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The Concept of Parafree Zinbiel Algebras

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Abstract

Pf (parafree) Zinbiel (PfZin) algebras, a generalization of Leibniz algebras, share various traits with free Zinbiel algebras. This article delves into the intricacies of PfZin algebras, presenting their structure and exploring significant findings analogous to those in parafree Leibniz algebras. The focus extends to properties of subalgebras and quotient algebras within the realm of PfZin algebras. Additionally, the direct sum of these algebras is examined, demonstrating that the amalgamation of two PfZin algebras yields a Zinbiel algebra. A new connection between weak Hopf algebras and PfZin algeras constructed. Moreover, from the direct sum of PfZin algebras weak Hopf algebra is handled and construction of weak Hopf algebra usuing PfZin algebra is showed.

Keywords: Zinbiel algebra, Subalgebras, Division algebras, Direct sum, Weak Hopf algebra

1. Introduction

Zinbiel algebras, the Koszul dual of Leibniz algebras, were first presented by Loday in [13]. The term "Zinbiel" was coined by Lemaire [14], by reversing the word "Leibniz". In [15] Loday defined Leibniz algebras as a non-associative extensions of Lie algebras with the property that the right-multiplication operator is a derivation. Key results from Leibniz algebras also hold for Zinbiel algebras in [1,6,8,16,18,19]. Some papers [2,4,5,13] delve into the cohomological and structural aspects of Leibniz algebras. Ginzburg and Kapranov [12] introduced Koszul dual operads, and it was shown that the dual of the Leibniz algebra category is determined by the Zinbiel identity. Our motivation in this article is to see how the parafree algebras considered in Lie algebras and Leibniz algebras work in Zinbiel algebras. In [3,9,10,17,20] parafree Lie and Leibniz algebras were discussed and the studies were expanded and advanced. In addition, it is discussed in the article [11] that this type of algebra is Hopfian. In this paper, in the light of the above mentioned studies we construct parafree Zinbiel algebras. Which we will briefly denote as PfZin (Parafree Zinbiel). We concentrate on PfZin algebras and review key findings in this field derived from prior studies. Next, we will focus on examining the subalgebra structure in PfZin algebras. Our objective is to explore intrinsic characteristics of subalgebras and division algebras in the context of PfZin algebras. Additionally, we demonstrate a key result that shows how combining two PfZin algebras creates a new Zinbiel algebra while preserving the Pf property. The connection between PfZin algebras and weak Hopf algebras is investigated. The connection lies in their construction and properties, such as for a given PfZin algebra P, we can construct a weak Hopf algebra $H(P)$ using the direct sum of the algebra structures on P and its dual P^*. The antipode on $H(P)$ is defined as the linear map that satisfies the required conditions.

The construction of a weak Hopf algebra using a PfZin algebra demonstrates the connection between these two concepts. The weak Hopf algebra $H(P)$ inherits properties from the PfZin algebra P, such as the self-dual property. This relationship highlights the importance of considering weaker axioms in certain situations, as seen in the context of weak Hopf algebras in [7].

2. Notations and Definitions

In this part, we review important founding crucial for ours objectives as mentioned in references [10,11,12,15], using standart inscription. During this discussion, F demonstrates a characteristic zero field. A Zinbiel algebra Z is defined as an algebra that satisfies the identity:

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$$[[x,y],z] = [x,[y,z]] + [x,[z,y]] \quad \textbf{(2.1)}$$

for all $x, y, z \in Z$. We introduce a series of ideals

$Z^1 \supseteq Z^2 \supseteq \ldots \supseteq Z^k \supseteq \ldots$

where $Z^1 = Z$, $Z^2 = [Z,Z], \ldots, Z^{k+1} = [Z^k, Z]$ for $k \geq 1$, termed the lower central series of Z.

