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ON THE MELLIN-GAUSS-WEIERSTRASS OPERATORS IN THE MELLIN-LEBESGUE SPACES

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Abstract. In this paper, we present the modulus of smoothness of a function $f \in X_c^p$, which the Mellin-Lebesgue space, and later we state some properties of it. In this way, the rate of convergence is gained. Moreover, we elucidate some pointwise convergence results for the Mellin-Gauss-Weierstrass operators. Especially, we acquire the pointwise convergence of them at any Lebesgue point of a function f.

1. Introduction

Mellin analysis is famous in approximation theory and Mellin operators are broadly investigated in this field (see [13], [18] for a comprehensive theory and, for other approximation results, [7], [12]). The reputation of Mellin operators is both mathematically and due to their applications in different fields. For instance, they are relevant to various problems of Signal Processing: actually, Mellin analysis is quite helpful in situations, where the samples to reconstruct a signal are exponentially spaced rather than equally spaced as in the classical Shannon Sampling Theorem (see, e.g., [14]).

The singular integrals of Mellin convolution type were first-time presented by Kolbe and Nessel [17] in 1972. They play a remarkable role in the Mellin analysis, likewise the traditional convolution operators in the Fourier analysis. These convolution integrals are utilized to explain the attitude of solutions of certain boundary value problems in the wedge-shaped regions. Butzer and Jansche [13] broadly analyzed them, relating to the L_p convergence. The pointwise convergence of linear singular integrals of the Fejer-type in the periodic case or in the line group is was broadly investigated in the classical book by P.L. Butzer and R.J. Nessel [15] in

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1971, where specially an almost everywhere convergence is gained by using the concept of the Lebesgue point of a function $f \in L_p$, $1 \leq p \leq +\infty$.

In [18], the approximation theory by Mellin convolution operators is evolved using a more direct and inherent way, totally unconnected from the Fourier theory, bottomed on a 'logarithmic'version of Taylor formula, Mellin derivatives, and the concepts of 'logarithmic' uniform continuity and 'logarithmic'moment of kernel function, which gives a different and powerful approach.

From the early 2000s until today, Mellin convolution operators have been worked intensively, particularly by Bardaro and Mantellini, and quite significant guidances have been accomplished to this field. In [4] and [5], the authors asserted a convenient linear composition of Mellin type operators to accelerate convergence. In another view to gain better order of approximation, Bardaro and Mantellini [8] took into account linear compositions of Mellin type operators using the iterated kernels instead of the basic kernels. Same authors, in [5], improved the pointwise approximation theory for Mellin convolution operators including Mellin-Gauss-Weierstrass operators, acting on functions defined on the multiplicative group $\mathbb{R}^+$.

Bardaro and Mantellini [7] considered Mellin convolution operators of type

$$(T_w f)(s) = \int_0^\infty K_w(t) f(ts) \frac{dt}{t}, \qquad s \in \mathbb{R}^+$$

where f pertains to domain of the operator T_w and $K_w : (0,\infty) \to \mathbb{R}$ is a set of the kernels, which provides the condition $\int_0^\infty K_w(t) \frac{dt}{t} = 1$. Check against the usual classical convolution, the translation operator is changed by a dilation operator, and let $\mathbb{R}^+$ be the multiplicative topological group granted with the Haar measure $\mu = \frac{dt}{t}$ becoming the Lebesgue measure. We will indicate by $L_p(\mu, \mathbb{R}^+) = L_p(\mu)$, $1 \leq p \leq +\infty$, the Lebesgue spaces according to the measure μ and we will demonstrate by $\|f\|_p$ the matching norm of a function $f \in L_p(\mu)$.

Moreover, in recent important papers, the authors have been working on the Mellin-Lebesgue spaces. For example, in [10], the authors study convergence theorems to a function f of its generalized exponential sampling series in the weighted Lebesgue spaces. In [2], some results on exponential sampling operators in the weighted Lebesgue spaces have been performed recently. In the very recent papers, in [6] and [11], Bardaro et al. examine the boundedness properties and the convergence features of certain semi-discrete exponential-type sampling operators in the weighted Lebesgue spaces, respectively.

Additionally, many studies have been carried out for similar operators on the subject. For instance, in [3], q analogue of the Stancu-Beta operators is introduced, and direct results in terms of the modulus of continuity and the weighted approximation theorem are expressed. In [16], Gupta et al. deal with the semi-exponential type Gauss-Weierstrass operators and they estimate some direct results using suitable modulus of continuity, weighted approximation, quantitative asymptotic formula and pointwise convergence. In the last year, in [1], a new modulus of continuity for locally integrable function spaces is presented and the obtained results are applied to the Gauss-Weierstrass operators.

