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# On Some New Generalized Gaussian Oresme Numbers

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**Abstract**

In this study, we defined and examined a new generalization to a special sequence of rational numbers. We gave the similarities and relationships of these newly defined Gaussian Oresme numbers with the generalized Fibonacci numbers and Lucas numbers existing in the literature. Moreover, we have obtained some important identities provided by these numbers we have discussed. We also obtained the relationship of these numbers with Gaussian Fibonacci numbers.

**Keywords:** fibonacci sequences, gaussian numbers, recurrence relation, oresme numbers

# Bazı Yeni Genelleştirilmiş Gaussian Oresme Sayıları

**Öz**

Bu çalışmada rasyonel sayıların özel bir dizisine yönelik yeni bir genelleme tanımladık ve inceledik. Yeni tanımlanan bu Gaussian Oresme sayılarının literatürde var olan genelleştirilmiş Fibonacci sayıları ve Lucas sayıları ile benzerliklerini ve ilişkilerini verdik. Üstelik ele aldığımız bu sayıların sağladığı bazı önemli kimlikleri de elde ettik. Bu sayıların Gaussian Fibonacci sayılarıyla ilişkisini de elde ettik.

**Anahtar Kelimeler:** fibonacci dizileri, gaussian sayıları, tekrarlama bağıntısı, oresme sayıları

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## Introduction

There are many number sequences studied in the literature. Some of these sequences are Fibonacci, Lucas, Jacobstal, Pell, Jacaobstal Lucas, Pell Lucas. The recurrence relation of Fibonacci sequences is as follows (Koshy, 2001).

$$F_{n+1} = F_n + F_{n-1},\ F_0 = 0, F_1 = 1. \tag{1}$$

Horadam was the first to examine complex numbers whose coefficients were selected from this famous number sequence (Horadam, 1963).

$$CF_n = F_n + iF_{n+1}. \tag{2}$$

In his work in (Berzsenyi, 1977) Berzsenyi investigated a natural extension of Fibonacci numbers to the complex plane and described the frequently studied numbers known as Gaussian numbers. Numerous authors, including Berzsenyi, have studied on Gaussian numbers and their applications. Gaussian numbers were defined by some authors in the literature. Some of these references are as follows: (Halici et al., 2023; Halici et al., 2016; Pethe et al., 1986; Yilmaz et al., 2023). Horadam and Pethe (Pethe et al., 1986) defined generalized Gaussian Fibonacci numbers. The authors used the recurrence relationship provided by these numbers to obtain some identities containing the products of combinations of Fibonacci, Pell and Chebyshev polynomials. Also, in (Cagman, 2021), (Cagman, 2023) and (Halıcı et al., 2020) the authors worked on different integer sequences. In (Ozkan et al., 2020), Ozkan and Tastan defined Gaussian Fibonacci polynomials and also introduced the matrices of Gaussian Fibonacci and Lucas polynomials. Yilmaz and Ertas examined the Gaussian Oresme numbers and used these numbers for quaternion sequence (Yilmaz et al., 2023) defined Gaussian Oresme sequence using Oresme number sequence and gave new results (Halici et al., 2023).

$G(n,m) = n + im$, where $n > m$ and $n, m \in \mathbb{Z}$, denotes Gaussian integers. For fixed real numbers $p_1, p_2$, $q_1$and $q_2$ initial conditions are

$$G(0,0) = 0, G(1,0) = 1, G(0,1) = i, G(1,1) = p_2 + ip_1 \tag{3}$$

The following equations are sufficient to obtain a single value for each Gaussian integer.

$$G(n+2,m) = p_1 G(n+1,m) - q_1 G(n,m), \tag{4}$$

$$G(n,m+2) = p_2 G(n,m+1) - q_2 G(n,m). \tag{5}$$

In 1963, Horadam introduced these numbers as the generalized complex Fibonacci sequence (Horadam, 1963).

$$GF_n = F_n + iF_{n-1},\ \ GF_0 = 0, GF_1 = 1,\ n \geq 1. \tag{6}$$

Some elements of equation (6) have in the table below.

