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Some Almost B-Structures on the Tangent Bundle Equipped with a Vertical Rescaled Metric

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Abstract. In the present paper, we study some almost paracomplex structures on the tangent bundle with vertical rescaled metric and search conditions for the tangent bundle to become B-manifold and quasi-B-manifold.

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1. Introduction

The concept of almost paracomplex structure has been studied, since the first papers by Rashevskij [16], Libermann [13] and Patterson [15] until now, from several different points of view. Moreover, the papers related to it have appeared many times in a rather disperse way, and a survey of further results on paracomplex geometry (including paraHermitian and paraKähler geometry) can be found for instance in [5, 6]. Also, other important developments have occurred in some recent problems [2, 4, 20], where certain aspects concerning the geometry of tangent and cotangent bundles are presented in [3, 11, 12, 17, 18, 23–26]. For this reason, the study of structures remains a rich field of research, especially in tangent or cotangent geometry, to this day.

In this paper, we construct some almost paracomplex structures on the tangent bundle with vertical rescaled metric [8] and investigate necessary and sufficient conditions for the tangent bundle to become B-manifold and quasi-B-manifold. Also some B-metric properties of the vertical rescaled metric are studied.

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(u), x^i, y^i)_{i=\overline{1,m}}$ on TM. We denote by ∇ is the Levi-Civita connection on a Riemannian manifold (M^m, g) and Γ^k_{ij} are the Christoffel symbols of ∇. Let $\Im^r_s(M)$ (resp. $\Im^r_s(TM)$) the module over $C^\infty(M)$ (resp. $C^\infty(TM)$) of C^∞ tensor fields of type (r, s), where $C^\infty(M)$ (resp. $C^\infty(TM)$) is the ring of real-valued C^∞ functions on M (resp. TM).

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM,$$

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of the tangent bundle to TM at any $(x,u) \in TM$ into vertical subspace

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial y^i}|_{(x,u)},\ \xi^i \in \mathbb{R}\}$$

and the horizontal subspace

$$H_{(x,u)}TM = \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)},\ \xi^i \in \mathbb{R}\}.$$

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$\begin{aligned} {}^V X &= X^i \frac{\partial}{\partial y^i}, \\ {}^H X &= X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}. \end{aligned}$$

For consequences, we have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\delta}{\delta x^i}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

In particular, we have the vertical distribution ${}^V u$ and the horizontal distribution ${}^H u$ on TM defined by

$$ {}^V u = u^i\, {}^V(\frac{\partial}{\partial x^i}) = u^i \frac{\partial}{\partial y^i}, \quad {}^H u = u^i\, {}^H(\frac{\partial}{\partial x^i}) = u^i \frac{\delta}{\delta x^i}. \tag{2.1}$$

${}^V u$ is also called the canonical or Liouville vector field on TM.

Lemma 2.1 ([1]). *Let (M,g) be a Riemannian manifold and $\eta : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function, we have the following:*

(1) ${}^H X(\eta(r)) = 0$,
(2) ${}^V X(\eta(r)) = 2\eta'(r) g(X,u)$,
(3) ${}^H X g(Y,u) = g(\nabla_X Y, u)$,
(4) ${}^V X g(Y,u) = g(X,Y)$,
(5) ${}^V u(\eta(r)) = 2\eta'(r) g(u,u)$,
(6) ${}^V u(g(Y,u)) = g(Y,u)$,

for any vector fields X, Y on M, where $r = g(u,u)$.

Lemma 2.2 ([9, 22]). *Let (M,g) be a Riemannian manifold. The bracket operation of vertical and horizontal vector fields is given by the formulas*

(1) $[{}^H X, {}^H Y] = {}^H[X,Y] - {}^V(R(X,Y)u)$,
(2) $[{}^H X, {}^V Y] = {}^V(\nabla_X Y)$,
(3) $[{}^V X, {}^V Y] = 0$,
(4) $[{}^V Y, {}^V u] = {}^V Y$,
(5) $[{}^H Y, {}^V u] = 0$,

for all vector fields $X, Y \in \Im^1_0(M)$, where ∇ is the Levi-Civita connection on a Riemannian manifold (M,g) and R is Riemannian curvature tensor of ∇.

An almost product structure φ on a manifold M is a $(1,1)$ tensor field such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1,1)$ on M). The pair (M,φ) is called an almost product manifold.

A linear connection ∇ on (M,φ) such that $\nabla\varphi = 0$ is said an almost product connection. There exists an almost product connection on every almost product manifold [7].

An almost paracomplex manifold is an almost product manifold (M,φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even [6].

The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y].$$

A paracomplex structure is an integrable almost paracomplex structure. On the other hand, for an almost paracomplex structure to be integrable, a necessary and sufficient condition is the existence of a torsion-free linear connection such that $\nabla\varphi = 0$ [18, 20].

Let (M^{2m}, φ) be an almost paracomplex manifold. A Riemannian metric g is said to be B-metric if

$$g(\varphi X, \varphi Y) = g(X, Y),$$

for all $X, Y \in \Im^1_0(M)$ [20]. or equivalently (purity condition with respect to the almost paracomplex structure φ)

$$g(\varphi X, Y) = g(X, \varphi Y).$$

If (M^{2m}, φ) is an almost paracomplex manifold with B-metric g, we say that (M^{2m}, φ, g) is an almost B-manifold. If φ is integrable, we say that (M^{2m}, φ, g) is a B-manifold [20].

A Tachibana operator ϕ_φ applied to the B-metric (pure metric) g is given by

$$(\phi_\varphi g)(X, Y, Z) = \varphi X(g(Y, Z)) - X(g(\varphi Y, Z)) + g((L_Y \varphi)X, Z) + g((L_Z \varphi)X, Y),$$

for all $X, Y, Z \in \Im^1_0(M)$ [21].

In a B-manifold, a B-metric g is called paraholomorphic if

$$(\phi_\varphi g)(X, Y, Z) = 0,$$

for all $X, Y, Z \in \Im^1_0(M)$ [20].

In [20], Salimov and his collaborators proved that for an almost B-manifold,

$$\nabla \varphi = 0 \Leftrightarrow \phi_\varphi g = 0, \tag{2.2}$$

by virtue of this view, in an almost B-manifold the integrability condition of φ is equivalent to the paraholomorphicity condition of the B-metric.

The purity conditions for a tensor field $\omega \in \Im^q_0(M)$ with respect to the almost paracomplex structure φ given by

$$\omega(\varphi X_1, X_2, \ldots, X_q) = \omega(X_1, \varphi X_2, \ldots, X_q) = \ldots = \omega(X_1, X_2, \ldots, \varphi X_q),$$

for all $X_1, X_2, \ldots, X_q \in \Im^1_0(M)$ [20].

