and the norm of q are given as $q^* = S_q - \boldsymbol{V}_q$ and $N(q) = \sqrt{{q_1}^2 + {q_2}^2 + {q_3}^2 + {q_4}^2}$. Let the norm of q be $N(q) = 1$, then q is defined as unit quaternion [33].

Vivarelli expressed a relationship between spinors and quaternions such that

$$\begin{aligned} &f : \mathbb{H} \to \mathbb{S} \\ &\quad q \to f(q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3) \cong \begin{bmatrix} q_3 + \mathbf{i}q_0 \\ q_1 + \mathbf{i}q_2 \end{bmatrix} \equiv \varpi \end{aligned} \tag{1.1}$$

where $q = q_0 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3$ is any real quaternion [19]. Then, Vivarelli gave a spinor representation of $q \times p$ such that

$$q \times p \to -\mathbf{i}\hat{\varpi}\rho. \tag{1.2}$$

where the spinor ρ corresponds to the real quaternion p with the aid of the transformation f in the equation (1.1) and the complex, unitary, square matrix $\hat{\varpi}$ can be written as

$$\hat{\varpi} = \begin{bmatrix} q_3 + \mathbf{i}q_0 & q_1 - \mathbf{i}q_2 \\ q_1 + \mathbf{i}q_2 & -q_3 + \mathbf{i}q_0 \end{bmatrix} \tag{1.3}$$

[19]. In addition, the spinor matrix $\varpi_L = -\mathbf{i}\widehat{\varpi}$, namely

$$\varpi_L = \begin{bmatrix} q_0 - \mathbf{i}q_3 & -q_2 - \mathbf{i}q_1 \\ q_2 - \mathbf{i}q_1 & q_0 + \mathbf{i}q_3 \end{bmatrix} \tag{1.4}$$

was called the left Hamilton spinor matrix or fundamental spinor matrix of q [19, 34].

Now, the some equalities about Pell and Pell-Lucas quaternions given in [17] can be expressed. But before that we would like to touch upon an important issue here. There are many studies in the literature about Pell and Pell Lucas number sequences and Pell and Pell-Lucas quaternion sequences. In these studies, while the initial conditions of Pell number sequences are taken as 0 and 1, there is an information confusion regarding the initial conditions of Pell-Lucas number sequences. That is, in some studies, the initial conditions of Pell-Lucas number sequences are taken as 1, 1, while in some studies, the initial conditions are taken as 2, 2. Additionally, in some studies, the expression "Modified Pell number sequence" was used in studies with initial conditions of 1, 1. Actually, there is no problem so far. The real problem is that if the initial conditions are taken differently, some formulas such as Binet, Cassini and sum formulas turn out to be different. Also, the relationships between Pell and Pell-Lucas are different. For example, if you take the initial condition of Pell-Lucas number sequence as 1, 1, you shouldn't use formulas in another study where the initial condition is 2, 2. Otherwise, an information confusion is created in the literature. In this study, the initial conditions of Pell-Lucas number sequence are taken as $Q_0 = 2$, $Q_1 = 2$ and the formulas are used accordingly. Now, we expressed Pell and Pell-Lucas quaternions.

For $n \geq 2$ the nth Pell quaternion and Pell-Lucas quaternion is defined that

$$QP_n = P_n + \boldsymbol{i}P_{n+1} + \boldsymbol{j}P_{n+2} + \boldsymbol{k}P_{n+3}$$

and

$$QPL_n = Q_n + \boldsymbol{i}Q_{n+1} + \boldsymbol{j}Q_{n+2} + \boldsymbol{k}Q_{n+3}$$

where the nth Pell number and Pell-Lucas number $P_n = 2P_{n-1} + P_{n-2}$ and $P_0 = 0$, $P_1 = 1$ and $Q_n = 2Q_{n-1} + Q_{n-2}$ and $Q_0 = 2$, $Q_1 = 2$ [17]. Moroever, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coincide with basis vectors given for real quaternions. Therefore, the recurrence relation of Pell and Pell-Lucas quaternions for $n \geq 2$ are

$$QP_n = 2QP_{n-1} + QP_{n-2}$$

with initial conditions $QP_0 = \boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$, $QP_1 = 1 + 2\boldsymbol{i} + 5\boldsymbol{j} + 12\boldsymbol{k}$ and

$$QPL_n = 2QPL_{n-1} + QPL_{n-2}$$

with initial conditions $QPL_0 = 2 + 2\boldsymbol{i} + 6\boldsymbol{j} + 14\boldsymbol{k}$, $QPL_1 = 2 + 6\boldsymbol{i} + 14\boldsymbol{j} + 34\boldsymbol{k}$ [17].

Now, we write the some relationship between Pell and Pell-Lucas quaternions with the aid of [5, 12, 14–16, 35, 36]. Therefore, we can write these relationships that

$$\begin{array}{l} i)\ QP_{n-1} + QP_{n+1} = QPL_n, \\ ii)\ QPL_n + QPL_{n+1} = 4QP_{n+1}, \\ iii)\ QPL_{n+1} + QPL_{n-1} = 8QP_n. \end{array}$$

Moreover, the Binet formula for Pell and Pell-Lucas quaternions are given that

$$QP_n = \frac{\gamma^n \underline{\gamma} - \mu^n \underline{\mu}}{\gamma - \mu}$$

and

$$QPL_n = \gamma^n \underline{\gamma} + \mu^n \underline{\mu}$$

where the quaternions $\underline{\gamma}$ and $\underline{\mu}$ are $\underline{\gamma} = 1 + \boldsymbol{i}\gamma + \boldsymbol{j}\gamma^2 + \boldsymbol{k}\gamma^3$ and $\underline{\mu} = 1 + \boldsymbol{i}\mu + \boldsymbol{j}\mu^2 + \boldsymbol{k}\mu^3$, $\gamma = 1 + \sqrt{2}$, $\mu = 1 - \sqrt{2}$ are roots of the characteristic equation $x^2 - 2x - 1 = 0$.