**Table 1.** Gaussian Fibonacci Numbers

| $n$ | $GF_n$ |
|---|---|
| 0 | 1 |
| 1 | $1 + i$ |
| 2 | $2 + i$ |
| 3 | $3 + 2i$ |
| 4 | $5 + 3i$ |
| 5 | $8 + 5\mathrm{i}$ |
| … | … |

Also, Gaussian numbers gave by the following equations in relation to generalized Fibonacci $U_n$ and Lucas numbers $V_n$ (Pethe et al., 1986). Where $G(n,0) = U_n$, $G(0,m) = iV_m$ these are

$$U_{n+2} = p_1 U_{n+1} - q_1 U_n;\ U_0 = 0, U_1 = 1, \tag{7}$$

$$V_{n+2} = p_2 V_{n+1} - q_2 V_n;\ V_0 = 0, V_1 = 1. \tag{8}$$

In (Pethe et al., 1986), the following equation gave for the numbers $G(n,m)$

$$G(n,m) = U_n V_{m+1} + iU_{n+1} V_m. \tag{9}$$

Many authors studied generalized Fibonacci sequences and Lucas sequences. In (Akyuz et al., 2013), Halici and Akyuz discussed and examined the relationships between these sequences, the identities formed by the sequences terms, and also the polynomials of these sequences. We use the following equations to define new concepts.

$$V_{m+1} = U_{m+2} - qU_m,\ V_m = U_{m+1} - qU_{m-1}. \tag{10}$$

From equations (9) and (10), we get

$$G(n,m) = U_n(U_{m+2} - qU_m) + iU_{n+1}(U_{m+1} - qU_{m-1}). \tag{11}$$

Another well-known sequence used to make generalizations in integer sequence studies is the Horadam sequence $w_n = w_n(w_0, w_1; p, q)$,

$$w_{n+2} = pw_{n+1} - qw_n\ ,\ \ n \geq 0. \tag{12}$$

It is possible to obtain different sequences by changing the initial conditions and coefficients of the Horadam sequence $w_n$. The Oresme sequence, which has rational coefficients, is one of these newly derived sequences. This sequence was described by N. Oresme and also studied by Horadam (Horadam, 1974).

For $O_0 = 0$ and $O_1 = \frac{1}{2}$, Oresme sequence is defined as

$$O_{n+2} = O_{n+1} - \frac{1}{4} O_n. \tag{13}$$

The closed formula for this sequence is

$$O_n = \frac{n}{2^n}. \tag{14}$$

The sequence $\{O_n\}$ is obtained by taking $p = 1$, $q = \frac{1}{4}$ in equation (12) and

$$O_n = \frac{1}{2} U_{n-1} \tag{15}$$

is obtained.

For $k \geq 2$, a generalization was made by Cook (Cook, 2004) is in the form

$$O_{n+2}^{(k)} = O_{n+1}^{(k)} - \frac{1}{k^2} O_n^{(k)}, O_0^{(k)} = 0, O_1^{(k)} = \frac{1}{k}. \tag{16}$$

In the table below we gave the first four terms of Oresme numbers and generalized Oresme numbers. Specifically, if we take $k = 2$ in $k$- Oresme numbers, we get Oresme numbers.

We gave the numbers $O_n$ and numbers $O_n^{(k)}$ in the next table.

**Table 2.** The Numbers $O_n$ and Numbers $O_n^{(k)}$

| $n$ | $O_n$ | $O_n^{(k)}$ |
|---|---|---|
| 0 | 0 | 0 |
| 1 | $\frac{1}{2}$ | $\frac{1}{k}$ |
| 2 | $\frac{1}{2}$ | $\frac{1}{k}$ |
| 3 | $\frac{3}{8}$ | $\frac{k^2-1}{k^3}$ |

In this work, the authors define new generalized Gaussian Oresme numbers using Oresme numbers. Halici et al. gave the relation of these numbers with Gaussian Fibonacci numbers.