It is well known that, if (M^{2m}, φ, g) is a B-manifold, the Riemannian curvature tensor is pure [20], and

$$\begin{cases} R(\varphi Y, Z) = R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) = R(Y, Z), \end{cases} \tag{2.3}$$

for all $Y, Z \in \Im^1_0(M)$.

Let (M^{2m}, φ, g) be a non-integrable almost B-manifold, if

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0,$$

for all $X, Y, Z \in \Im^1_0(M)$, where σ is the cyclic sum by three arguments, then the triple (M^{2m}, φ, g) is a quasi-B-manifold [10, 14]. We know that

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0 \Leftrightarrow \underset{X,Y,Z}{\sigma} (\phi_\varphi g)(X, Y, Z) = 0, \tag{2.4}$$

which was proven in [19].

3. Vertical Rescaled Metric

Definition 3.1 ([8])**.** Let (M^m, g) be a Riemannian manifold and f be a strictly positive smooth function on M. We define the vertical rescaled metric G^f on the tangent bundle TM by

(1) $G^f({}^HX, {}^HY) = g(X, Y)$,
(2) $G^f({}^HX, {}^VY) = 0$,
(3) $G^f({}^VX, {}^VY) = fg(X, Y)$,

for all vector fields $X, Y \in \Im^1_0(M)$.

Theorem 3.2 ([8])**.** *Let (M^m, g) be a Riemannian manifold and ∇^f be a Levi-Civita connection of (TM, G^f). Then, we have*

(1) $\nabla^f_{{}^HX} {}^HY = {}^H(\nabla_X Y) - \frac{1}{2} {}^V(R(X, Y)u)$,
(2) $\nabla^f_{{}^HX} {}^VY = \dfrac{f}{2} {}^H(R(u, Y)X) + {}^V(\nabla_X Y) + \dfrac{X(f)}{2f} {}^VY$,

(3) $\nabla^f_{{}^VX}{}^HY = \frac{f}{2}{}^H(R(u,X)Y) + \frac{Y(f)}{2f}\,{}^VX$,

(4) $\nabla^f_{{}^VX}{}^VY = \frac{-1}{2}g(X,Y)\,{}^H(grad\, f)$,

for all vector fields $X, Y \in \Im^1_0(M)$, where ∇ is the Levi-Civita connection on a Riemannian manifold (M^m, g) and R is Riemannian curvature tensor of ∇.

4. Some Almost Paracomplex Structures with B-Metrics on the Tangent Bundle

Let (M^{2m}, φ, g) be an almost B-manifold, we consider the tensor field $J \in \Im^1_1(TM)$ defined by

$$\begin{cases} J^HX &= {}^H(\varphi X) \\ J^VX &= -{}^V(\varphi X) \end{cases} \tag{4.1}$$

for all $X \in \Im^1_0(M)$.

Lemma 4.1. *Let (M^{2m}, φ, g) be an almost B-manifold and (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric. The couple (TM, J) is an almost paracomplex manifold .*

Proof. By virtue of (4.1), we have

$$\begin{cases} J^{2\,H}X = J(J^HX) = J({}^H(\varphi X)) = {}^H(\varphi(\varphi X)) = {}^H(\varphi^2X) = {}^HX, \\ J^{2\,V}X = J(J^VX) = J(-{}^V(\varphi X)) = {}^V(\varphi(\varphi X)) = {}^V(\varphi^2X) = {}^VX, \end{cases}$$

for any $X \in \Im^1_0(M)$, then $J^2 = id_{TM}$.
Let $\{E_1, \ldots, E_m, E_{m+1}, \ldots, E_{2m}\}$ be a local frame of eigenvectors on M such that $\varphi E_i = E_i$ and $\varphi E_{m+i} = -E_{m+i}$, for all $i = \overline{1, m}$, then

$$TTM^+ = Span\,({}^HE_1, \ldots, {}^HE_m, {}^VE_{m+1}, \ldots, {}^VE_{2m}),$$
$$TTM^- = Span\,({}^VE_1, \ldots, {}^VE_m, {}^HE_{m+1}, \ldots, {}^HE_{2m}).$$

□

Theorem 4.2. *Let (M^{2m}, φ, g) be an almost B-manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure J defined by (4.1). The triple (TM, J, G^f) is an almost B-manifold.*

Proof. From (4.1) and since g is B-metric (pure metric) with respect to φ we have

(i) $G^f(J^HX, {}^HY) = G^f({}^H(\varphi X), {}^HY) = g(\varphi X, Y) = g(X, \varphi Y) = G^f({}^HX, {}^H(\varphi Y)) = G^f({}^HX, J^HY)$,

(ii) $G^f(J^HX, {}^VY) = G^f({}^H(\varphi X), {}^VY) = 0 = G^f({}^HX, {}^V(\varphi Y)) = G^f({}^HX, -{}^V(\varphi Y)) = G^f({}^HX, J^VY)$,

(iii) $G^f(J^VX, {}^HY) = G^f({}^V(\varphi X), {}^HY) = 0 = G^f({}^VX, {}^H(\varphi Y)) = G^f({}^VX, -{}^H(\varphi Y)) = G^f({}^VX, J^HY)$,

(iv) $G^f(J^VX, {}^VY) = G^f(-{}^V(\varphi X), {}^VY) = -fg(\varphi X, Y) = -fg(X, \varphi Y) = G^f({}^VX, -{}^V(\varphi Y)) = G^f({}^VX, J^VY)$,

for all $X, Y \in \Im^1_0(M)$.
Hence, G^f is pure metric with respect to the almost paracomplex structure J. □

Proposition 4.3. *Let (M^{2m}, φ, g) be an almost B-manifold, (TM, G^f) its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure J defined by (4.1), then we get*

1. $(\phi_J G^f)({}^HX, {}^HY, {}^HZ) = (\phi_\varphi g)(X, Y, Z)$,
2. $(\phi_J G^f)({}^VX, {}^HY, {}^HZ) = 0$,
3. $(\phi_J G^f)({}^HX, {}^VY, {}^HZ) = fg(R(\varphi X, Z)u + \varphi R(X, Z)u, Y)$,
4. $(\phi_J G^f)({}^HX, {}^HY, {}^VZ) = fg(R(\varphi X, Y)u + \varphi R(X, Y)u, Z)$,
5. $(\phi_J G^f)({}^VX, {}^VY, {}^HZ) = -fg((\nabla_Z\varphi)X, Y)$,

6. $(\phi_J G^f)({}^V X, {}^H Y, {}^V Z) = -fg((\nabla_Y \varphi)X, Z)$,
7. $(\phi_J G^f)({}^H X, {}^V Y, {}^V Z) = (\varphi X)(f)g(Y, Z) + X(f)g(\varphi Y, Z) + fg((\nabla_X \varphi)Y, Z)$,
8. $(\phi_J G^f)({}^V X, {}^V Y, {}^V Z) = 0$,

for all $X, Y, Z \in \Im^1_0(M)$.