On the other hand, we give the generating functions of Pell and Pell-Lucas quaternions such that

$$G_P(t) = \frac{QP_0 + (QP_1 - 2QP_0)\,t}{1 - 2t - t^2}$$

and

$$G_{PL}(t) = \frac{QPL_0 + (QPL_1 - 2QPL_0)\,t}{1 - 2t - t^2},$$

respectively. In addition to that, Cassini formula for Pell and Pell-Lucas quaternions can be given that

$$QP_{n-1}QP_{n+1} - (QP_n)^2 = (-1)^n \left(\frac{\gamma \underline{\mu}\,\underline{\gamma} - \mu \underline{\gamma}\,\underline{\mu}}{\gamma - \mu} \right)$$

and

$$QPL_{n-1}QPL_{n+1} - (QPL_n)^2 = (-1)^{n-1} (\gamma - \mu)(\gamma \underline{\mu}\,\underline{\gamma} - \mu \underline{\gamma}\,\underline{\mu}),$$

respectively.

2. Main Theorems and Results

We know that there is a spinor for every real quaternion by means of the transformation f in the equation (1.1). Considering this information, a new transformation between Pell and Pell-Lucas quaternions and spinors can be defined and the spinors corresponding to Pell and Pell-Lucas quaternions can be given. Therefore, these spinors associated with Pell and Pell-Lucas quaternions are called as Pell and Pell-Lucas spinors. Then, some formulas such that Binet, Cassini, sum formulas and generating functions for these quaternions spinors and theorems are given.

Definition 2.1. Let $QP_n = P_n + \boldsymbol{i}P_{n+1} + \boldsymbol{j}P_{n+2} + \boldsymbol{k}P_{n+3}$ be nth Pell quaternion where P_n is nth Pell number and the set of Pell quaternions be $\mathbb{Q}_P$. Therefore, the following linear transformation is defined as

$$\begin{aligned} f_P :\ & \mathbb{Q}_P \to \mathbb{S} \\ & QP_n \mapsto f_P(QP_n) \cong SP_n = \begin{bmatrix} P_{n+3} + \mathbf{i}P_n \\ P_{n+1} + \mathbf{i}P_{n+2} \end{bmatrix} \end{aligned} \tag{2.1}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coincide with basis vectors in $\mathbb{R}^3$ and $\mathbf{i}^2 = -1$. So, a new sequence for the spinors related with Pell quaternions is defined and this sequence is called as *"Pell Spinor Sequence"* defined as

$$\{SP_n\}_{n\in\mathbb{N}}^{\infty} = \left\{ \begin{bmatrix} 5 \\ 1 + 2\mathbf{i} \end{bmatrix}, \begin{bmatrix} 12 + \mathbf{i} \\ 2 + 5\mathbf{i} \end{bmatrix}, \begin{bmatrix} 29 + 2\mathbf{i} \\ 5 + 12\mathbf{i} \end{bmatrix}, \begin{bmatrix} 70 + 5\mathbf{i} \\ 12 + 29\mathbf{i} \end{bmatrix}, \ldots \right\}$$

where $SP_n = \begin{bmatrix} P_{n+3} + \mathbf{i}P_n \\ P_{n+1} + \mathbf{i}P_{n+2} \end{bmatrix}$ is nth Pell spinor and P_n is nth Pell number.

Similarly, we can give the following definition of Pell-Lucas spinor sequence.

Definition 2.2. Let $QPL_n = Q_n + \boldsymbol{i}Q_{n+1} + \boldsymbol{j}Q_{n+2} + \boldsymbol{k}Q_{n+3}$ be nth Pell-Lucas quaternion where Q_n is nth Pell-Lucas number and the set of Pell-Lucas quaternions be $\mathbb{Q}_{PL}$. Therefore, the following linear transformation is defined as

$$\begin{aligned} f_{PL} :\ & \mathbb{Q}_{PL} \to \mathbb{S} \\ & QPL_n \mapsto f_{PL}(QPL_n) \cong SPL_n = \begin{bmatrix} Q_{n+3} + \mathbf{i}Q_n \\ Q_{n+1} + \mathbf{i}Q_{n+2} \end{bmatrix}. \end{aligned}$$

Therefore, a new sequence for the spinors related with Pell-Lucas quaternions is called as ***"Pell-Lucas Spinor Sequence"*** where

$$\{SPL_n\}_{n\in\mathbb{N}}^{\infty} = \left\{ \begin{bmatrix} 14 + 2\mathbf{i} \\ 2 + 6\mathbf{i} \end{bmatrix}, \begin{bmatrix} 34 + 2\mathbf{i} \\ 6 + 14\mathbf{i} \end{bmatrix}, \begin{bmatrix} 82 + 6\mathbf{i} \\ 14 + 34\mathbf{i} \end{bmatrix}, \begin{bmatrix} 198 + 14\mathbf{i} \\ 34 + 82\mathbf{i} \end{bmatrix}, \ldots \right\}$$

where $SPL_n = \begin{bmatrix} Q_{n+3} + \mathbf{i}Q_n \\ Q_{n+1} + \mathbf{i}Q_{n+2} \end{bmatrix}$ is nth Pell-Lucas spinor and Q_n is nth Pell-Lucas number.

Definition 2.3. The conjugate of Pell quaternion QP_n is QP_n^*, and Pell spinor corresponding to this conjugate is defined as

$$SP_n^* = \begin{bmatrix} -P_{n+3} + \mathbf{i}P_n \\ -P_{n+1} - \mathbf{i}P_{n+2} \end{bmatrix}.$$

Similarly, Pell Lucas spinor corresponding to the conjugate of Pell-Lucas quaternion QPL_n is defined as

$$SPL_n^* = \begin{bmatrix} -Q_{n+3} + \mathbf{i}Q_n \\ -Q_{n+1} - \mathbf{i}Q_{n+2} \end{bmatrix}.$$

Definition 2.4. Pell spinor representation of the norm of Pell quaternion QP_n is

$$\overline{SP_n}^t SP_n.$$

Similarly, Pell-Lucas spinor representation of the norm of Pell-Lucas quaternion QPL_n is

$$\overline{SPL_n}^t SPL_n.$$

Now, the recurrence relations of Pell and Pell-Lucas spinor sequences with the following equations can be obtained.