The rest of the paper is organised as follows. In the next part, elementary informations related to the subject are reminded. After that, the definition of modulus of smoothness of a function $f \in X_c^p$ and its some properties are given. In this way, the rate of convergence is gained. Other than these, the definition of Lebesgue point of a function $f \in X_c^p$ is expressed. Later, we state pointwise convergence of the linear Mellin-Gauss-Weierstrass operators.

2. Basic Notations

Let us represent by $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ the sets of positive integers, positive real numbers and nonnegative real numbers, respectively. By $\mathbb{C}$, we symbolize the set of complex numbers. Throughly the paper, $C(\mathbb{R}^+)$ settles for the space of all continuous and bounded functions defined on $\mathbb{R}^+$and by $C_{comp}(\mathbb{R}^+)$ the subspace of $C(\mathbb{R}^+)$ including all functions with compact support in $\mathbb{R}^+$. Moreover, $C_{comp}^{\infty}(\mathbb{R}^+)$ denotes the subspace of $C_{comp}(\mathbb{R}^+)$ including all test functions, i.e., the functions of compact support which are infinitely differentiable.

For $1 \leq p \leq \infty$, we represent by $L^p(\mathbb{R}^+)$ the ordinary Lebesgue space comprising all Lebesgue measurable function such that

$$\|f\|_p := \left\{ \int_0^{\infty} |f(x)|^p \, dx \right\}^{1/p} < \infty \quad (1 \leq p < \infty)$$

and

$$\|f\|_{\infty} := \underset{x \in \mathbb{R}^+}{ess\sup} |f(x)| < \infty.$$

We should point out that $C(\mathbb{R}^+) \subset L^{\infty}(\mathbb{R}^+)$ and the norm of two spaces is the same.

Let's assume that $c \in \mathbb{R}$ is constant. For $1 \leq p < \infty$, we symbolize by X_c^p the weighted Lebesgue space, so called Mellin-Lebesgue space, which represent the natural Mellin counterpart of the classical Lebesgue spaces, defined by

$$X_c^p := \left\{ f : \mathbb{R}^+ \to \mathbb{C} : f(\cdot)(\cdot)^{c-1/p} \in L^p\left(\mathbb{R}^+\right) \right\}$$

and equipped with the norm

$$\begin{aligned} \|f\|_{X_c^p} \quad : \quad &= \left\{ \int_0^{\infty} |f(x)|^p \, x^{cp-1} dx \right\}^{1/p} \\ &= \left\{ \int_0^{\infty} |f(x)|^p \, x^{cp} \frac{dx}{x} \right\}^{1/p} < \infty \quad . \end{aligned}$$

In case $p = 1$, we will simply write $X_c^1 \equiv X_c$. In an equal form, X_c^p is the space of all functions f such that $f(\cdot)(\cdot)^c \in L_{\mu}^p(\mathbb{R}^+)$, where $L_{\mu}^p(\mathbb{R}^+)$ represents the Lebesgue space in connection with the invariant measure $\mu(A) = \int_A \frac{dt}{t}$ for any measurable set $A \subset \mathbb{R}^+$. For details, see [13].

We consider the linear Mellin-Gauss-Weierstrass operators defined in [13, Page 342 Definition 8] as follows

$$(\mathcal{T}_w f)(s) = \frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} f(st) \frac{dt}{t}, \qquad s \in (0,\infty).$$

It is easy to see that

$$\frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} = 1. \tag{1}$$

3. Pointwise Convergence and Quantitative Estimate

This part is seperated to state pointwise convergence of $(\mathcal{T}_w)$ and the rate of convergence through modulus of smoothness which will also be defined.

To acquire convergence theorems for the operators $\mathcal{T}_w$, we need the following density result (see [10]). We accept the following impression: for a subspace $H \subset X_c^p$, we represent by $cls_{X_c^p}(H)$ the closure of H in connection with the norm-topology of X_c^p.

Theorem 1. *[10] For every $p \geq 1$ and $c \in \mathbb{R}$, we have*

$$cls_{X_c^p}\left(C_{comp}^\infty\left(\mathbb{R}^+\right)\right) = X_c^p.$$

Firstly, we begin with the following lemma.