## Generalized Gaussian Oresme Numbers

In this section we define generalized Gaussian Oresme numbers. We gave the Binet formula for these numbers and gave some basic identities and the similarities and relationships between Gaussian Oresme numbers and Gaussian Fibonacci numbers.

**Definition 1.** For $n, m \geq 0$, using by the equation (11) generalized Gaussian Oresme numbers

are defined as

$$GO(n,m) = (GF_2 + i)O_{n+1}(2O_{m+1} - O_m) + iO_n\left(\frac{1}{2}O_m - O_{m+1}\right). \tag{17}$$

It is shown as $GO(n,m) = GO_{n,m}$.

Since the Binet formula was discovered by the French mathematician Jacques Philipe Marie Binet in 1843, it is known as the Binet formula in the literature. Binet formula allows us to find the $nth$ term.

In the next theorem we give a closed formula for the numbers $GO_{n,m}$.

**Theorem 2.** For $n, m \geq 0$, we get

$$GO_{n,m} = \frac{O_{n+m+1}}{n+m+1}[1 + GF_2(n+1)]. \tag{18}$$

**Proof.** In the definition of generalized Gaussian Oresme numbers, we can use the closed formula

gave for Oresme numbers.

$$GO_{n,m} = (1+2i)\frac{n+1}{2^{n+1}}\left(2\frac{m+1}{2^{m+1}} - \frac{m}{2^m}\right) + i\frac{n}{2^n}\left(\frac{1}{2}\frac{m}{2^m} - \frac{m+1}{2^{m+1}}\right),$$

$$GO_{n,m} = (1+2i)\frac{n+1}{2^{n+m+1}} - i\frac{n}{2^{n+m+1}},$$

$$GO_{n,m} = \frac{1}{2^{n+m+1}}[n(1+i) + (1+2i)],$$

$$GO_{n,m} = \frac{O_{n+m+1}}{n+m+1}[1 + GF_2(n+1)].$$

Thus, the proof is completed.

Now, we give an important equation, which is one of the important identities and represents

the matrix form of the recursive relation in the following Theorem.

**Theorem 3.** For $n, m \geq 1$, we have

$GO_{n+1,m+1}\,GO_{n-1,m-1} - \,GO_{n,m}{}^2 = -\frac{O_{2n+2m+1}}{2(n+m)+1}(GF_2 - GF_1).$ (19)

**Proof.** If we substitute $n+1, m+1$ and $n-1, m-1$ in equation (18), we obtain the following equalities.

$$GO_{n,+1,m+1} = \frac{1}{2^{n+m+3}}[i(n+3)+(n+2)],$$

$$GO_{n,m} = \frac{1}{2^{n+m+1}}[i(n+2)+(n+1)]$$

and

$$GO_{n-1,m-1} = \frac{1}{2^{n+m-1}}[i(n+1)+n].$$

We can write the left side of the equation.

$LHS = \frac{1}{2^{2n+2m+2}}\big[[i(n+3)+(n+2)][i(n+1)+n]-[i(n+2)+(n+1)]^2\big],$

$= \frac{1}{2^{2n+2m+2}}[-(n+3)(n+1)+in(n+3)+i(n+2)(n+1)+n(n+2)+(n+2)^2 \;-(n+1)^2 - 2i(n+1)(n+2)],$

$LHS = -\frac{O_{2n+2m+1}}{2n+2m+1}i.$

Thus, the proof is completed.

Now, using the closed formula in (18), we also give the following theorem.

**Theorem 4.** For $n-r \geq 0$, we get

$GO_{n+r,m+r}\,GO_{n-r,m-r} - \,GO_{n,m}{}^2 = -\frac{O_{2n+2m+1}}{2n+2m+1}r^2(GF_2 - GF_1).$ (20)

**Proof.** Let us substitute $GO_{n+r,m+r}$, $GO_{n,m}$ and $GO_{n-r,m-r}$ on the left side of the equation.

$LHS = \frac{1}{2^{2n+2m+2}}\big[[i(n+r+2)+(n+r+1)][i(n-r+2)+(n-r+1)]-[i(n+2)+\,(n+1)]^2\big].$

If the necessary arithmetic operations are performed on this equation, then the following equation is obtained.

$$GO_{n+r,m+r}\,GO_{n-r,m-r} - \,GO_{n,m}{}^2 = -\frac{1}{2^{n+m+2}}2ir^2,$$

$$GO_{n+r,m+r}\,GO_{n-r,m-r} - \,GO_{n,m}{}^2 = -\frac{O_{2n+2m+1}}{2n+2m+1}r^2(GF_2 - GF_1).$$

Thus, the proof is completed.