Theorem 2.1. *The recurrence relation of Pell spinors for $n \geqslant 2$ is*

$$SP_n = 2SP_{n-1} + SP_{n-2}$$

where nth, $(n-1)th$ and $(n+1)th$ Pell spinors are SP_n, SP_{n-1} and SP_{n-2}, respectively. The recurrence relation for Pell-Lucas spinor for $n \geqslant 2$ is

$$SPL_n = 2SPL_{n-1} + SPL_{n-2}$$

where nth, $(n-1)th$ and $(n+1)th$ Pell-Lucas spinors are SPL_n, SPL_{n-1} and SPL_{n-2}, respectively.

Proof. Firstly, we show the recurrence relation for Pell spinors. Therefore, if we calculate $2SP_{n-1} + SP_{n-2}$, then we obtain

$$2SP_{n-1} + SP_{n-2} = 2\begin{bmatrix} P_{n+2} + \mathbf{i}P_{n-1} \\ P_n + \mathbf{i}P_{n+1} \end{bmatrix} + \begin{bmatrix} P_{n+1} + \mathbf{i}P_{n-2} \\ P_{n-1} + \mathbf{i}P_n \end{bmatrix}$$

$$= \begin{bmatrix} 2P_{n+2} + P_{n+1} + \mathbf{i}(2P_{n-1} + P_{n-2}) \\ 2P_n + P_{n-1} + \mathbf{i}(2P_{n+1} + P_n) \end{bmatrix}.$$

Since the recurrence relation for Pell number sequence is $P_n = 2P_{n-1} + P_{n-2}$, we have

$$2SP_{n-1} + SP_{n-2} = \begin{bmatrix} P_{n+3} + \mathbf{i}P_n \\ P_{n+1} + \mathbf{i}P_{n+2} \end{bmatrix} = SPn.$$

Similarly, we can easily obtain for Pell-Lucas spinor sequence such that

$$2SPL_{n-1} + SPL_{n-2} = 2\begin{bmatrix} Q_{n+2} + \mathbf{i}Q_{n-1} \\ Q_n + \mathbf{i}Q_{n+1} \end{bmatrix} + \begin{bmatrix} Q_{n+1} + \mathbf{i}Q_{n-2} \\ Q_{n-1} + \mathbf{i}Q_n \end{bmatrix}$$

$$= \begin{bmatrix} 2Q_{n+2} + Q_{n+1} + \mathbf{i}(2Q_{n-1} + Q_{n-2}) \\ 2Q_n + Q_{n-1} + \mathbf{i}(2Q_{n+1} + Q_n) \end{bmatrix}$$

$$= \begin{bmatrix} Q_{n+3} + \mathbf{i}Q_n \\ Q_{n+1} + \mathbf{i}Q_{n+2} \end{bmatrix} = SPLn$$

where the recurrence relation $Q_n = 2Q_{n-1} + Q_{n-2}$ of Pell Lucas number sequence is used $(n \geqslant 2)$. ☐

Now, the some relations between Pell and Pell-Lucas spinors can be given.

Theorem 2.2. *Let nth Pell and Pell-Lucas spinors be SP_n and SPL_n, respectively. In this case, for $n \geqslant 2$ there are the following relations between these spinors;*

$$\begin{aligned} i)&\ \ SP_{n-1} + SP_{n+1} = SPL_n, \\ ii)&\ \ SPL_n + SPL_{n+1} = 4SP_{n+1}, \\ iii)&\ \ SPL_{n+1} + SPL_{n-1} = 8SP_n, \\ iv)&\ \ 2SP_n + 2SP_{n-1} = SPL_n. \end{aligned}$$

Proof. i) Let $(n-1)th$ and $(n+1)th$ Pell spinors be SP_{n-1} and SP_{n+1}, respectively. Then, we can write the equation

$$SP_{n-1}+SP_{n+1}=\begin{bmatrix}P_{n+2}+\mathbf{i}P_{n-1}\\ P_n+\mathbf{i}P_{n+1}\end{bmatrix}+\begin{bmatrix}P_{n+4}+\mathbf{i}P_{n+1}\\ P_{n+2}+\mathbf{i}P_{n+3}\end{bmatrix}$$

$$=\begin{bmatrix}P_{n+2}+P_{n+4}+\mathbf{i}(P_{n-1}+P_{n+1})\\ P_n+P_{n+2}+\mathbf{i}(P_{n+1}+P_{n+3})\end{bmatrix}.$$

On the other hand, we know that the relationship between Pell and Pell-Lucas numbers is $Q_n=P_{n-1}+P_{n+1}$ from [35]. If we use this relationship we can write

$$SP_{n-1}+SP_{n+1}=\begin{bmatrix}Q_{n+3}+\mathbf{i}Q_n\\ Q_{n+1}+\mathbf{i}Q_{n+2}\end{bmatrix}=SPL_n.$$

This completes the proof.

ii) Assume that nth and $(n+1)th$ Pell-Lucas spinors are SPL_n and SPL_{n+1}. Therefore, we have

$$SPL_n+SPL_{n+1}=\begin{bmatrix}Q_{n+3}+\mathbf{i}Q_n\\ Q_{n+1}+\mathbf{i}Q_{n+2}\end{bmatrix}+\begin{bmatrix}Q_{n+4}+\mathbf{i}Q_{n+1}\\ Q_{n+2}+\mathbf{i}Q_{n+3}\end{bmatrix}$$

$$=\begin{bmatrix}Q_{n+3}+Q_{n+4}+\mathbf{i}(Q_n+Q_{n+1})\\ Q_{n+1}+Q_{n+2}+\mathbf{i}(Q_{n+2}+Q_{n+3})\end{bmatrix}.$$