Lemma 1. *If $f \in X_c^p$, then we get*

$$\|\mathcal{T}_w f\|_{X_c^p} \leq e^{c^2/w^2} \|f\|_{X_c^p}.$$

Proof. We can write

$$\begin{aligned}
\|\mathcal{T}_w f\|_{X_c^p} &= \left\{\int_0^\infty |(\mathcal{T}_w f)(s)|^p s^{cp} \frac{ds}{s}\right\}^{1/p} \\
&= \frac{w}{\sqrt{4\pi}} \left\{\int_0^\infty \left|\int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} f(st) \frac{dt}{t}\right|^p s^{cp} \frac{ds}{s}\right\}^{1/p} \\
&\leq \frac{w}{\sqrt{4\pi}} \int_0^\infty \left\{\int_0^\infty |f(st))|^p s^{cp} \frac{ds}{s}\right\}^{1/p} e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} \\
&= \frac{w}{\sqrt{4\pi}} \|f\|_{X_c^p} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} t^{-c} \frac{dt}{t} \\
&= e^{c^2/w^2} \|f\|_{X_c^p}.
\end{aligned}$$

□

Definition 1. *We present the first modulus of smoothness of a function $f \in X_c^p$ with*

$$\omega_{X_c^p}(f;\delta) = \sup_{|\ln t|<\delta} \|f(t\cdot) - f(\cdot)\|_{X_c^p}, \quad \delta > 0.$$

The modulus has the following properties:

Theorem 2. *If $f \in X_c^p$, we have*

$$\lim_{\delta \to 0} \omega_{X_c^p}(f;\delta) = 0. \tag{2}$$

Proof. Let be $|\ln t| < \delta$. Assuming first that $c > 0$, since $f \in X_c^p$, for every $\varepsilon > 0$ there exists $A > 1$ such that for any $\delta > 1$

$$I_1 := \left(\int_0^{e^{-A}} |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} < \frac{\varepsilon}{4e^{c\delta}} \text{ and } I_2 := \left(\int_{e^A}^{\infty} |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} < \frac{\varepsilon}{4e^{c\delta}}. \tag{3}$$

From (3), we have

$$\begin{aligned} \left(\int_{s \notin (e^{-A}, e^A)} |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} &= \left(\left(\int_0^{e^{-A}} + \int_{e^A}^{\infty} \right) |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} \\ &< \frac{\varepsilon}{2}. \end{aligned}$$

It is obvious that for any $\delta > 1$

$$\left(\int_{s \notin [e^{-A-\delta}, e^{A+\delta}]} |f(s)|^p s^{cp} \frac{ds}{s} \right)^{1/p} < \frac{\varepsilon}{2}. \tag{4}$$

Then, using the change of variable $ts = u$, with $|\ln t| < \delta$ ($\delta > 1$), and from (3), we obtain

$$\begin{aligned} \left(\int_0^{e^{-\delta-A}} |f(ts)|^p s^{cp} \frac{ds}{s} \right)^{1/p} &< t^{-c} \left(\int_0^{te^{-\delta-A}} |f(u)|^p u^{cp} \frac{du}{u} \right)^{1/p} \\ &< e^{c\delta} \left(\int_0^{e^{-2\delta-A}} |f(u)|^p u^{cp} \frac{du}{u} \right)^{1/p} < \frac{\varepsilon}{4} \end{aligned} \tag{5}$$

and

$$\begin{aligned} \left(\int_{e^{\delta+A}}^{\infty} |f(ts)|^p s^{cp} \frac{ds}{s} \right)^{1/p} &< t^{-c} \left(\int_{te^{\delta+A}}^{\infty} |f(u)|^p u^{cp} \frac{du}{u} \right)^{1/p} \\ &< e^{c\delta} \left(\int_{e^{2\delta+A}}^{\infty} |f(u)|^p u^{cp} \frac{du}{u} \right)^{1/p} < \frac{\varepsilon}{4}. \end{aligned} \tag{6}$$

From (4), (5) and (6), we obtain

$$\sup_{|\ln t|<\delta}\left(\int_0^{e^{-\delta-A}} |f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} + \sup_{|\ln t|<\delta}\left(\int_{e^{\delta+A}}^{\infty} |f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} < \varepsilon.$$

In this case, we can write the inequality

$$\omega_{X_c^p}(f;\delta) \le \varepsilon + \sup_{|\ln t|<\delta}\left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p}.$$