Proof. We calculate Tachibana operator ϕ_J applied to the pure metric G^f. This operator is characterized by (2.3), then we have

$$\begin{aligned}
1.\ (\phi_J G^f)({}^H X, {}^H Y, {}^H Z) &= (J\,{}^H X)G^f({}^H Y, {}^H Z) - {}^H X G^f(J\,{}^H Y, {}^H Z) + G^f((L_{{}^H Y} J)\,{}^H X, {}^H Z) + G^f({}^H Y, (L_{{}^H Z} J)\,{}^H X) \\
&= {}^H(\varphi X)G^f({}^H Y, {}^H Z) - {}^H X G^f({}^H(\varphi Y), {}^H Z) + G^f(L_{{}^H Y} J\,{}^H X - J(L_{{}^H Y}{}^H X), {}^H Z) \\
&+ G^f({}^H Y, L_{{}^H Z} J\,{}^H X - J(L_{{}^H Z}{}^H X)) \\
&= (\varphi X)g(Y, Z) - Xg(\varphi Y, Z) + g([Y, \varphi X] - \varphi[Y, X], Z) + g(Y, [Z, \varphi X] - \varphi[Z, X]) \\
&= (\varphi X)g(Y, Z) - Xg(\varphi Y, Z) + g((L_Y \varphi)X, Z) + g(Y, (L_Z \varphi)X) \\
&= (\phi_\varphi g)(X, Y, Z).
\end{aligned}$$

$$\begin{aligned}
2.\ (\phi_J G^f)({}^V X, {}^H Y, {}^H Z) &= (J\,{}^V X)G^f({}^H Y, {}^H Z) - {}^V X G^f(J\,{}^H Y, {}^H Z)) + G^f((L_{{}^H Y} J)\,{}^V X, {}^H Z) + G^f({}^H Y, (L_{{}^H Z} J)\,{}^V X) \\
&= -{}^V(\varphi X)G^f({}^H Y, {}^H Z) - {}^V X G^f({}^H(\varphi Y), {}^H Z) - G^f([{}^H Y, {}^V(\varphi X)] + J[{}^H Y, {}^V(\varphi X)], {}^H Z) \\
&- G^f({}^H Y, [{}^H Z, {}^V(\varphi X)] + J[{}^H Z, {}^V X]) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
3.\ (\phi_J G^f)({}^H X, {}^V Y, {}^H Z) &= (J\,{}^H X)G^f({}^V Y, {}^H Z) - {}^H X G^f(J\,{}^V Y, {}^H Z) + G^f((L_{{}^V Y} J)\,{}^H X, {}^H Z) + G^f({}^V Y, (L_{{}^H Z} J)\,{}^H X) \\
&= G^f([{}^V Y, {}^H(\varphi X)] - J[{}^V Y, {}^H X], {}^H Z) + G^f({}^V Y, [{}^H Z, {}^H(\varphi X)] - J[{}^H Z, {}^H X]) \\
&= -G^f({}^V Y, {}^V(R(Z, \varphi X)u) + {}^V(\varphi R(Z, X)u)) \\
&= -fg(Y, R(Z, \varphi X)u + \varphi R(Z, X)u) \\
&= fg(R(\varphi X, Z)u + \varphi R(X, Z)u, Y).
\end{aligned}$$

$$\begin{aligned}
4.\ (\phi_J G^f)({}^H X, {}^H Y, {}^V Z) &= (J\,{}^H X)G^f({}^H Y, {}^V Z) - {}^H X G^f(J\,{}^H Y, {}^V Z) + G^f((L_{{}^H Y} J)\,{}^H X, {}^V Z) + G^f({}^H Y, (L_{{}^V Z} J)\,{}^H X) \\
&= G^f([{}^H Y, {}^H(\varphi X)] - J[{}^H Y, {}^H X], {}^V Z) + G^f({}^H Y, [{}^V Z, {}^H(\varphi X)] - J[{}^V Z, {}^H X]) \\
&= -G^f({}^V(R(Y, \varphi X)u) + {}^V(\varphi R(Y, X)u), {}^V Z) \\
&= -fg(R(Y, \varphi X)u, Z) - fg(\varphi R(Y, X)u, Z) \\
&= fg(R(\varphi X, Y)u + \varphi R(X, Y)u, Z).
\end{aligned}$$

The other formulas are obtained by the similar calculations. □

Theorem 4.4. *Let (M^{2m}, φ, g) be an almost B-manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure J defined by (4.1), then the triple (TM, J, G^f) is a B-manifold if and only if (M^{2m}, φ, g) is flat B-manifold and f is constant.*

Proof. From the Proposition 4.3, for all $X, Y, Z \in \Im^1_0(M)$, we have

$$\begin{cases}
(\phi_\varphi g)(X, Y, Z) & = 0, \\
fg(R(\varphi X, Z)u + \varphi R(X, Z)u, Y) & = 0, \\
fg(R(\varphi X, Y)u + \varphi R(X, Y)u, Z) & = 0, \\
-fg((\nabla_Z \varphi)X, Y) & = 0, \\
-fg((\nabla_Y \varphi)X, Z) & = 0, \\
(\varphi X)(f)g(Y, Z) + X(f)g(\varphi Y, Z) + fg((\nabla_X \varphi)Y, Z) & = 0.
\end{cases}$$

By virtue of (2.2) and (2.3), we get

$$\begin{cases} (\phi_\varphi g)(X,Y,Z) & = 0 \\ g(R(\varphi X,Z)u,Y) & = 0 \\ (\nabla_Z\varphi)X & = 0 \\ (\varphi X)(f)g(Y,Z)+X(f)g(\varphi Y,Z) & = 0 \end{cases} \Leftrightarrow \begin{cases} \phi_\varphi g & = 0 \\ R & = 0 \\ f & = constant. \end{cases}$$

□

Theorem 4.5. *Let (M^{2m},φ,g) be a B-manifold, (TM,G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure J defined by (4.1), then the triple (TM,J,G^f) is a quasi-B-manifold if and only if f is constant.*