In addition to that, we know that there is the relationship $4P_{n+1}=Q_n+Q_{n+1}$ between Pell and Pell-Lucas numbers from [37]. So, we get

$$SPL_n+SPL_{n+1}=\begin{bmatrix}4P_{n+4}+\mathbf{i}4P_{n+1}\\ 4P_{n+2}+\mathbf{i}4P_{n+3}\end{bmatrix}=4SP_{n+1}$$

iii) Suppose that $(n-1)th$ and $(n+1)th$ Pell-Lucas spinors are SPL_{n-1} and SPL_{n+1}, respectively. Then, we get

$$SPL_{n+1}+SPL_{n-1}=\begin{bmatrix}Q_{n+4}+\mathbf{i}Q_{n+1}\\ Q_{n+2}+\mathbf{i}Q_{n+3}\end{bmatrix}+\begin{bmatrix}Q_{n+2}+\mathbf{i}Q_{n-1}\\ Q_n+\mathbf{i}Q_{n+1}\end{bmatrix}$$

$$=\begin{bmatrix}Q_{n+4}+Q_{n+2}+\mathbf{i}(Q_{n+1}+Q_{n-1})\\ Q_{n+2}+Q_n+\mathbf{i}(Q_{n+3}+Q_{n+1})\end{bmatrix}=\begin{bmatrix}8P_{n+3}+\mathbf{i}8P_n\\ 8P_{n+1}+\mathbf{i}8P_{n+2}\end{bmatrix}=8P_n$$

where $8P_n=Q_{n+1}+Q_{n-1}$.

iv) This proof is clear that $SPL_n=SP_{n-1}+SP_{n+1}$ from option i). Moreover, we know that $SP_{n+1}=2SP_n+SP_{n-1}$. Consequently,

$$SPL_n=SP_{n-1}+2SP_n+SP_{n-1}=2SP_n+2SP_{n-1}.$$

This completes the proof. □

Theorem 2.3. *Assume that nth Pell and Pell-Lucas spinors are SP_n and SPL_n, respectively. Therefore, the Binet Formulas for these spinors are the following equations. The Binet formula for Pell spinors is*

$$SP_n=\frac{1}{\gamma-\mu}\Big(\gamma^nS_\gamma-\mu^nS_\mu\Big),$$

the Binet formula for Pell-Lucas spinors is

$$SPL_n=\gamma^nS_\gamma+\mu^nS_\mu$$

where $\gamma=1+\sqrt{2}$, $\mu=1-\sqrt{2}$ are the roots of characteristic equation $x^2-2x-1=0$ and $S_\gamma=\begin{bmatrix}\gamma^3+\boldsymbol{i}\\ \gamma+\boldsymbol{i}\gamma^2\end{bmatrix}$ and $S_\mu=\begin{bmatrix}\mu^3+\boldsymbol{i}\\ \mu+\boldsymbol{i}\mu^2\end{bmatrix}$.

Proof. First, we prove it for Pell spinors. We know that the Binet formula for Pell number sequence is

$$P_n = \frac{\gamma^n - \mu^n}{\gamma - \mu}$$

where $\gamma = 1 + \sqrt{2},\ \mu = 1 - \sqrt{2}$. Therefore, if we write the last equation in the nth Pell spinor we obtain

$$SP_n = \begin{bmatrix} P_{n+3} + \mathbf{i}P_n \\ P_{n+1} + \mathbf{i}P_{n+2} \end{bmatrix} = \frac{1}{\gamma - \mu}\begin{bmatrix} \gamma^{n+3} - \mu^{n+3} + \mathbf{i}(\gamma^n - \mu^n) \\ \gamma^{n+1} - \mu^{n+1} + \mathbf{i}(\gamma^{n+2} - \mu^{n+2}) \end{bmatrix}$$

$$SP_n = \frac{1}{\gamma - \mu}\left(\begin{bmatrix} \gamma^{n+3} + \mathbf{i}\gamma^n \\ \gamma^{n+1} + \mathbf{i}\gamma^{n+2} \end{bmatrix} - \begin{bmatrix} \mu^{n+3} + \mathbf{i}\mu^n \\ \mu^{n+1} + \mathbf{i}\mu^{n+2} \end{bmatrix} \right)$$

$$SP_n = \frac{1}{\gamma - \mu}\left(\gamma^n \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix} - \mu^n \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix} \right)$$

or

$$SP_n = \frac{1}{\gamma - \mu}\left(\gamma^n S_\gamma - \mu^n S_\mu \right)$$

where $S_\gamma = \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix}$ and $S_\mu = \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$.

Now, we give the Binet formula for Pell-Lucas spinors. We know that the Binet formula for Pell-Lucas number sequence is $Q_n = \gamma^n + \mu^n$. In this case, we can obtain

$$SPL_n = \begin{bmatrix} Q_{n+3} + \mathbf{i}Q_n \\ Q_{n+1} + \mathbf{i}Q_{n+2} \end{bmatrix} = \begin{bmatrix} \gamma^{n+3} + \mu^{n+3} + \mathbf{i}(\gamma^n + \mu^n) \\ \gamma^{n+1} + \mu^{n+1} + \mathbf{i}(\gamma^{n+2} + \mu^{n+2}) \end{bmatrix}$$

$$SPL_n = \begin{bmatrix} \gamma^{n+3} + \mathbf{i}\gamma^n \\ \gamma^{n+1} + \mathbf{i}\gamma^{n+2} \end{bmatrix} + \begin{bmatrix} \mu^{n+3} + \mathbf{i}\mu^n \\ \mu^{n+1} + \mathbf{i}\mu^{n+2} \end{bmatrix}$$

$$SPL_n = \gamma^n \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix} + \mu^n \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$$

or

$$SPL_n = \gamma^n S_\gamma + \mu^n S_\mu$$

where $S_\gamma = \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix}$ and $S_\mu = \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$. □