Note that, when r = 1 in the last equation, the Cassini identity for the numbers $GO_{n,m}$ is obtained.

Now, let us give d'Ocagne's identity for this new sequence.

**Theorem 5.** For $a, b, n, m \in \mathbb{Z}^+$, the following equation is satisfied.

$GO_{n+1,m+1}\,GO_{a,b} - \,GO_{n,m}\,GO_{a+1,b+1} = \frac{O_{(n+m+a+b)}O_3}{3(n+m+a+b)}(GF_2 - GF_1)(a-n).$ (21)

**Proof.** Let us calculate the left side of the equation (21). From the Binet formula, we write $LHS = \frac{1}{2^{n+m+a+b+4}}\big[[i(n+3)+(n+2)][i(a+2)+(a+1)]-[i(n+2)+(n+1)][i(a+3)+(a+2)]\big].$

If we rearrange this equation, then the following equation is obtained.

$GO_{n+1,m+1}\,GO_{a,b} - \,GO_{n,m}\,GO_{a+1,b+1} = \frac{1}{2^{n+m+a+b+4}}(2ai - 2in),$

$GO_{n+1,m+1}\,GO_{a,b} - \,GO_{n,m}\,GO_{a+1,b+1} = \frac{O_{(n+m+a+b)}O_3}{3(n+m+a+b)}(GF_2 - GF_1)(a-n).$

Thus, the desired result is obtained.

In the following theorem, we give the Vajda identity.

**Theorem 6.** For $a, b, n, m \in \mathbb{Z}^+$, the following equation is true.

$$GO_{n+a,m+a}\, GO_{n+b,m+b} - GO_{n,m}\, GO_{n+a+b,m+a+b} = \frac{O_{2(n+m+a+b)}O_3}{3(n+m+a+b)} b(2ai + GF_1) \quad (22)$$

**Proof.** From the closed formula, we can write

$$GO_{n+a,m+a} = \frac{1}{2^{n+m+2a+1}}[i(n+a+2) + (n+a+1)],$$

$$GO_{n+b,m+b} = \frac{1}{2^{n+m+2b+1}}[i(n+b+2) + (n+b+1)],$$

$$GO_{n+a+b,m+a+b} = \frac{1}{2^{n+m+2a+2b+1}}[i(n+a+b+2) + (n+a+b+1)].$$

If these values are substituted on the left side of equation (22) and used the equation (18), then the following equation is obtained.

$$LHS = \frac{1}{2^{2(n+m+a+b+1)}}\begin{bmatrix} -(n+a+2)(n+b+2) + i(n+a+2)(n+b+1) + i(n+a+1)(n+b+1) + \\ (n+2)(n+a+b+2) - i(n+2)(n+a+b+1) \\ -i(n+1)(n+a++b+2) - (n+1)(n+a+b+1) \end{bmatrix},$$

$$GO_{n+a,m+a}\, GO_{n+b,m+b} - GO_{n,m}\, GO_{n+a+b,m+a+b} = \frac{O_{2(n+m+a+b)}}{8(n+m+a+b)} b(1+2ai),$$

$$GO_{n+a,m+a}\, GO_{n+b,m+b} - GO_{n,m}\, GO_{n+a+b,m+a+b} = \frac{O_3 O_{2(n+m+a+b)}}{3(n+m+a+b)} b(2ai + GF_1).$$

Thus, the proof is completed.

We gave some important identities above. We introduce a new and useful identity in the next theorem.