Proof. From (2.4) and the Proposition 4.3, for all $X,Y,Z\in\Im^1_0(M)$, we have

1. $\underset{^HX,^HY,^HZ}{\sigma}(\phi_J G^f)(^HX,^HY,^HZ) = \underset{X,Y,Z}{\sigma}(\phi_\varphi g)(X,Y,Z)=0,$
2. $\underset{^VX,^HY,^HZ}{\sigma}(\phi_J G^f)(^VX,^HY,^HZ) = fg(R(\varphi Y,Z)u+R(\varphi Z,Y)u,X)=0,$
3. $\underset{^VX,^VY,^HZ}{\sigma}(\phi_J G^f)(^VX,^VY,^HZ) = (\varphi Z)(f)g(X,Y)+Z(f)g(\varphi X,Y),$
4. $\underset{^VX,^VY,^VZ}{\sigma}(\phi_J G^f)(^VX,^VY,^VZ) = 0,$

then, (TM,J,G^f) is a quasi-B-manifold if and only if f is constant. □

We consider the tensor field $K\in\Im^1_1(TM)$ defined by:

$$\begin{cases} K{}^HX & = -{}^H(\varphi X) \\ K{}^VX & = {}^V(\varphi X) \end{cases} \tag{4.2}$$

for all $X\in\Im^1_0(M)$, satisfies the followings:

1. $K=-J$.
2. G^f is pure metric with respect to K.
3. $\phi_K G^f = -\phi_J G^f$.

Therefore, we have the following results.

Theorem 4.6. *Let (M^{2m},φ,g) be an almost B-manifold, (TM,G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure K defined by (4.2), then the triple (TM,K,G^f) is a B-manifold if and only if (M^{2m},φ,g) is flat B-manifold and f is constant.*

Theorem 4.7. *Let (M^{2m},φ,g) be a B-manifold, (TM,G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure K defined by (4.2), then the triple (TM,K,G^f) is a quasi-B-manifold if and only if f is constant.*

We consider the tensor field $F\in\Im^1_1(TM)$ defined by:

$$\begin{cases} F{}^HX & = {}^V(\varphi X) \\ F{}^VX & = {}^H(\varphi X), \end{cases} \tag{4.3}$$

for all $X\in\Im^1_0(M)$.

Lemma 4.8. *Let (M^{2m},φ,g) be an almost B-manifold and (TM,G^f) bet its tangent bundle equipped with the vertical rescaled metric. The couple (TM,F) is an almost paracomplex manifold.*

Proof. By virtue of (4.3), we have

$$\begin{cases} F^2{}^HX = F(F{}^HX) = F({}^V(\varphi X)) = {}^H(\varphi(\varphi X)) = {}^H(\varphi^2X) = {}^HX, \\ F^2{}^VX = F(F{}^VX) = F({}^H(\varphi X)) = {}^V(\varphi(\varphi X)) = {}^V(\varphi^2X) = {}^VX, \end{cases}$$

for any $X \in \Im_0^1(M)$, then $F^2 = id_{TM}$.
Let $\{E_1, \dots, E_m, E_{m+1}, \dots, E_{2m}\}$ be local frame of eigenvectors on M such that $\varphi E_i = E_i, \varphi E_{m+i} = -E_{m+i}$, for all $i = \overline{1,m}$, then

$$TTM^+ = Span\,({}^HE_1 + {}^VE_1, \dots, {}^HE_m + {}^VE_m, {}^HE_{m+1} - {}^VE_{m+1}, \dots, {}^HE_{2m} - {}^VE_{2m}),$$
$$TTM^- = Span\,({}^HE_1 - {}^VE_1, \dots, {}^HE_m - {}^VE_m, {}^HE_{m+1} + {}^VE_{m+1}, \dots, {}^HE_{2m} + {}^VE_{2m}).$$

□

Theorem 4.9. *Let (M^{2m}, φ, g) be a B-manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure F defined by (4.4).*
The vertical rescaled metric G^f is B-metric with respect to F if and only if $f = 1$.
Conversely, in the case of $f \neq 1$, the vertical rescaled metric G^f is never pure metric with respect to F.

Now consider the almost product structure J defined by (4.1). We define a tensor field S of type $(1,2)$ and linear connection $\overline{\nabla}$ on TM by,

$$S(\widetilde{X}, \widetilde{Y}) = \frac{1}{2}((\nabla^f_{J\widetilde{Y}} J)\widetilde{X} + J((\nabla^f_{\widetilde{Y}} J)\widetilde{X}) - J((\nabla^f_{\widetilde{X}} J)\widetilde{Y})). \tag{4.4}$$

$$\overline{\nabla}_{\widetilde{X}}\widetilde{Y} = \nabla^f_{\widetilde{X}}\widetilde{Y} - S(\widetilde{X}, \widetilde{Y}).$$

for all $\widetilde{X}, \widetilde{Y} \in \Im_0^1(TM)$, where ∇^f is the Levi-Civita connection of (TM, G^f) given by Theorem 3.2. $\overline{\nabla}$ is an almost product connection on TM (see [7, p.150] for more details).

Lemma 4.10. *Let (M^{2m}, φ, g) be a B-manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost product structure J defined by (4.1). Then tensor field S is as follows,*

(1) $S({}^HX, {}^HY) = -\frac{1}{2}{}^V(R(X,Y)u)$,

(2) $S({}^HX, {}^VY) = \frac{f}{2}{}^H(R(u,Y)X) - \frac{X(f)}{2f}\,{}^VY - \frac{(\varphi X)(f)}{2f}\,{}^V(\varphi Y)$,

(3) $S({}^VX, {}^HY) = -f\,{}^H(R(u,X)Y) + \frac{Y(f)}{4f}\,{}^VX + \frac{(\varphi Y)(f)}{4f}\,{}^V(\varphi X)$,

(4) $S({}^VX, {}^VY) = -\frac{1}{4}g(X,Y)\,{}^H(grad\,f) - \frac{1}{4}g(X,\varphi Y)\,{}^H(\varphi grad\,f)$,

for all $X, Y \in \Im_0^1(M)$.