Theorem 2.4. *Let nth Pell and Pell-Lucas spinors be SP_n and SPL_n, respectively. The sum formulas for Pell spinors are the following options;*

$$\text{i)}\quad \sum_{t=0}^{n} SP_t = \frac{1}{4}\Big[SPL_{n+1} - SPL_0 \Big],$$

$$\text{ii)}\quad \sum_{t=0}^{n} SP_{2t} = \frac{1}{2}\Big[SP_{2n+1} - SP_{-1} \Big],$$

$$\text{iii)}\quad \sum_{t=0}^{n} SP_{2t-1} = \frac{1}{2}\Big[SP_{2n} - SP_{-2} \Big].$$

Proof. i) We know that for Pell spinors the Binet formula is $SP_n = \frac{1}{\gamma-\mu}(\gamma^n S_\gamma - \mu^n S_\mu)$. Therefore, we can write

$$\begin{aligned} \sum_{t=0}^{n} SP_t &= \sum_{t=0}^{n} \frac{1}{\gamma - \mu}(\gamma^t S_\gamma - \mu^t S_\mu) \\ &= \frac{1}{\gamma - \mu}(\sum_{t=0}^{n} \gamma^t S_\gamma - \sum_{t=0}^{n} \mu^t S_\mu). \end{aligned} \tag{2.2}$$

On the other hand, we know that $\sum_{t=0}^{n}\gamma^t=\frac{1-\gamma^{n+1}}{1-\gamma}$ and $\sum_{t=0}^{n}\mu^t=\frac{1-\mu^{n+1}}{1-\mu}$. If we use these information in the last equation then, we get

$$\sum_{t=0}^{n} SP_t=\frac{1}{4}\Bigg((\gamma^{n+1}S_\gamma+\mu^{n+1}S_\mu)-(S_\gamma+S_\mu)\Bigg)$$

where $\gamma-\mu=2\sqrt{2}$. Moreover, for Pell-Lucas spinors the Binet formula is $SPL_n=\gamma^n S_\gamma+\mu^n S_\mu$. So, we can obtain that

$$\sum_{t=0}^{n} SP_t=\frac{1}{4}(SPL_{n+1}-SPL_0)$$

and this completes the proof.

ii) Similarly, if we use the Binet formula for Pell spinors then we easily get

$$\begin{aligned}\sum_{t=0}^{n} SP_{2t}&=\sum_{t=0}^{n}\frac{1}{\gamma-\mu}(\gamma^{2t}S_\gamma-\mu^{2t}S_\mu)\\&=\frac{1}{\gamma-\mu}\Bigg(\sum_{t=0}^{n}\gamma^{2t}S_\gamma-\sum_{t=0}^{n}\mu^{2t}S_\mu\Bigg).\end{aligned}$$

Moreover, we know that $\sum_{t=0}^{n}\gamma^{2t}=\frac{1-\gamma^{2n+2}}{1-\gamma^2}$ and $\sum_{t=0}^{n}\mu^{2t}=\frac{1-\mu^{2n+2}}{1-\mu^2}$. Therefore, we have

$$\begin{aligned}\sum_{t=0}^{n} SP_{2t}&=\frac{1}{2(\gamma-\mu)}\Bigg(\frac{1-\mu^{2n+2}}{\mu}S_\mu-\frac{1-\gamma^{2n+2}}{\gamma}S_\gamma\Bigg)\\&=\frac{1}{2(\gamma-\mu)}\Bigg(\mu S_\gamma-\gamma S_\mu+\gamma^{2n+1}S_\gamma-\mu^{2n+1}S_\mu\Bigg)\end{aligned}$$

where $\gamma\mu=-1$. Then, we obtain

$$\sum_{t=0}^{n} SP_{2t}=\frac{1}{2}(SP_{2n+1}+SP_0-\frac{1}{2}SPL_0).$$

In addition to that, if we use $SPL_0=2SP_0+SP_{-1}$ from Theorem (2.2) , we easily get

$$\sum_{t=0}^{n} SP_{2t}=\frac{1}{2}(SP_{2n+1}-SP_{-1}).$$

iii) We use the Binet formula for Pell spinors. So, we can write

$$\begin{aligned}\sum_{t=0}^{n} SP_{2t-1}&=\sum_{t=0}^{n}\frac{1}{\gamma-\mu}(\gamma^{2t-1}S_\gamma-\mu^{2t-1}S_\mu)\\&=\frac{1}{\gamma-\mu}\Bigg(\sum_{t=0}^{n}\gamma^{2t-1}S_\gamma-\sum_{t=0}^{n}\mu^{2t-1}S_\mu\Bigg).\end{aligned}$$

Similar to the other options i) and ii) we can easily obtain that

$$\begin{aligned}\sum_{t=0}^{n} SP_{2t-1}&=\frac{1}{2(\gamma-\mu)}\Bigg(\gamma^2(1-\mu^{2n+2})S_\mu-\mu^2(1-\gamma^{2n+2})S_\gamma\Bigg)\\&=\frac{1}{2(\gamma-\mu)}\Bigg(\gamma^{2n}S_\gamma-\mu^{2n}S_\mu+2\sqrt{2}(S_\gamma+S_\mu)-3(S_\gamma-S_\mu)\Bigg).\end{aligned}$$

If we use Binet formulas for Pell and Pell-Lucas spinors then, we get

$$\sum_{t=0}^{n} SP_{2t-1}=\frac{1}{2}(SP_{2n}-3SP_0+SPL_0)$$

and consequently

$$\sum_{t=0}^{n} SP_{2t-1} = \frac{1}{2}(SP_{2n} - SP_{-2})$$

where we know that $SPL_0 = 2SP_0 + 2SP_{-1}$ and $SP_0 = 2SP_{-1} + SP_{-2}$. This proof is completed. □

Now, considering [18, 34] we express the following definition.