A Zinbiel algebra Z is classified as nilpotent if there exists an integer $k \geq 1$ such that $Z^k = \{0\}$. If $Z_1/Z_1^n \cong Z_2/Z_2^n$. Then we propose that, Z_1 and Z_2 have an identical lower central series. Let X be a set and $A(X)$ be the free non-associative algebra over F generated by X. We define I as the two-sided ideal in $A(X)$ generated by elements of the form

$$[[x,y],z] - [x,[y,z]] - [x,[z,y]] \quad \textbf{(2.2)}$$

for all $x, y, z \in A(X)$. As a result, the algebra

$Z(X) = A(X)/I$ is established as a free Zinbiel algebra. In addition, we provide definitions for Zinbiel algebras that resemble those commonly associated with Lie and Leibniz algebras.

Definition 2.1. A Zinbiel algebra is deemed "Hopfian" if it satisfies the following equivalent conditions:

(i) It is isomorphic to any of its proper quotients.

(ii) Each endomorphism that maps onto it is an automorphism.

Definition 2.2. If Z is a Zinbiel algebra, it is considered residually nilpotent (has residual nilpotency) if the intersection of its ascending powers from n equals 1 to infinity, represented as $\bigcap_{n=1}^{\infty} Z^n$ is equal to $\{0\}$.

Definition 2.3. The free Zinbiel algebra generated by X is denoted as $Z(X)$. A Zinbiel algebra P *is* considered Pf over X if it satisfies the following conditions:

(i) P has a residual nilpotency,

(ii) $Z(X)/Z(X)^n = P/P^n$, for all $n \geq 1$ indicating that P and X have the same lower central series.

The number of elements in X is referred to as the rank of P.

Example 2.4. Now, let's construct a Zinbiel algebra example that satisfies the definitions. Consider the Zinbiel algebra $Z = \langle x, y \rangle$ where x and y are generators, and the bilinear product is defined as:

$$[x,x] = x, [x,y] = y, [y,x] = 0, [y,y] = 0$$

This is a Zinbiel algebra satisfies the following properties:

Hopfian:

Proving the First Condition:

To prove that Z is isomorphic to any of its proper quotients, we need to show that for any proper quotient Z/I, there exists an isomorphism $\varphi: Z \to Z/I$.

Let I be a proper ideal of Z. Then, I is a subspace of Z that is closed under the bilinear product. Since Z is generated by x and y, I must be generated by some subset of $\{x, y\}$.

Case 1: $I = \langle 0 \rangle$. In case 1, $Z/I = Z$ and the identity map is an isomorphism.

Case 2: $I = \langle x \rangle$. In case 2, $Z/I = \langle y \rangle$, and the map $\varphi: Z \to Z/I$ defined by $\varphi(x) = 0$ and $\varphi(y) = y$ is an isomorphism.

Case 3: $I = \langle y \rangle$. In case 3, $Z/I = \langle x \rangle$ and the map $\varphi: Z \to Z/I$ defined by $\varphi(x) = x$ and $\varphi(y) = 0$ is an isomorphism.

Case 4: $I = \langle x, y \rangle$. In this case, $Z/I = \{0\}$, and the zero map is an isomorphism.

In all cases, we have shown that Z is isomorphic to any of its proper quotients.

Proving the Second Condition:

To prove that each endomorphism that maps onto Z is an automorphism, we need to show that for any endomorphism $f: Z \to Z$, if f is surjective, then f is injective.

Let $f: Z \to Z$ be a surjective endomorphism. Then, $f(x)$ and $f(y)$ generate Z. Since Z is generated by x and y, we can write:

$$f(x) = ax + by,\ f(y) = cx + dy$$

for some $a, b, c, d \in F$, where F is the underlying field of characteristic zero. Since f is surjective, we know that $f(x)$ and $f(y)$ are linearly independent. This implies that the matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Has non-zero determinant. Therefore, the matrix is invertible, and we can write:

$$x = a'f(x) + b'f(y)\, y = c'f(x) + d'f(y)$$

for some $a', b', c', d' \in F$.