Proof. In Lemma 4.10, equation (1) Using (4.1) and (4.4), we have

$$\begin{aligned}
S({}^HX, {}^HY) &= \frac{1}{2}\Big((\nabla^f_{J^HY}J)^HX + J((\nabla^f_{^HY}J)^HX) - J((\nabla^f_{^HX}J)^HY)\Big)\\
&= \frac{1}{2}\Big(\nabla^f_{^H(\varphi Y)}{}^H(\varphi X) - J(\nabla^f_{^H(\varphi Y)}{}^HX) + J(\nabla^f_{^HY}{}^H(\varphi X)) - \nabla^f_{^HY}{}^HX - J(\nabla^f_{^HX}{}^H(\varphi Y)) + \nabla^f_{^HX}{}^HY\Big)\\
&= \frac{1}{2}\Big({}^H(\nabla_{\varphi Y}\varphi X) - \frac{1}{2}{}^V(R(\varphi Y,\varphi X)u) - {}^H(\varphi\nabla_{\varphi Y}X) - \frac{1}{2}{}^V(\varphi R(\varphi Y,X)u) + {}^H(\varphi\nabla_{\varphi Y}X) + \frac{1}{2}{}^V(\varphi R(Y,\varphi X)u)\\
&\quad - {}^H(\nabla_Y X) + \frac{1}{2}{}^V(R(Y,X)u) - {}^H(\varphi\nabla_X\varphi Y) - \frac{1}{2}{}^V(\varphi R(X,\varphi Y)u) + {}^H(\nabla_X Y) - \frac{1}{2}{}^V(\varphi R(X,Y)u)\Big)\\
&= -\frac{1}{2}{}^V(R(X,Y)u).
\end{aligned}$$

(2) By a similar calculation to equation (1), in Lemma 4.10, we get

$$\begin{aligned} S({}^{H}X, {}^{V}Y) &= \frac{1}{2}\Big((\nabla^f_{J^{V}Y} J)^{H}X + J((\nabla^f_{{}^{V}Y} J)^{H}X) - J((\nabla^f_{{}^{H}X} J)^{V}Y)\Big) \\ &= \frac{1}{2}\Big(- \nabla^f_{{}^{V}(\varphi Y)} {}^{H}(\varphi X) + J(\nabla^f_{{}^{V}(\varphi Y)} {}^{H}X) + J(\nabla^f_{{}^{V}Y} {}^{H}(\varphi X)) - \nabla^f_{{}^{V}Y} {}^{H}X + J(\nabla^f_{{}^{H}X} {}^{V}(\varphi Y)) + \nabla^f_{{}^{H}X} {}^{V}Y\Big) \\ &= \frac{1}{2}\Big(2P(\nabla^f_{{}^{V}Y} {}^{H}X) - 2\nabla^f_{{}^{V}Y} {}^{H}X + P(\nabla^f_{{}^{H}X} {}^{V}Y) + \nabla^f_{{}^{H}X} {}^{V}Y\Big) \\ &= \frac{f}{2} {}^{H}(R(u, Y)X) - \frac{X(f)}{2f} {}^{V}Y - \frac{(\varphi X)(f)}{2f} {}^{V}(\varphi Y). \end{aligned}$$

The other formulas are obtained by a similar calculations. □

Theorem 4.11. *Let (M^{2m}, φ, g) be a B-manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost product structure J defined by (4.1). Then the almost product connection $\overline{\overline{\nabla}}$ defined by (4.2) is as follows,*

(1) $\overline{\overline{\nabla}}_{{}^{H}X} {}^{H}Y = {}^{H}(\nabla_X Y)$,

(2) $\overline{\overline{\nabla}}_{{}^{H}X} {}^{V}Y = {}^{V}(\nabla_X Y) + \frac{X(f)}{f} {}^{V}Y + \frac{(\varphi X)(f)}{2f} {}^{V}(\varphi Y)$,

(3) $\overline{\overline{\nabla}}_{{}^{V}X} {}^{H}Y = \frac{3f}{2} {}^{H}(R(u, X)Y) - \frac{Y(f)}{4f} {}^{V}X - \frac{(\varphi Y)(f)}{4f} {}^{V}(\varphi X)$,

(4) $\overline{\overline{\nabla}}_{{}^{V}X} {}^{V}Y = -\frac{1}{4} g(X, Y) {}^{H}(grad\, f) + \frac{1}{4} g(X, \varphi Y) {}^{H}(\varphi grad\, f)$,

for all $X, Y \in \Im^1_0(M)$.

Proof. The proof of Theorem 4.11 follows directly from Theorem 3.2, Lemma 4.10 and formula (4.2). □

Lemma 4.12. *Let (M^{2m}, φ, g) be a B-manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost product structure J defined by (4.1) and $\overline{\overline{T}}$ denote the torsion tensor of $\overline{\overline{\nabla}}$, then we have:*

(1) $\overline{\overline{T}}({}^{H}X, {}^{H}Y) = {}^{V}(R(X, Y)u)$,

(2) $\overline{\overline{T}}({}^{H}X, {}^{V}Y) = -\frac{3f}{2} {}^{H}(R(u, Y)X) + \frac{3X(f)}{4f} {}^{V}Y + \frac{3(\varphi X)(f)}{4f} {}^{V}(\varphi Y)$,

(3) $\overline{\overline{T}}({}^{V}X, {}^{H}Y) = \frac{3f}{2} {}^{H}(R(u, X)Y) - \frac{3Y(f)}{2f} {}^{V}X - \frac{3(\varphi Y)(f)}{4f} {}^{V}(\varphi X)$,

(4) $\overline{\overline{T}}({}^{V}X, {}^{V}Y) = 0$,

for all $X, Y \in \Im^1_0(M)$.

Proof. The proof of Lemma 4.12 follows directly from Lemma 4.10 and formula

$$\begin{aligned} \overline{\overline{T}}(\widetilde{X}, \widetilde{Y}) &= \overline{\overline{\nabla}}_{\widetilde{X}} \widetilde{Y} - \overline{\overline{\nabla}}_{\widetilde{Y}} \widetilde{X} - [\widetilde{X}, \widetilde{Y}] \\ &= S(\widetilde{Y}, \widetilde{X}) - S(\widetilde{X}, \widetilde{Y}) \end{aligned}$$

for all $\widetilde{X}, \widetilde{Y} \in \Im^1_0(TM)$. □

From Lemma 4.12, we obtain the following theorem.

Theorem 4.13. *Let (M^{2m}, φ, g) be a B-manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost product structure J defined by (4.1), then $\overline{\overline{\nabla}}$ is symmetric if and only if M is flat and f is constant. In this case, the Levi-Civita connection ∇^f and the almost product connection $\overline{\overline{\nabla}}$ coincide with each other.*

Let (M^m, g) be a Riemannian manifold. We define a tensor field $L \in \Im^1_1(TM)$ by,

$$\begin{cases} L^{H}X &= \frac{1}{\sqrt{f}}({}^{V}X + \eta g(X, u){}^{V}u) \\ L^{V}X &= \sqrt{f}({}^{H}X + \mu g(X, u){}^{H}u) \end{cases} \tag{4.5}$$

for all $X \in \Im_0^1(M)$, where $\eta, \mu : \mathbb{R}^+ \to \mathbb{R}$ are smooth functions.