**Theorem 7.** For $n, m, z \in \mathbb{Z}^+$the following equation is satisfied.

$$GO_{n+z,m}\, GO_{n,m+z+1} - \frac{1}{4} GO_{n+z-1,m}\, GO_{n,m+z} = \frac{O_4 O_{2(n+m+z)}}{4(n+m+z)} i(n'+2) - n'z', \quad (23)$$

where $n' = n + 1$ and $z' = z + 1$.

**Proof.** Let us calculate the left side of equation (23) using the Binet formula. Then, we write

$$LHS = \frac{1}{2^{2(n+m+z)+3}}\begin{bmatrix} -(n+z+2)(n+2) + i(n+z+2)(n+1) + i(n+2)(n+z+1) + \\ (n+1)(n+z+1) + (n+2)(n+z+1) \\ -i(n+2)(n+z) - i(n+z+1)(n+1) - (n+z)(n+1) \end{bmatrix},$$

$$GO_{n+z,m}\, GO_{n,m+z+1} - \frac{1}{4} GO_{n+z-1,m}\, GO_{n,m+z} = \frac{1}{2^{2(n+m+z)+3}}[i(n+3) - (n+1)(z+1)].$$

Substituting the values $n'$ and $z'$, we get

$$GO_{n+z,m}\, GO_{n,m+z+1} - \frac{1}{4} GO_{n+z-1,m}\, GO_{n,m+z} = \frac{O_4 O_{2(n+m+z)}}{4(n+m+z)} i(n'+2) - n'z'.$$

Thus, the proof is completed.

It is useful to give arithmetic operations for some consecutive terms.

$$GO_{n,m} + GO_{n+1,m+1} = \frac{O_{n+m}}{8(n+m)}[i(5n+11) + (5n+6)]. \quad (24)$$

$$GO_{(n,m)} - GO_{n+1,m+1} = \frac{O_{n+m}}{8(n+m)}[i(3n+5) + (3n+2)]. \quad (25)$$

$$GO_{n,m}\, GO_{n+1,m+1} = \frac{O_5 O_{2n+2m}}{5(n+m)}[i(2n^2 + 8n + 7) - 2(n+2)]. \quad (26)$$

$$\frac{GO_{n+1,m+1}}{GO_{n,m}} = \frac{[i(n+3) + (n+2)]}{4[i(n+2) + (n+1)]}. \quad (27)$$

In the next theorem, we give the generating function of new sequence.

**Theorem 8.** For $n \in \mathbb{Z}^{+}$, the following equation is true.

$$\sum_{n=0}^{\infty} GO_{n,m}x^{n} = \frac{2GF_3 - xGF_2}{4-x}; \quad n = m. \tag{28}$$

**Proof.** The following equations can be written using the definition of generating function.

$$g(x) = GO(0,0) + GO(1,1)x + GO(2,2)x^2 + \cdots$$

$$\left[-\frac{g(x)+(i+1)}{4}\right]x = -\frac{1}{8}(3i+2)x - \frac{1}{32}(4i+3)x^2 - \frac{1}{128}(5i+4)x^3 + \cdots$$

$$g(x) - \left[\frac{g(x)+(i+1)}{4}\right]x = \frac{1}{2}(2i+1),$$

$$g(x)\left(1-\frac{1}{4}x\right) = \frac{1}{2}(2i+1) + \frac{1}{4}(i+1)x,$$

$$g(x) = \frac{2GF_3 + xGF_2}{4-x}.$$

Thus, the desired result is obtained.

## Conclusion

In this study, a new generalization of Gaussian Oresme numbers is obtained. Some identities provided by these new numbers, which have an important place in the literature, are shown. In addition, the relation of these newly defined numbers with the Gaussian Fibonacci sequence is examined.

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### Author Contribution

*Elifcan Sayın* and *Serpil Halıcı* wrote the project and conducted and directed the studies. The authors have read and approved the article.

### Ethics Statement

There are no ethical issues with the publication of this article.

### Conflict of Interest

The authors state that there is no conflict of interest.

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