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Research Article

A Study on Maximal Embedding Dimension Numerical Semigroups

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Abstract: In this article, it is examine some the numerical semigroups W and $\frac{W}{2}$ such that $W=<p,q>$ and $\frac{W}{2}=<p,\frac{p+q}{2},q>$ where $p<q$ and p,q are odd natural numbers.

Keywords: Pseudo symmetric; symmetric; numerical semigroup; maximal embedding dimension

Maksimal Gömme Boyutlu Sayısal Yarıgruplar Hakkında Bir Çalışma

Özet: Bu makalede, $p<q$ için p,q tek doğal sayılar olmak üzere,

$W=<p,q>$ ve $\frac{W}{2}=<p,\frac{p+q}{2},q>$ şeklindeki bazı sayısal yarıgrupları incelenmektedir.

Anahtar Kelimeler: Simetrikimsi; simetrik; sayısal yarıgrup; maksimal gömme boyut

1. Introduction

Numerical semigroups first appeared in problems in classical number theory posed by J. J. Sylvester and Ferdinand Georg Frobenius in the late 19th century. Numerical semigroups have re-mained the focus of extensive study due to the abundance of open problems and applications in other fields such as electrical engineering, differential equations, and algebraic geometry.

Let $\mathbb{N} = \{u \in \mathbb{Z} : u \geq 0\}$ and $\mathbb{Z}$ be integer set. $\Omega \subseteq \mathbb{N}$ is called a numerical semigroup if

(i) $z_1 + z_2 \in d$, for $z_1, z_2 \in \Omega$

(ii) $\mathbb{N} \backslash \Omega$ is finite

(iii) $0 \in \Omega$.

A numerical semigroup Ω can be written that

$$\Omega = < z_1, z_2, ..., z_k > = \{z_1u_1 + z_2u_2 + ... + z_ku_k : u_i \in \mathbb{N}, 1 \leq i \leq k\}.$$

$L \subset \mathbb{N}$ is minimal system of generators of Ω if $< L > = \Omega$ and there isn't any subset $T \subset L$ such that $< T > = \Omega$. Also, $\lambda(\Omega) = \min\{z \in \Omega : z > 0\}$ and $d(\Omega) = k$ are called multiplicity and embedding dimension of Ω, respectively. It is know that $d(\Omega) \leq \lambda(\Omega)$. If $d(\Omega) = \lambda(\Omega)$ then Ω is called maximal embedding dimension (MED) (See [1,5,6]). Also, $\mu(\Omega) = \max(\mathbb{Z} \backslash \Omega)$ is called Frobenius number of Ω, $q(\Omega) = Card(\{0,1,2,...,\mu(\Omega)\} \cap \Omega)$ is called as the determine number of Ω. Here, we will indicate the number of elements of the set Ω by $Card(\Omega)$.

For Ω is a numerical semigroup such that $\Omega = < z_1, z_2, ..., z_k >$, then we write that

$\Omega = < z_1, z_2, ..., z_k > = \{l_0 = 0, l_1, l_2, ..., l_{q-1}, l_q = \mu(\Omega) + 1, \rightarrow ...\}$, where $z_i < z_{i+1}$, $q = q(\Omega)$ and the arrow means that every integer greater than $\mu(\Omega) + 1$ belongs to Ω for $i = 1, 2, ..., q = q(\Omega)$.

Let Ω be a numerical semigroup. If $u \in \mathbb{N}$ and $u \notin \Omega$, then u is called gap of Ω. We denote the set of gaps of Ω, by $Y(\Omega)$, i.e, $Y(\Omega) = \mathbb{N} \backslash \Omega = \{u \in \mathbb{N} : u \notin \Omega\}$. The number $P(\Omega) = Card(Y(\Omega))$ is called the genus of Ω, and we note that $P(\Omega) + q(\Omega) = \mu(\Omega) + 1$ (see [8]).