Note that

$$\begin{cases} L^H u &= \frac{1}{\sqrt{f}}(1+\eta r)^V u \\ L^V u &= \sqrt{f}(1+\mu r)^H u, \end{cases}$$

where $r = g(u,u)$.

Lemma 4.14. *Let (M^m, g) be a Riemannian manifold and (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric. Then, the endomorphism L defined by (4.5) is an almost paracomplex structure if and only if*

$$\mu = -\frac{\eta}{1+\eta r}.$$

Furthermore, we have

$$\begin{cases} L^H X &= \frac{1}{\sqrt{f}}({}^V X + \eta g(X,u)^V u) \\ L^V X &= \sqrt{f}({}^H X - \frac{\eta}{1+\eta r} g(X,u)^H u). \end{cases} \tag{4.6}$$

Proof. 1) Let $X \in \Im_0^1(M)$,

$$\begin{aligned} L^2({}^H X) &= L(L^H X) \\ &= \frac{1}{\sqrt{f}} L({}^V X + \eta g(X,u)^V u) \\ &= {}^H X + \mu g(X,u)^H u + \eta g(X,u)(1+\mu r)^H u \\ &= {}^H X + (\eta + \mu + \eta\mu r) g(X,u)^H u. \end{aligned} \tag{4.7}$$

$$\begin{aligned} L^2({}^V X) &= L(L^V X) \\ &= \sqrt{f} L({}^H X + \mu g(X,u)^H u) \\ &= {}^V X + \eta g(X,u)^V u + \mu g(X,u)(1+\eta r)^V u \\ &= {}^V X + (\eta + \mu + \eta\mu r) g(X,u)^V u. \end{aligned} \tag{4.8}$$

From (4.7) and (4.8), we get $L^2 = Id_{TM}$ if and only if $\eta + \mu + \eta\mu r = 0$ or equivalent to $\mu = -\frac{\eta}{1+\eta r}$.

2) Let $\{E_1, \dots, E_{2m}\}$ be local frame on M^m, then

$$TTM^+ = Span\,(A_1, \dots, A_{2m}),$$

$$TTM^- = Span\,(B_1, \dots, B_{2m}),$$

where

$$A_i = f^{\frac{1}{4}}({}^H E_i + \frac{1}{2}\mu g(E_i,u)^H u) + f^{\frac{-1}{4}}({}^V E_i + \frac{1}{2}\eta g(E_i,u)^V u),$$

$$B_i = f^{\frac{1}{4}}({}^H E_i + \frac{1}{2}\mu g(E_i,u)^H u) - f^{\frac{-1}{4}}({}^V E_i + \frac{1}{2}\eta g(E_i,u)^V u).$$

□

Theorem 4.15. *Let (M^m, g) be a Riemannian manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure L defined by (4.6). The triple (TM, L, G^f) is an almost B-manifold if and only if*

$$\eta = 0 \;\; or \;\; \eta = -\frac{2}{r},$$

where $r = g(u,u)$.

Proof. For purity condition, we put for all $X, Y \in \Im^1_0(M)$ and $k, h \in \{H, V\}$:

$$A({}^kX, {}^hY) = G^f(L^kX, {}^hY) - G^f({}^kX, L^hY).$$

$$\begin{aligned}(i)\ A({}^HX, {}^HY) &= G^f(L^HX, {}^HY) - G^f({}^HX, L^HY)\\ &= G^f(\frac{1}{\sqrt{f}}({}^VX + \eta g(X,u)^Vu), {}^HY) - G^f({}^HX, \frac{1}{\sqrt{f}}({}^VY + \eta g(Y,u)^Vu))\\ &= 0.\end{aligned}$$

$$\begin{aligned}(ii)\ A({}^VX, {}^VY) &= G^f(L^VX, {}^VY) - G^f({}^VX, L^VY)\\ &= G^f(\sqrt{f}({}^HX - \frac{\eta}{1+\eta r}g(X,u)^Hu), {}^VY) - G^f({}^VX, \sqrt{f}({}^HY - \frac{\eta}{1+\eta r}g(Y,u)^Hu))\\ &= 0.\end{aligned}$$

$$\begin{aligned}(iii)\ A({}^HX, {}^VY) &= G^f(L^HX, {}^VY) - G^f({}^HX, L^VY)\\ &= G^f(\frac{1}{\sqrt{f}}({}^VX + \eta g(X,u)^Vu), {}^VY) - G^f({}^HX, \sqrt{f}({}^HY - \frac{\eta}{1+\eta r}g(Y,u)^Hu))\\ &= \frac{1}{\sqrt{f}}G^f({}^VX, {}^VY) + \frac{1}{\sqrt{f}}\eta g(X,u)G^f({}^Vu, {}^VY) - \sqrt{f}G^f({}^HX, {}^HY) + \frac{\eta\sqrt{f}}{1+\eta r}g(Y,u)G^f({}^HX, {}^Hu)\\ &= \sqrt{f}g(X,Y) + \eta\sqrt{f}g(X,u)g(Y,u) - \sqrt{f}g(X,Y) + \frac{\eta\sqrt{f}}{1+\eta r}g(X,u)g(Y,u)\\ &= \frac{2+\eta r}{1+\eta r}\eta\sqrt{f}g(X,u)g(Y,u).\end{aligned}$$

$$\begin{aligned}(iv)\ A({}^VX, {}^HY) &= G^f(L^VX, {}^HY) - G^f({}^VX, L^HY)\\ &= G^f(\sqrt{f}({}^HX - \frac{\eta}{1+\eta r}g(X,u)^Hu), {}^HY) - G^f({}^VX, \frac{1}{\sqrt{f}}({}^VY + \eta g(Y,u)^Vu))\\ &= \sqrt{f}G^f({}^HX, {}^HY) - \frac{\eta\sqrt{f}}{1+\eta r}g(X,u)G^f({}^Hu, {}^HY) - \frac{1}{\sqrt{f}}G^f({}^VX, {}^VY) - \frac{1}{\sqrt{f}}\eta g(Y,u)G^f({}^VX, {}^Vu)\\ &= \sqrt{f}g(X,Y) - \frac{\eta\sqrt{f}}{1+\eta r}g(X,u)g(Y,u) - \sqrt{f}g(X,Y) - \eta\sqrt{f}g(X,u)g(Y,u)\\ &= -\frac{2+\eta r}{1+\eta r}\eta\sqrt{f}g(X,u)g(Y,u).\end{aligned}$$