Definition 2.5. Suppose that SP_n and SPL_n are nth Pell and Pell-Lucas spinors. The fundamental Pell and Pell-Lucas spinor matrices are

$$(SP_n)_L = \begin{bmatrix} P_n - \mathbf{i}P_{n+3} & -P_{n+2} - \mathbf{i}P_{n+1} \\ P_{n+2} - \mathbf{i}P_{n+1} & P_n + \mathbf{i}P_{n+3} \end{bmatrix}$$

and

$$(SPL_n)_L = \begin{bmatrix} Q_n - \mathbf{i}Q_{n+3} & -Q_{n+2} - \mathbf{i}Q_{n+1} \\ Q_{n+2} - \mathbf{i}Q_{n+1} & Q_n + \mathbf{i}Q_{n+3} \end{bmatrix}.$$

The fundamental Pell and Pell-Lucas spinor matrices are also called as left Hamilton Pell and Pell Lucas spinor matrices, respectively.

Now, we express the Cassini Formula for Pell and Pell-Lucas spinors.

Theorem 2.5. *The similar formula replacing Cassini formula for Pell spinors is*

$$(SP_{n-1})_L SP_{n+1} - (SP_n)_L SP_n = (-1)^n \frac{1}{\gamma - \mu}(\gamma(S_\mu)_L S_\gamma - \mu(S_\gamma)_L S_\mu)$$

and for Pell-Lucas spinors the similar formula is

$$(SPL_{n-1})_L SPL_{n+1} - (SPL_n)_L SPL_n (-1)^{n-1}(\gamma - \mu)(\gamma(S_\mu)_L S_\gamma - \mu(S_\gamma)_L S_\mu)$$

where $S_\mu = \begin{bmatrix} \mu^3 + \boldsymbol{i} \\ \mu + \boldsymbol{i}\mu^2 \end{bmatrix}$, $(S_\mu)_L = \begin{bmatrix} 1 - \boldsymbol{i}\mu^3 & -\mu^2 - \boldsymbol{i}\mu \\ \mu^2 - \boldsymbol{i}\mu & 1 + \boldsymbol{i}\mu^3 \end{bmatrix}$, $S_\gamma = \begin{bmatrix} \gamma^3 + \boldsymbol{i} \\ \gamma + \boldsymbol{i}\gamma^2 \end{bmatrix}$, $(S_\gamma)_L = \begin{bmatrix} 1 - \boldsymbol{i}\gamma^3 & -\gamma^2 - \boldsymbol{i}\gamma \\ \gamma^2 - \boldsymbol{i}\gamma & 1 + \boldsymbol{i}\gamma^3 \end{bmatrix}$.

Proof. Pell spinor product corresponding to the product of Pell quaternions $QP_{n-1}QP_{n+1} - (QP_n)^2$ is $(SP_{n-1})_L SP_{n+1} - (SP_n)_L SP_n$. In this case, if we use the Binet formula in Theorem (2.3) for Pell spinors $SP_n = \frac{1}{\gamma-\mu}(\gamma^n S_\gamma - \mu^n S_\mu)$, then we get

$$(SP_n)_L = \frac{1}{\gamma - \mu}(\gamma^n L_{S_\gamma} - \mu^n L_{S_\mu}).$$

Therefore, we obtain

$$\begin{aligned} (SP_{n-1})_L SP_{n+1} - (SP_n)_L SP_n =& \frac{1}{\gamma - \mu}(\gamma^{n-1}(S_\gamma)_L - \mu^{n-1}(S_\mu)_L)\frac{1}{\gamma - \mu}(\gamma^{n+1}S_\gamma - \mu^{n+1}S_\mu) \\ &- \frac{1}{\gamma - \mu}(\gamma^n(S_\gamma)_L - \mu^n(S_\mu)_L)\frac{1}{\gamma - \mu}(\gamma^n S_\gamma - \mu^n S_\mu) \\ =& \frac{1}{(\gamma - \mu)^2}\Big((-\gamma^{n-1}\mu^{n+1} + \gamma^n\mu^n)(S_\gamma)_L S_\mu + (-\gamma^{n+1}\mu^{n-1} + \gamma^n\mu^n)(S_\mu)_L S_\gamma\Big) \\ =& (-1)^{n-1}\frac{1}{\gamma - \mu}(\mu(S_\gamma)_L S_\mu - \gamma(S_\mu)_L S_\gamma) \\ =& (-1)^n \frac{1}{\gamma - \mu}(\gamma(S_\mu)_L S_\gamma - \mu(S_\gamma)_L S_\mu) \end{aligned}$$

where $S_\mu = \begin{bmatrix} \mu^3 + \mathbf{i} \\ \mu + \mathbf{i}\mu^2 \end{bmatrix}$, $(S_\mu)_L = \begin{bmatrix} 1 - \mathbf{i}\mu^3 & -\mu^2 - \mathbf{i}\mu \\ \mu^2 - \mathbf{i}\mu & 1 + \mathbf{i}\mu^3 \end{bmatrix}$, $S_\gamma = \begin{bmatrix} \gamma^3 + \mathbf{i} \\ \gamma + \mathbf{i}\gamma^2 \end{bmatrix}$, $(S_\gamma)_L = \begin{bmatrix} 1 - \mathbf{i}\gamma^3 & -\gamma^2 - \mathbf{i}\gamma \\ \gamma^2 - \mathbf{i}\gamma & 1 + \mathbf{i}\gamma^3 \end{bmatrix}$.
Similarly, for Pell-Lucas Spinors considering $SPL_n = \gamma^n S\gamma + \mu^n S_\mu$ and $(SPL_n)_L = \gamma^n (S\gamma)_L + \mu^n (S_\mu)_L$ we have

$$\begin{aligned} (SPL_{n-1})_L SPL_{n+1} - (SPL_n)_L SPL_n =& (\gamma^{n-1}(S_\gamma)_L + \mu^{n-1}(S_\mu)_L)(\gamma^{n+1}S_\gamma + \mu^{n+1}S_\mu) \\ &- (\gamma^n(S_\gamma)_L + \mu^n(S_\mu)_L)(\gamma^n S_\gamma + \mu^n S_\mu) \\ =& (\gamma^{n-1}\mu^{n+1} - \gamma^n\mu^n)(S_\gamma)_L S_\mu + (\gamma^{n+1}\mu^{n-1} - \gamma^n\mu^n)(S_\mu)_L S_\gamma \\ =& (-1)^{n-1}(\gamma - \mu)(\gamma(S_\mu)_L S_\gamma - \mu(S_\gamma)_L S_\mu) \end{aligned}$$