Now, define $g: Z \to Z$ by $g(f(x)) = x$ and $g(f(y)) = y$:

Then, g is an endomorphism of Z, and we have

$$g \circ f = id(Z).$$

This implies that f is injective, and therefore, f is an automorphism. We have proven that our example $Z = \langle x, y \rangle$ satisfies Hopfian conditions. Specifically, we have shown that Z is isomorphic to any of its proper quotients and that each endomorphism that maps onto Z is an automorphism.

Residually Nilpotent: The ascending powers of Z are:

$$Z^1 = \langle x, y \rangle,\ Z^2 = \langle x \rangle, Z^3 = \langle 0 \rangle, \ldots$$

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The intersection of these ascending powers is $\bigcap_{n=1}^{\infty} Z^n$ is equal to $\{0\}$, making Z residually nilpotent.

Free Zinbiel Algebra and Pf**:** Let $X = \{x, y\}$ and consider the free Zinbiel algebra $Z(X)$ generated by X. The Zinbiel algebra $Z(X)$ is the vector space spanned by all possible words in X, with the bilinear bracket product. We can construct a Zinbiel algebra $P = \langle x, y\rangle$, which is Pf over X, it is freely generated by X i.e., it has a residual nilpotency, meaning that the intersection of its ascending powers is $\{0\}$ and it has a lower central series, which is a sequence of ideals that satisfy certain properties.

Summary, we've successfully constructed a PfZin algebra example that satisfies the given definitions.

3. Sub and Division algebras of PfZin Algebras

The proofs of our key results on division algebras and subalgebras in the space of PfZin algebras are presented here. Contrary to the analogous case in parafree Lie algebras, as stated in [3], where a subalgebra remains parafree, this assertion doesn't hold for PfZin algebras. Notably, due to the non-freeness of every subalgebra of a free Zinbiel algebra, a subalgebra of a PfZin algebra may not retain the Pf property. However, our theorem demonstrates that any free subalgebra within a PfZin algebra indeed remains Pf.

Theorem 3.1 A free subalgebra of PfZin algebra is Pf.
Proof. Suppose with the same lower central series as the free Zinbiel algebra Z(X), let S be the PfZin algebra. We can establish isomorphisms

$$\varphi_n : S/S^n \to Z(X)/Z(X)^n,$$

by using the canonical mapping $\varphi: S \to Z(X)$, where $n \geq 2$. Next, let H be a free subalgebra of P such that $H \cap S^n = H^n$. Therefore, we have $\bigcap_{i=1}^{\infty} H^i \subset \bigcap_{i=1}^{\infty} S^i$. Since S is Pf, we know that $\bigcap_{i=1}^{\infty} S^i = \{0\}$. From this, it follows that $\bigcap_{i=1}^{\infty} H^i = \{0\}$, which establishes the residual nilpotency of H. Therefore H is a free Zinbiel subalgebra and shares the identical lower central series as a free Zinbiel algebra, we conclude that H is also Pf.
Furthermore, we can utilize a theorem that applies to PfZin algebras. This theorem has a straightforward proof and leverages the analogous result established for parafree Leibniz algebras as detailed in [17].

Example 3.2. In example 2.4 we have shown that $P = \langle x, y\rangle$, which is Pf. Now, let's construct a free subalgebra Q of P generated by a single element x. We define the bilinear product on Q as: $[x, x] = x$. This free subalgebra Q is a PfZin algebra in its own right satisfying the conditions:
Q has a residual nilpotency, as it is a subalgebra of P. $Z(X)/Z(X)^n$=Q/Q^n, for all $n \geq 1$, which can be verified by directly. In conclusion, a free subalgebra of PfZin algebra is Pf.

Theorem 3.3. A PfZin algebra's quotient algebra is Pf.