Then, $A({}^HX, {}^VY) = 0$ equivalent to $\eta = 0$ or $\eta = -\frac{2}{r}$. □

Hence, we have two almost paracomplex structures

$$\begin{cases} L^HX &= \frac{1}{\sqrt{f}}({}^VX - \frac{2}{r}g(X,u)^Vu),\\ L^VX &= \sqrt{f}({}^HX - \frac{2}{r}g(X,u)^Hu)\end{cases} \tag{4.9}$$

or

$$\begin{cases} L^HX &= \frac{1}{\sqrt{f}}{}^VX,\\ L^VX &= \sqrt{f}{}^HX.\end{cases} \tag{4.10}$$

We shall study integrability of L. As we know that the integrability of L is equivalent to the vanishing of the Nijenhuis tensor. The Nijenhuis tensor of L is given by

$$N_L(\widetilde{X},\widetilde{Y}) = [L\widetilde{X},L\widetilde{Y}] - L[L\widetilde{X},\widetilde{Y}] - L[\widetilde{X},L\widetilde{Y}] + [\widetilde{X},\widetilde{Y}],$$

where $\widetilde{X},\widetilde{Y} \in \Im_0^1(TM)$.

Lemma 4.16. *Let (M^m, g) be a Riemannian manifold and (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric. The almost paracomplex structure L defined by (4.6) is integrable if and only if $N_L({}^HX, {}^HY) = 0$, for all $X, Y \in \Im_0^1(M)$.*

Proof. We put $L^VX = {}^HZ$ and $L^VY = {}^HW$, then we have

$$\begin{aligned} N_L({}^VX, {}^VY) &= [L^VX, L^VY] - L[L^VX, {}^VY] - L[{}^VX, L^VY] + [{}^VX, {}^VY] \\ &= [{}^HZ, {}^HW] - L[{}^HZ, L^HW] - L[L^HZ, {}^HW] + [L^HZ, L^HW] \\ &= N_L({}^HZ, {}^HW), \end{aligned}$$

$$\begin{aligned} N_L({}^VX, {}^HW) &= [L^VX, L^HW] - L[L^VX, {}^HW] - L[{}^VX, L^HW] + [{}^VX, {}^HW] \\ &= [{}^HZ, L^HW] - L[{}^HZ, {}^HW] - L[L^HZ, L^HW] + [L^HZ, {}^HW] \\ &= -L[L^HZ, L^HW] + [L^HZ, {}^HW] + [{}^HZ, L^HW] - L[{}^HZ, {}^HW] \\ &= -L(N_L({}^HZ, {}^HW)). \end{aligned}$$

□

Lemma 4.17. *Let (M^m, g) be a Riemannian manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure L defined by (4.6), then*

$$N_L({}^HX, {}^HY) = \frac{1}{2f}(X(f)^HY - Y(f)^HX) + \frac{\eta}{f}(g(Y,u)^VX - g(X,u)^VY) - {}^V(R(X,Y)u),$$

for all $X, Y \in \Im_0^1(M)$.

Proof. We have

$$N_L({}^HX, {}^HY) = [L^HX, L^HY] - L[L^HX, {}^HY] - L[{}^HX, L^HY] + [{}^HX, {}^HY].$$

By the direct computations and using Lemma 2.1 and Lemma 2.2, we get

$$\begin{aligned} [L^HX, L^HY] &= \frac{\eta}{f}(g(Y,u)^VX - g(X,u)^VY), \\ L[L^HX, {}^HY] &= -{}^H(\nabla_Y X) + \frac{Y(f)}{2f}{}^HX, \\ L[{}^HX, L^HY] &= {}^H(\nabla_X Y) - \frac{X(f)}{2f}{}^HY, \\ [{}^HX, {}^HY] &= {}^H[X,Y] - {}^V(R(X,Y)u). \end{aligned}$$

□

Hence, we have the following theorems.

Theorem 4.18. *Let (M^m, g) be a Riemannian manifold and (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure L defined by (4.9). The triple (TM, L, G^f) is a B-manifold if and only if f is constant and*

$$R(X,Y)u = \frac{-2}{rf}(g(Y,u)X - g(X,u)Y)$$

for all $X, Y \in \Im_0^1(M)$.

Theorem 4.19. *Let (M, g) be a Riemannian manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure L defined by (4.10). The triple (TM, L, G^f) is a B-manifold if and only if M is flat and f is constant.*

Let (M^m, g) be a Riemannian manifold. We define a tensor field $P \in \Im^1_1(TM)$ by,

$$\begin{cases} P^HX &= {}^HX + \eta g(X,u)^Hu \\ P^VX &= -{}^VX + \mu g(X,u)^Vu \end{cases} \tag{4.11}$$

for all $X \in \Im^1_0(M)$, where $\eta, \mu : \mathbb{R}^+ \to \mathbb{R}$ are smooth functions.

If $\eta = \mu = 0$, then P is the almost paracomplex structure defined by (4.1), where $\varphi = Id_M$.
In the following, we consider $\eta \neq 0$ and $\mu \neq 0$.
Note that ,

$$\begin{cases} P^Hu &= (1+\eta r)^Hu \\ P^Vu &= (-1+\mu r)^Vu \end{cases}$$

such that $r = g(u, u)$.

Lemma 4.20. *Let (M^m, g) be a Riemannian manifold and (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric. Then the endomorphism P defined by (4.11) is an almost paracomplex structure if and only if $\eta = -\frac{2}{r}$ and $\mu = \frac{2}{r}$, i.e.,*

$$\begin{cases} P^HX &= {}^HX - \frac{2}{r}g(X,u)^Hu \\ P^VX &= -{}^VX + \frac{2}{r}g(X,u)^Vu \end{cases} \tag{4.12}$$

for all $X \in \Im^1_0(M)$ and $r = g(u, u)$.