and consequently

$$(SPL_{n-1})_L SPL_{n+1} - (SPL_n)_L SPL_n = (-1)^{n-1}(\gamma-\mu)(\gamma(S_\mu)_L S_\gamma - \mu(S_\gamma)_L S_\mu)$$

where $S_\mu = \begin{bmatrix} \mu^3+\mathbf{i} \\ \mu+\mathbf{i}\mu^2 \end{bmatrix}$, $(S_\mu)_L = \begin{bmatrix} 1-\mathbf{i}\mu^3 & -\mu^2-\mathbf{i}\mu \\ \mu^2-\mathbf{i}\mu & 1+\mathbf{i}\mu^3 \end{bmatrix}$, $S_\gamma = \begin{bmatrix} \gamma^3+\mathbf{i} \\ \gamma+\mathbf{i}\gamma^2 \end{bmatrix}$, $(S_\gamma)_L = \begin{bmatrix} 1-\mathbf{i}\gamma^3 & -\gamma^2-\mathbf{i}\gamma \\ \gamma^2-\mathbf{i}\gamma & 1+\mathbf{i}\gamma^3 \end{bmatrix}$. □

Conclusion 2.1. The Cassini formulas for Pell and Pell-Lucas spinors can be obtained that

$$for\ Pell\ spinors\ (SP_{n-1})_L SP_{n+1} - (SP_n)_L SP_n = (-1)^{n-1} \begin{bmatrix} 12+2\mathbf{i} \\ 4+10\mathbf{i} \end{bmatrix},$$

$$for\ Pell-Lucas\ spinors\ (SPL_{n-1})_L SPL_{n+1} - (SPL_n)_L SPL_n = 8(-1)^{n-1} \begin{bmatrix} 12+2\mathbf{i} \\ 4+10\mathbf{i} \end{bmatrix}.$$

Theorem 2.6. *The generator function for Pell spinors is*

$$G_{SP}(t) = \frac{1}{1-2t-t^2} \begin{bmatrix} 5+2t+\boldsymbol{i}t \\ 1+\boldsymbol{i}(2+t) \end{bmatrix}$$

and the generator function for Pell-Lucas spinors is

$$G_{SPL}(t) = \frac{1}{1-2t-t^2} \begin{bmatrix} 14+6t+\boldsymbol{i}(2-2t) \\ 2+2t+\boldsymbol{i}(6+2t) \end{bmatrix}$$

Proof. We take nth Pell spinor is SP_n. Therefore, for nth Pell spinor the generator function is calculated with the aid of the equation $G_{SP}(t) = \sum_{n=0}^{\infty} SP_n t^n$. In this case, using $G_{SP}(t)$, $2tG_{SP}(t)$ and $t^2 G_{SP}(t)$ we obtain that

$$\begin{aligned} G_{SP}(t) &= SP_0 + SP_1 t + SP_2 t^2 + SP_3 t^3 + SP_4 t^4 + SP_5 t^5 + ... \\ -2tG_{SP}(t) &= -2SP_0 t - 2SP_1 t^2 - 2SP_2 t^3 - 2SP_3 t^4 - 2SP_4 t^5 - 2SP_5 t^6 + ... \\ -t^2 G_{SP}(t) &= -SP_0 t^2 - SP_1 t^3 - SP_2 t^4 - SP_3 t^5 - SP_4 t^6 - SP_5 t^7 + ... \end{aligned}$$

and

$$G_{SP}(t) = \frac{1}{(1-2t-t^2)}(SP_0 + (SP_1 - 2SP_0)t)$$

where

$$\begin{aligned} SP_0 + (SP_1 - 2SP_0) &= \begin{bmatrix} P_3+\mathbf{i}P_0 \\ P_1+\mathbf{i}P_2 \end{bmatrix} + \left(\begin{bmatrix} P_4+\mathbf{i}P_1 \\ P_2+\mathbf{i}P_3 \end{bmatrix} - \begin{bmatrix} 2P_3+2\mathbf{i}P_0 \\ 2P_1+2\mathbf{i}P_2 \end{bmatrix} \right) t \\ &= \begin{bmatrix} 5 \\ 1+2\mathbf{i} \end{bmatrix} + \left(\begin{bmatrix} 12+\mathbf{i} \\ 2+5\mathbf{i} \end{bmatrix} - \begin{bmatrix} 10 \\ 2+4\mathbf{i} \end{bmatrix} \right) t = \begin{bmatrix} 5+2t+\mathbf{i}t \\ 1+\mathbf{i}(2+t) \end{bmatrix}. \end{aligned}$$

Consequently, we get

$$G_{SP}(t) = \frac{1}{1-2t-t^2} \begin{bmatrix} 5+2t+\mathbf{i}t \\ 1+\mathbf{i}(2+t) \end{bmatrix}.$$

Now, we calculate the generator function for Pell-Lucas spinors. Therefore, if we consider the function $G_{SPL}(t) = \sum_{n=0}^{\infty} SPL_n t^n$, we have

$$G_{SPL}(t) = \frac{1}{1-2t-t^2}(SPL_0 + (SPL_1 - 2SPL_0)t)$$

using $G_{SPL}(t)$, $2tG_{SPL}(t)$ and $t^2 G_{SPL}(t)$. Finally, we obtain

$$G_{SPL}(t) = \frac{1}{1-2t-t^2} \begin{bmatrix} 14+6t+\mathbf{i}(2-2t) \\ 2+2t+\mathbf{i}(6+2t) \end{bmatrix}.$$