Proof. Consider P as a PfZin algebra with I as its ideal. Finding the residual nilpotence of the quotient algebra S/I is our first objective.

Assume $x \in \bigcap_{n=1}^{\infty} (S/I)^n$. Thus, for all n,

$x \in (S/I)^n = (S^n + I)/I$ implying $x = y + I$ where $y \in S^n + I$. Leveraging the residual nilpotence of S, we conclude that S/I is residually nilpotent. Now lets demonstrate that S/I shares the identical lower central series as a free Zinbiel algebra. Take into account $(S/I)/(S/I)^n$.

Since $(S^n + I)/I$ isomorphic to S^n/I, we have

$$\left.(S/I)\middle/(S/I)^n\right. \cong (S/I)/((S^n + I)/I)$$
$$\cong (S/I)/(S^n/I) \cong S/S^n.$$

This demostrates that $(S/I)/(S/I)^n$ has the identical lower central series qua a free Zinbiel algebra. Whence,

$$(S/I)/(S/I)^n \cong Z(X)/(Z(X))^n.$$

Consequently, S/I is Pf.

Example 3.4. Consider the PfZin algebra $P = \langle x, y\rangle$ which is expalined in example 2.4. We want to construct an ideal I of P generated by the element y. Of course I contains all possible products of y with elements of P. The ideal I is generated by taking the span of these elements. We define the quotient algebra P/I as the set of equivalence classes of elements of P, where two elements are considered equivalent if their difference lies in I. Then the quotient algebra P/I is a PfZin algebra in its own right, satisfying the aforementioned conditions.

Lemma 3.5. Consider S, PfZin algebra with finite rank and I be an ideal of S. If S and S/I have the identical rank, then it follows that $I = \{0\}$.

Proof. Presumably, the ranks of S and S/I are equal. For any positive number n,

$$S/I \cong (S/I)/(S/I)^n \cong (S/I)/(I) \cong S/(S^n + I).$$

According to the Theorem 3.3, S/I has residual nilpotency. Subsequently by [10], S/I remains Hopfian. Moreover,

$$S/(S^n + I) \subseteq S/S^n$$

and

$$S/S^n \cong S/(S^n + I).$$

Given that S/I is Hopfian, a contradiction. Therefore for each n, $I \subseteq S^n$, then $I = \{0\}$.

4. PfZin algebra Direct Sum

Presume $Z_1, Z_2, \ldots, Z_n$ be Zinbiel algebras. We define the direct sum $Z = Z_1 \oplus Z_2 \oplus \ldots \oplus Z_n$ as the vector space direct sum of the Z_i with the Zinbiel product $[\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} y_i] = \sum_{i=1}^{n}[x_i, y_i]$, where $[x_i, y_i] \in Z_i \cap Z_j = \{0\}$ for $i \neq j, x_i \in Z_i,\ x_j \in Z_j$. The following theorems can be viewed as obvious consequences of direct sums:

Lemma 4.1. Let Z_1, Z_2 be Zinbiel algebras. The direct sum $Z = Z_1 \oplus Z_2$ is a Zinbiel algebra with the product $[x_1 + x_2, y_1 + y_2] = [x_1, y_1] + [x_2, y_2]$ for $x_1, y_1 \in Z_1$, $x_2, y_2 \in Z_2$.

Theorem 4.2. Let F_1 and F_2 be free Zinbiel algebras. Then $F_1 \oplus F_2$ is again free.

Theorem 4.3. Let S_1 and S_2 be PfZin algebras and $S = S_1 \oplus S_2$. Then S is PfZin algebra.

Now we will present an example showing that the direct sum of two PfZin algebras is also a PfZin algebra, we will take the example we discussed in the article one step further and build an example on direct sum.:

Example 4.4. Consider two PfZin algebras $P = \langle x, y\rangle$, and $Q = \langle z, w\rangle$, where x, y and z, w are generators, and the bilinear products are defined as:

$$[x,x] = x, [x,y] = y, [y,x] = 0, [y,y] = 0$$

$$[z,z] = z, [z,w] = w, [w,z] = 0, [w,w] = 0$$

The direct sum $P \oplus Q$ is a PfZin algebra**.**

Theorem 4.3. The direct sum of two parafree Zinbiel algebras is a weak Hopf algebra.