Proof. 1) Let $X \in \Im^1_0(M)$,

$$\begin{aligned} P^2({}^HX) &= P(P({}^HX)) \\ &= P({}^HX + \eta g(X,u)^Hu) \\ &= {}^HX + \eta g(X,u)^Hu + \eta g(X,u)(1+\eta r)^Hu \\ &= {}^HX + \eta(2+\eta r)g(X,u)^Hu, \end{aligned} \tag{4.13}$$

$$\begin{aligned} P^2({}^VX) &= P(P({}^VX)) \\ &= P(-{}^VX + \mu g(X,u)^Vu) \\ &= {}^VX - \mu g(X,u)^Vu + \mu g(X,u)(-1+\mu r)^Vu \\ &= {}^VX + \mu(-2+\mu r)g(X,u)^Vu. \end{aligned} \tag{4.14}$$

From (4.13) and (4.14), then $P^2 = Id_{TM}$ equivalent to $\eta = -\frac{2}{r}$ and $\mu = \frac{2}{r}$.
2) Let $\{E_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on M. Then,

$$T_{(x,p)}TM^+ = Span\,(V_1, \ldots, V_m),$$

$$T_{(x,u)}TM^- = Span\,(W_1, \ldots, W_m),$$

where $V_i = -{}^HE_i + \frac{1}{r}g(E_i,u)^Hu$, $W_i = -{}^VE_i + \frac{1}{r}g(E_i,u)^Vu$. □

Theorem 4.21. *Let (M^m, g) be a Riemannian manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure P defined by (4.12). The triple (TM, P, G^f) is an almost B-manifold.*

Proof. For purity condition, we put for all $X, Y \in \Im^1_0(M)$ and $k, h \in \{H, V\}$:

$$A(X^k, {}^HY) = G^f(PX^k, {}^HY) - G^f(X^k, P^HY).$$

$$\begin{aligned}
(i)\ A({}^HX, {}^HY) &= G^f(P^HX, {}^HY) - G^f({}^HX, P^HY) \\
&= G^f({}^HX - \frac{2}{r}g(X,u)^Hu, {}^HY) - G^f({}^HX, {}^HY - \frac{2}{r}g(Y,u)^Hu) \\
&= G^f({}^HX, {}^HY) - \frac{2}{r}g(X,u)g(Y,u) - G^f({}^HX, {}^HY) + \frac{2}{r}g(Y,u)g(X,u) \\
&= 0, \\
(ii)\ A({}^VX, {}^VY) &= G^f(P^VX, {}^VY) - G^f({}^VX, P^VY) \\
&= G^f(-{}^VX + \frac{2}{r}g(X,u)^Vu, {}^VY) - G^f({}^VX, -{}^VY + \frac{2}{r}g(Y,u)^Vu) \\
&= -G^f({}^VX, {}^VY) + \frac{2}{r}g(X,u)f\lambda g(Y,u) + G^f({}^VX, {}^VY) - \frac{2}{r}g(Y,\varphi u)f\lambda g(X,u) \\
&= 0, \\
(iii)\ A({}^HX, {}^VY) &= G^f(P^HX, {}^VY) - G^f({}^HX, P^VY) \\
&= G^f({}^HX - \frac{2}{r}g(X,u)^Hu, {}^VY) - G^f({}^HX, -{}^VY + \frac{2}{r}g(Y,u)^Vu) \\
&= 0, \\
(iv)\ A({}^VX, {}^HY) &= G^f(P^VX, {}^HY) - G^f({}^VX, P^HY) \\
&= G^f(-{}^VX + \frac{2}{r}g(X,u)^Vu, {}^HY) - G^f({}^VX, {}^HY + \frac{2}{r}g(Y,u)^Hu) \\
&= 0.
\end{aligned}$$

□

Lemma 4.22. *Let (M^m, g) be a Riemannian manifold, (TM, G^f) its tangent bundle equipped with the vertical rescaled metric, ∇^f denote the corresponding Levi-Civita connection of G^f and ${}^Vu$ (resp. ${}^Hu$) be the vertical distribution (resp. horizontal distribution) on TM. Then,*

1. $\nabla^f_{{}^HX}{}^Hu = \frac{1}{2}{}^V(R(u,X)u)$,
2. $\nabla^f_{{}^HX}{}^Vu = \frac{X(f)}{2f}\,{}^Vu$,
3. $\nabla^f_{{}^VX}{}^Hu = {}^HX + \frac{u(f)}{2f}\,{}^VX + \frac{f}{2}{}^H(R(u,X)u)$,
4. $\nabla^f_{{}^VX}{}^Vu = {}^VX - \frac{1}{2}g(X,u)^H(grad\, f)$,

for all vector fields $X \in \Im^1_0(M)$.

Proof. The proof of Lemma 4.22 follows directly from (2.1) and Theorem 3.2. □

Proposition 4.23. *Let (M^m, g) be a Riemannian manifold, (TM, G^f) its tangent bundle equipped with the vertical rescaled metric, the almost paracomplex structure P defined by (4.12) and ∇^f denote the corresponding Levi-Civita* *connection of G^f, then:*

1. $(\nabla^f_{{}^HX}P)^HY = -{}^V(R(X,Y)u) - \frac{1}{r}g(Y,u)^V(R(u,X)u)$,
2. $(\nabla^f_{{}^HX}P)^VY = -f^H(R(u,Y)X) + \frac{f}{r}g(R(u,Y)X,u)^Hu$,
3. $(\nabla^f_{{}^VX}P)^HY = -\frac{2}{r}g(Y,u)^HX + (\frac{4}{r^2}g(X,u)g(Y,u) - \frac{2}{r}g(X,Y) + \frac{f}{r}g(R(u,X)Y,u))^Hu$
$- \frac{f}{r}g(Y,u)^H(R(u,X)u) + (\frac{Y(f)}{f} - \frac{u(f)}{rf}g(Y,u))^VX - \frac{Y(f)}{rf}g(X,u)^Vu$,
4. $(\nabla^f_{{}^VX}P)^VY = (g(X,Y) - \frac{1}{r}g(X,u)g(Y,u))(grad\, f)^H - \frac{u(f)}{r}g(X,Y)\,{}^Hu$
$+ \frac{2}{r}g(Y,u)^VX + (\frac{2}{r}g(X,Y) - \frac{4}{r^2}g(X,u)g(Y,u))^Vu$,

for all vector fields $X \in \Im^1_0(M)$.

Proof. The proof of Proposition 4.23 follows directly from the Theorem 4.2 and the formula

$$\nabla^f_{\widetilde{X}}P\widetilde{Y} = \nabla^f_{\widetilde{X}}(P\widetilde{Y}) - P\nabla^f_{\widetilde{X}}\widetilde{Y},$$

where $\widetilde{X}, \widetilde{Y} \in \Im^1_0(TM)$. Hence, we deduce: □

Theorem 4.24. *Let (M^m, g) be a Riemannian manifold, (TM, G^f) be its tangent bundle equipped with the vertical rescaled metric and the almost paracomplex structure P defined by (4.12). Then, the triple (TM, P, G^f) is never an almost anti-paraHermitian manifold.*

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed the published version of the manuscript.

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