Ω is called symmetric numerical semigroup if $\mu(\Omega) - m \in \Omega$, for $m \in \mathbb{Z} \backslash \Omega$. It is know the numerical semigroup $\Omega = < q_1, q_2 >$ is symmetric and $\mu(\Omega) = q_1q_2 - q_1 - q_2$. Thus,

$q(\Omega) = P(\Omega) = \frac{m(\Omega)+1}{2}$. Ω is called pseudo symmetric numerical semigroup if $m(\Omega)$ is even and $m(\Omega) - f \notin \Omega$, $f = \frac{m(\Omega)}{2}$, for $f \in \mathbb{Z} \backslash \Omega$ (For details see [2,3,4,9,10]).

Let Ω be a numerical semigroup and $u \in \Omega, u > 0$. The set $Ap(\Omega, u) = \{z \in \Omega : z - u \notin \Omega\}$ is called the Apery set of Ω according u, and $Ap(\Omega, u) \subseteq \Omega$ ([8]).

For numerical semigroup let's the set , for . If then the set is called the half of ([7]). We note that while is symmetric (pseudo symmetric) numerical semigroup, need not be symmetric (pseudo symmetric). For example is symmetric numerical semigroup but is not symmetric numerical semigroup (is pseudo symmetric but is not pseudo symmetric) .

In this article, it is examine some results on the symmetric numerical semigroup Ω but $\frac{\Omega}{2}$ is pseudo symmetric numerical semigroup such that $\Omega = < p, q >$ and $\frac{\Omega}{2} = < p, \frac{p+q}{2}, q >$ where $p < q$ and p, q are odd natural numbers .

2. MAIN RESULTS

Proposition 1. ([8]) If $\Omega = < z_1, z_2, ..., z_k >$ is MED numerical semigroup then $Ap(\Omega, z_1) = \{0, z_2, ..., z_k\}$.

Proposition 2. ([8]) If $\Omega = < z_1, z_2, ..., z_k >$ is a numerical semigroup then $m(\Omega) = \max(Ap(\Omega, z_1)) - z_1$.

Proposition 3. ([7]) Ω is pseudo symmetric numerical semigroup $\Leftrightarrow$ $q(\Omega) = \frac{m(\Omega)}{2}$.

Proposition 4. ([8]) Ω is pseudo symmetric numerical semigroup $\Leftrightarrow$ $P(\Omega) = \frac{m(\Omega)+2}{2}$.

Proposition 5. ([8]) If $\Omega = < z_1, z_2, ..., z_k >$ is a MED numerical semigroup then $m(\Omega) = z_k - z_1$.

Proposition 6. ([8]) Let $\Omega = < z_1, z_2, ..., z_k >$ be a numerical semigroup. Then the following conditions are equalities:

(1) Ω is MED

(2) $P(\Omega) = \frac{1}{z_1} \sum_{i=2}^{k} z_i - \frac{z_1 - 1}{2}$.

Theorem 7. Let $W=<p,q>$ be numerical semigroups , where $p<q$ and p,q are odd integers. Then, $\frac{W}{2}=<p,\frac{p+q}{2},q>$.

Proof. Let $W=<p,q>$ be numerical semigroups , where $p<q$ and p,q are odd integers. Then, we write

$$x\in<p,\frac{p+q}{2},q>\Leftrightarrow \exists w_1,w_2,w_3\in\mathbb{N},\ x=pw_1+(\frac{p+q}{2})w_2+qw_3$$

$$\Leftrightarrow 2x=p(2w_1)+(p+q)w_2+q(2w_3)$$

$$\Leftrightarrow 2x=p(\underbrace{2w_1+w_2}_{n_1\in\mathbb{N}})+q(\underbrace{w_2+2w_3}_{n_2\in\mathbb{N}})$$

$$\Leftrightarrow 2x=pn_1+qn_2\in<p,q>=\Omega$$

$$\Leftrightarrow x\in\frac{\Omega}{2}.$$

In this theorem, if we put $p=3$ then we obtain followings:

Corollary 8. Let $W=<3,q>$ be numerical semigroups, where $q>3$ is odd integer. Then, the numerical semigroup $\frac{W}{2}=<3,\frac{3+q}{2},q>$ is MED, and $m(\frac{W}{2})=q-3$.

Theorem 9. Let $W=<3,q>$ be numerical semigroups, where $q>3$ is odd integer. Then, the numerical semigroup $\frac{W}{2}=<3,\frac{3+q}{2},q>$ is pseudo symmetric.