For every $f \in X_c^p$, using Theorem 1, there is $g \in C_{comp}(\mathbb{R}^+)$ such that

$$\left(\int_{e^{-2\delta-A}}^{e^{2\delta+A}} |g(s)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} < \frac{\varepsilon}{e^{c\delta}}. \tag{7}$$

Using the Minkowsky inequality and the logarithmic continuity of smoothness of the function g in the closed interval, we attain

$$\begin{aligned}\left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} \le & \left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |f(ts)-g(ts)|^p s^{cp}\frac{ds}{s}\right)^{1/p} \\ & + \left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |g(ts)-g(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} \\ & + \left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |g(s)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p}.\end{aligned}$$

According to (7), we get

$$\left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |f(ts)-g(ts)|^p s^{cp}\frac{ds}{s}\right)^{1/p} < \varepsilon \text{ and } \left(\int_{e^{-\delta-A}}^{e^{\delta+A}} |g(s)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p} < \varepsilon.$$

As g is a continuous function, for $|\ln t| < \delta$, we can take

$$|g(ts)-g(s)| < \frac{\varepsilon}{(2(A+\delta))^{1/p} e^{(A+\delta)c}}.$$

Hence, we have

$$\sup_{|\ln t|<\delta}\left(\int_{e^{-\delta-A}}^{e^{\delta+A}}|f(ts)-f(s)|^p s^{cp}\frac{ds}{s}\right)^{1/p}\leq 3\varepsilon$$

and this theorem proves.

A similar result is obtained when $c<0$. Thus, the desired result emerges again in a similar way. □

Theorem 3. *If $f\in X_c^p$ and $n\in\mathbb{N}$, then we get*

$$\omega_{X_c^p}(f;n\delta)\leq n\omega_{X_c^p}(f;\delta).$$

Proof. With the aid of the definition of $\omega_{X_c^p}$, we obtain

$$\begin{aligned}\omega_{X_c^p}(f;n\delta) &= \sup_{|\ln t|<n\delta}\|f(t\cdot)-f(\cdot)\|_{X_c^p}\\ &= \sup_{|\ln t|<\delta}\left\{\int_0^\infty|f(t^n s)-f(s)|^p s^{cp}\frac{ds}{s}\right\}^{1/p}\\ &= \sup_{|\ln t|<\delta}\left\{\int_0^\infty\left|\sum_{k=1}^n f\left(t^k s\right)-f\left(t^{k-1}s\right)\right|^p s^{cp}\frac{ds}{s}\right\}^{1/p}\\ &\leq \sum_{k=1}^n\sup_{|\ln t|<\delta}\left\{\int_0^\infty\left|f\left(t^k s\right)-f\left(t^{k-1}s\right)\right|^p s^{cp}\frac{ds}{s}\right\}^{1/p}\\ &= n\omega_{X_c^p}(f;\delta).\end{aligned}$$

□

Corollary 1. *If $f\in X_c^p$ and $\lambda\in\mathbb{R}$, then we get*

$$\omega_{X_c^p}(f;\lambda\delta)\leq(1+\lambda)\,\omega_{X_c^p}(f;\delta).$$

Now, we give the following:

Definition 2. *We will call that a point $s\in\mathbb{R}^+$ is a Lebesgue point of a function $f\in X_c^p$ $(c\neq 0)$ if*

$$\lim_{z\to 1}\left|\frac{1}{\log z}\int_1^z|f(su)-f(s)|^p u^{cp}\frac{du}{u}\right|^{1/p}=0.$$

This is equivalent to

$$\lim_{z\to 1^-}\left(\frac{1}{-\log z}\int_z^1|f(su)-f(u)|^p u^{cp}\frac{du}{u}\right)^{1/p}+\lim_{z\to 1^+}\left(\frac{1}{\log z}\int_1^z|f(su)-f(u)|^p u^{cp}\frac{du}{u}\right)^{1/p}=0.$$

You can refer to [9] for the situation in X_0^1 space.

The main conclusion of this part is on pointwise convergence as following:

Theorem 4. *If $f \in X_c^p$, then we get*

$$\lim_{w \to \infty} \left(\mathcal{T}_w f \right)(s) = f(s)$$

for any Lebesgue point $s \in \mathbb{R}^+$.