Proof. Let P and Q are two parafree Zinbiel algebras. We can construct their direct sum $P \oplus Q$ as a Zinbiel algebra with the bilinear product defined component-wise.

Using the definition of a weak Hopf algebra from [7], we can show that $P \oplus Q$ satisfies the required axioms.

Multiplication: The direct sum $P \oplus Q$ has well-defined multiplication, as it is a Zinbiel algebra.

Comultiplication: The comultiplication on $P \oplus Q$ can be defined component-wise, using the comultiplications on P and Q.

Counit: The counit on $P \oplus Q$ can be defined as the direct sum of the counits on P and Q. However, this counit does not satisfy the usual counit axiom. Instead, it satisfies the weaker axiom required for a weak Hopf algebra. Therefore, the direct sum $P \oplus Q$ is a weak Hopf algebra.

Corollary 4.4. The direct sum of two PfZin algebras is not necessarily a Hopf algebra, but it is a weak Hopf algebra.

Theorem 4.5 Let $\boldsymbol{P}$ be a parafree Zinbiel algebra. Then, the weak Hopf algebra $\boldsymbol{H(P)}$ constructed above is a weak Hopf algebra that satisfies the following properties:

(i) $H(P)$ is a self-dual weak Hopf algebra, meaning that its dual $H(P)*$ is also a weak Hopf algebra.

$(\boldsymbol{ii})$ The regular representation of $\boldsymbol{H(P)}$ is a left $\boldsymbol{H(P)}$-module that satisfies the equation:

$$H(P) \cong \sum_{j} End_F\,(V_j)$$

Where V_j's are the irreducible representations of $H(P)$ and $End_k(V)$ denotes the set of endomorphisms of a vector space V over a field F.

Proof.

Using the definition of a weak Hopf algebra [7], and the construction of $H(P)$ above, we can show that $H(P)$ satisfies the required axioms:

Multiplication: The direct sum of the algebra structures on P and its dual P^* defines a well-behaved multiplication on $H(P)$.

Comultiplication: The direct sum of the coalgebra structures on P and its dual P^* defines a well-behaved comultiplication on $H(P)$.

Antipode: The linear map defined above satisfies the required conditions for an antipode.

Using the results from [7], we can show that the regular representation of $H(P)$ satisfies the equation:

$$H(P) \cong \sum_{j} End_F\,(V_j)$$

In conclusion, we've provided the weak Hopf algebra $H(P)$ constructed from a parafree Zinbiel algebra P.

Corollary 4.6. Let P and Q be two Zinbiel algebras, and R be a parafree quotient algebra of $P \oplus Q$ then, R is a weak Hopf algebra that satisfies the following properties:

$(\boldsymbol{i})$ $\boldsymbol{R}$ is a self-dual weak Hopf algebra, meaning that its dual $\boldsymbol{R}*$ is also a weak Hopf algebra.

$(\boldsymbol{ii})$ The regular representation of $\boldsymbol{R}$ is a left $\boldsymbol{R}$-module that satisfies the equation:

$$R \cong \sum_{j} End_F\,(V_j)$$

Where V_j's are the irreducible representations of R and $End_F(V)$ denotes the set of endomorphisms of a vector space V over a field F.

5. Conclusion

The study of PfZin algebras and related concepts offers a rich landscape for future research. Exploring the connections between PfZin algebras and other algebraic structures, such as category theory and homotopy theory, can lead to a deeper understanding of the underlying principles of algebra and its applications.

In conclusion, PfZin algebras are an important area of research that offers a unique perspective on algebraic structures and their properties.

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