Proof. Let $W=<3,q>$ be numerical semigroups, where $q>3$ is odd integer. We write the numerical semigroups $\frac{W}{2}=<3,\frac{3+q}{2},q>$ is MED from Corollary 8. Then, we obtain $P(\frac{W}{2})==\frac{1}{3}(\frac{3+q}{2}+q)-1=\frac{q+1}{2}-1=\frac{q-1}{2}$ from Proposition 6. So, we have the numerical semigroup $\frac{W}{2}=<3,\frac{3+q}{2},q>$ is pseudo symmetric since $P(\frac{W}{2})=\frac{q-1}{2}=\frac{(q-3)+2}{2}=\frac{m(\frac{W}{2})+2}{2}$ and Proposition 4.

The following Corollary is clear since the numerical semigroup $W=<3,q>$ is symmetric.

Corollary 10. Let's $\Omega = <3, q>$, where $q > 3$ is odd integer. Then,

(1) $m(\Omega) = 2q - 3$
(2) $q(\Omega) = P(\Omega) = q - 1$.

Proposition 11. Let $\Omega = <3, q>$ be numerical semigroups and its half numerical semigroup is $\frac{\Omega}{2} = <3, \frac{3+q}{2}, q>$, where $q > 3$ is odd integer. Then, we have

(a) $m(\Omega) = m(\frac{\Omega}{2}) + q$

(b) $q(\Omega) = q(\frac{\Omega}{2}) + \frac{q+1}{2}$

(c) $P(\Omega) = 2P(\frac{\Omega}{2})$.

Proof. Let $\Omega = <3, q>$ be numerical semigroups and its half numerical semigroup is $\frac{\Omega}{2} = <3, \frac{3+q}{2}, q>$, where $q > 3$ is odd integer. Then,

(a) $m(\frac{\Omega}{2}) + q = (q - 3) + q = 2q - 3 = m(\Omega)$.

(b) $q(\frac{\Omega}{2}) = \frac{m(\frac{\Omega}{2})}{2} = \frac{q-3}{2}$ since $\frac{\Omega}{2}$ is pseudo symmetric from Proposition 3.

Thus, we obtain $q(\frac{\Omega}{2}) + \frac{q+1}{2} = \frac{q-3}{2} + \frac{q+1}{2} = q - 1 = q(\Omega)$.

(c) $2P(\frac{\Omega}{2}) = 2(\frac{q-1}{2}) = q - 1 = P(\Omega)$.

Example 12. Let's $\Omega = <3, 7> = \{3k_1 + 7k_2 : k_1, k_2 \in \mathbb{N}\} = \{0, 3, 6, 7, 9, 10, 12, \rightarrow ...\}$. Then $m(\Omega) = 11$, $q(\Omega) = 6$, $Y(\Omega) = \mathbb{N} \backslash \Omega = \{1, 2, 4, 5, 8, 11\}$ and $P(\Omega) = Card(Y(\Omega)) = 6$. In this case, we have $\frac{\Omega}{2} = \{u \in \mathbb{N} : 2u \in \Omega\} = \{0, 3, 5, \rightarrow ...\} = <3, 5, 7>$ is MED since $d(\frac{\Omega}{2}) = l(\frac{\Omega}{2}) = 3$.

Also, we write that $q(\frac{W}{2}) = 2$, we find $m(\frac{W}{2}) = 7 - 3 = 4$ from Proposition 5. So, $\frac{W}{2}$ is pseudo symmetric numerical semigroup since $q(\frac{W}{2}) = \frac{m(\frac{W}{2})}{2} = \frac{4}{2} = 2$ from Proposition 3.

On the other hand, we obtain

$m(\frac{W}{2}) + 7 = 4 + 7 = 11 = m(W)$,

$q(\frac{W}{2}) + \frac{7+1}{2} = 2 + 4 = 6 = q(W)$ and

$2P(\frac{W}{2}) = 2.3 = 6 = P(W)$ from Proposition 11.

Conflict of interest

The author reports no conflict of interest relevant to this article.

Research and Publication Ethics Statement

The author declares that this study complies with research and publication ethics.

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