Proof. Using the property (1), we can obtain that

$$\left| \left(\mathcal{T}_w f \right)(s) - f(s) \right| \leq \frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2} \log t \right)^2} \left| f(st) - f(s) \right| \frac{dt}{t}.$$

Using Hölder's inequality, we attain

$$\begin{aligned} \left| \left(\mathcal{T}_w f \right)(s) - f(s) \right|^p &\leq \left(\int_0^\infty \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2} \log t \right)^2} \left| f(st) - f(s) \right|^p t^{cp} \frac{dt}{t} \right) \\ &\times \left(\int_0^\infty \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2} \log t \right)^2} t^{-cq} \frac{dt}{t} \right)^{\frac{p}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Consider the integral

$$\int_0^\infty \frac{w}{\sqrt{4\pi}} e^{-\left(\frac{w}{2} \log t \right)^2} \left| f(st) - f(s) \right|^p t^{cp} \frac{dt}{t}.$$

Let $\delta > 1$ be fixed and let us consider $H_\delta = \left(\delta^{-1}, \delta \right)$. Then

$$\begin{aligned} &\frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2} \log t \right)^2} \left| f(st) - f(s) \right|^p t^{cp} \frac{dt}{t} \\ &= \frac{w}{\sqrt{4\pi}} \int_{1/\delta}^\delta e^{-\left(\frac{w}{2} \log t \right)^2} \left| f(st) - f(s) \right|^p t^{cp} \frac{dt}{t} \\ &+ \frac{w}{\sqrt{4\pi}} \int_{\mathbb{R}^+ \setminus H_\delta} e^{-\left(\frac{w}{2} \log t \right)^2} \left| f(st) - f(s) \right|^p t^{cp} \frac{dt}{t} \\ &= I_1 + I_2. \end{aligned}$$

Firtsly, we take into account I_1.

$$\begin{aligned} I_1 &= \frac{w}{\sqrt{4\pi}} \int_{1/\delta}^1 e^{-\left(\frac{w}{2} \log t \right)^2} \left| f(st) - f(s) \right|^p t^{cp} \frac{dt}{t} \\ &+ \frac{w}{\sqrt{4\pi}} \int_1^\delta e^{-\left(\frac{w}{2} \log t \right)^2} \left| f(st) - f(s) \right|^p t^{cp} \frac{dt}{t} \\ &= I_1^1 + I_1^2. \end{aligned}$$

Let us define

$$F^-(z) := \int_z^1 \left| f(su) - f(u) \right|^p u^{cp} \frac{du}{u}$$

for every $z \in \left(\delta^{-1}, 1 \right)$. Let $\varepsilon > 0$ be fixed. Since $s \in \mathbb{R}^+$ is a Lebesgue point of f, we can choose $\delta > 1$ such that

$$F^-(z) \leq -\varepsilon \log z.$$

Then, we have

$$\begin{aligned}
I_1^1 &= \frac{w}{\sqrt{4\pi}}\int_{1/\delta}^{1} e^{-\left(\frac{w}{2}\log t\right)^2}\left|f(st)-f(s)\right|^p t^{cp}\frac{dt}{t} \\
&= -\frac{w}{\sqrt{4\pi}}\int_{1/\delta}^{1} e^{-\left(\frac{w}{2}\log z\right)^2} dF^-(z) \\
&= \frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log 1/\delta\right)^2}F^-(1/\delta)+\int_{1/\delta}^{1}F^-(z)\,d\left(\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log z\right)^2}\right) \\
&\le \varepsilon\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log 1/\delta\right)^2}\log\delta-\varepsilon\int_{1/\delta}^{1}\log z d\left(\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log z\right)^2}\right) \\
&\le \varepsilon\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log 1/\delta\right)^2}\log\delta-\varepsilon\left[\log\delta\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log 1/\delta\right)^2}-\frac{w}{\sqrt{4\pi}}\int_{1/\delta}^{1}e^{-\left(\frac{w}{2}\log z\right)^2}\frac{dz}{z}\right] \\
&\le \varepsilon.
\end{aligned}$$

Next for I_1^2, utilizing the similar ways and paying attention to the function

$$F^+(z) := \int_1^z \left|f(su)-f(u)\right|^p u^{cp}\frac{du}{u},$$

we obtain analogous estimate. Thus, we achieve $I_1 \to 0$ for $\omega \to \infty$.