This completes the proof. □

Theorem 2.7. *Assume that $-nth$ Pell and Pell-Lucas spinors are SP_{-n} and SPL_{-n}. In this case these spinors are calculated as follows; for Pell spinors*

$$SP_{-n}=(-1)^n\begin{bmatrix}P_{n-3}-\boldsymbol{i}P_n\\ P_{n-1}-\boldsymbol{i}P_{n-2}\end{bmatrix}$$

for Pell-Lucas spinors

$$SPL_{-n}=(-1)^n\begin{bmatrix}-Q_{n-3}+\boldsymbol{i}Q_n\\ -Q_{n-1}-\boldsymbol{i}Q_{n-2}\end{bmatrix}.$$

Proof. We know that the Binet formula for nth Pell spinor is $SP_n=\frac{1}{\gamma-\mu}(\gamma^nS_\gamma-\mu^nS_\mu)$ where $S_\mu=\begin{bmatrix}\mu^3+\mathbf{i}\\ \mu+\mathbf{i}\mu^2\end{bmatrix}$, $S_\gamma=\begin{bmatrix}\gamma^3+\mathbf{i}\\ \gamma+\mathbf{i}\gamma^2\end{bmatrix}$. On the other hand, we can write the equation $\gamma\mu=-1\Longrightarrow\gamma=(-1)\mu^{-1}$. If we take n powers of both sides then, we get $\gamma^{-n}=(-1)^n\mu^n$. Similarly, we easily see that $\mu^{-n}=(-1)^n\gamma^n$. In this case, considering the Binet formula for $-nth$ Pell spinor $SP_{-n}=\frac{1}{\gamma-\mu}(\gamma^{-n}S_\gamma-\mu^{-n}S_\mu)$ we calculate as

$$SP_{-n}=\frac{1}{\gamma-\mu}((-1)^n\mu^nS_\gamma-(-1)^n\gamma^nS_\mu)$$

and

$$SP_{-n}=\frac{(-1)^n}{\gamma-\mu}(\mu^nS_\gamma-\gamma^nS_\mu).$$

If we make this equation even more detailed, we get

$$SP_{-n}=\frac{(-1)^n}{\gamma-\mu}\begin{bmatrix}\mu^n\gamma^3-\gamma^n\mu^3+\mathbf{i}(\mu^n-\gamma^n)\\ \mu^n\gamma-\gamma^n\mu+\mathbf{i}(\mu^n\gamma^2-\gamma^n\mu^2)\end{bmatrix}\tag{2.3}$$

where $\gamma=2-\mu$ and $\mu=2-\gamma$. Additionally, if the characteristic equation $x^2-2x-1=0$ of Pell number sequence is used, the equations $\gamma^2=5-2\mu$, $\mu^2=5-2\gamma$, $\gamma^3=12-5\mu$ and $\mu^3=12-5\gamma$ are obtained. Therefore, we obtain the Eq (2.3) as

$$SP_{-n}=\frac{(-1)^n}{\gamma-\mu}\begin{bmatrix}\mu^n(12-5\mu)-\gamma^n(12-5\gamma)+\mathbf{i}(\mu^n-\gamma^n)\\ \mu^n(2-\mu)-\gamma^n(2-\gamma)+\mathbf{i}(\mu^n(5-2\gamma)-\gamma^n(5-2\gamma))\end{bmatrix}$$

$$=(-1)^n\begin{bmatrix}-12(\frac{\gamma^n-\mu^n}{\gamma-\mu})+5(\frac{\gamma^{n+1}-\mu^{n+1}}{\gamma-\mu})-\mathbf{i}(\frac{\gamma^n-\mu^n}{\gamma-\mu})\\ -2(\frac{\gamma^n-\mu^n}{\gamma-\mu})+(\frac{\gamma^{n+1}-\mu^{n+1}}{\gamma-\mu})+\mathbf{i}(-5(\frac{\gamma^n-\mu^n}{\gamma-\mu})+2(\frac{\gamma^{n+1}-\mu^{n+1}}{\gamma-\mu}))\end{bmatrix}$$

$$=(-1)^n\begin{bmatrix}-12P_n+5P_{n+1}-\mathbf{i}P_n\\ -2P_n+P_{n+1}-\mathbf{i}(-5P_n+2P_{n+1})\end{bmatrix}=(-1)^n\begin{bmatrix}P_{n-3}-\mathbf{i}P_n\\ P_{n-1}-\mathbf{i}P_{n-2}\end{bmatrix}$$

$$=\begin{cases}\text{If } n \text{ is even number,} & \begin{bmatrix}P_{n-3}-\mathbf{i}P_n\\ P_{n-1}-\mathbf{i}P_{n-2}\end{bmatrix}\\ \text{If } n \text{ odd number,} & \begin{bmatrix}P_{n-3}-\mathbf{i}P_n\\ P_{n-1}-\mathbf{i}P_{n-2}\end{bmatrix}\end{cases}.$$

Now, we calculate for Pell-Lucas spinors. Considering Binet formula $SPL_n=\gamma^nS_\gamma+\mu^nS_\mu$ for Pell-Lucas spinor sequence and we can write for $-n$

$$SPL_{-n}=\gamma^{-n}S_\gamma+\mu^{-n}S_\mu.$$

If we use again the equations $\gamma^{-n}=(-1)^n\mu^n$ and $\mu^{-n}=(-1)^n\gamma^n$ then, we have

$$SPL_{-n}=(-1)^n(\mu^nS_\gamma+\gamma^nS_\mu)$$

and

$$SPL_{-n}=(-1)^n\begin{bmatrix}\mu^n\gamma^3+\gamma^n\mu^3+\mathbf{i}(\mu^n+\gamma^n)\\ \mu^n\gamma+\gamma^n\mu+\mathbf{i}(\mu^n\gamma^2+\gamma^n\mu^2)\end{bmatrix}.$$

Finally, we get

$$SPL_{-n} = (-1)^n \begin{bmatrix} -Q_{n-3} + \mathbf{i}Q_n \\ -Q_{n-1} + \mathbf{i}Q_{n-2} \end{bmatrix}.$$

□

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