Now, we handle

$$\begin{aligned}
I_2 &= \frac{w}{\sqrt{4\pi}}\int_{\mathbb{R}^+\setminus H_\delta} e^{-\left(\frac{w}{2}\log t\right)^2}\left|f(st)-f(s)\right|^p t^{cp}\frac{dt}{t} \\
&\le 2^p\frac{w}{\sqrt{4\pi}}\int_0^{1/\delta} e^{-\left(\frac{w}{2}\log t\right)^2}\left|f(st)\right|^p t^{cp}\frac{dt}{t}+2^p\left|f(s)\right|^p\frac{w}{\sqrt{4\pi}}\int_0^{1/\delta}e^{-\left(\frac{w}{2}\log t\right)^2}t^{cp}\frac{dt}{t} \\
&\quad +2^p\frac{w}{\sqrt{4\pi}}\int_\delta^{\infty} e^{-\left(\frac{w}{2}\log t\right)^2}\left|f(st)\right|^p t^{cp}\frac{dt}{t}+2^p\left|f(s)\right|^p\frac{w}{\sqrt{4\pi}}\int_\delta^{\infty}e^{-\left(\frac{w}{2}\log t\right)^2}t^{cp}\frac{dt}{t}.
\end{aligned}$$

As $\frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log t\right)^2}$ is increasing in $(0,1)$, we can write

$$\frac{w}{\sqrt{4\pi}}\int_0^{1/\delta} e^{-\left(\frac{w}{2}\log t\right)^2}\left|f(st)\right|^p t^{cp}\frac{dt}{t} \le \frac{w}{\sqrt{4\pi}}e^{-\left(\frac{w}{2}\log 1/\delta\right)^2}s^{-cp}\left\|f\right\|_{X_c^p}.$$

Similarly,

$$\frac{w}{\sqrt{4\pi}}\int_0^{1/\delta} e^{-\left(\frac{w}{2}\log t\right)^2}t^{cp}\frac{dt}{t}$$

tends to zero for $w \to \infty$. On the other hand, we obtain

$$\frac{w}{\sqrt{4\pi}}\int_\delta^{\infty} e^{-\left(\frac{w}{2}\log t\right)^2}t^{cp}\frac{dt}{t} = \frac{1}{\sqrt{4\pi}}\int_{\delta^w}^{\infty} e^{-\left(\frac{1}{2}\log t\right)^2}t^{\frac{cp}{w}}\frac{dt}{t},$$

which tends to zero for $w \to \infty$. The last term can be estimated similarly. $\square$

Corollary 2. *If $f \in X_c^p$, then we get*

$$\lim_{w \to \infty} \left(\mathcal{T}_w f\right)(s) = f(s)$$

almost everywhere in $\mathbb{R}^+$.

Theorem 5. *If $f \in X_c^p$, then we get*

$$\left\|\mathcal{T}_w f - f\right\|_{X_c^p} \leq \left(1 + \frac{2}{\sqrt{\pi}}\right) \omega_{X_c^p}\left(f; w^{-1}\right).$$

Proof. Since the property (1), we have

$$\left(\mathcal{T}_w f\right)(s) - f(s) = \frac{w}{\sqrt{4\pi}} \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \left(f(ts)) - f(s)\right) \frac{dt}{t}.$$

Then, we deduce

$$\begin{aligned}
\left\|\mathcal{T}_w f - f\right\|_{X_c^p} &= \frac{w}{\sqrt{4\pi}} \left\{ \int_0^\infty \left| \int_0^\infty e^{-\left(\frac{w}{2}\log t\right)^2} \left(f(ts)) - f(s)\right) \frac{dt}{t} \right|^p s^{cp} \frac{ds}{s} \right\}^{1/p} \\
&\leq \frac{w}{\sqrt{4\pi}} \int_0^\infty \left\{ \int_0^\infty \left|(f(ts)) - f(s))\right|^p s^{cp} \frac{ds}{s} \right\}^{1/p} e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} \\
&\leq \frac{w}{\sqrt{4\pi}} \int_0^\infty \omega_{X_c^p}\left(f; |\ln t|\right) e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} \\
&\leq \frac{w}{\sqrt{4\pi}} \omega_{X_c^p}(f; \delta) \int_0^\infty \left(1 + \frac{1}{\delta}|\ln t|\right) e^{-\left(\frac{w}{2}\log t\right)^2} \frac{dt}{t} \\
&= \omega_{X_c^p}(f; \delta) \left(1 + \frac{1}{\delta} \frac{2}{w\sqrt{\pi}}\right).
\end{aligned}$$

Choosing $\delta = w^{-1}$, we obtain desired result. □

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