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A Note on Uniformly Convergence for Positive Linear Operators Involving Euler Type Polynomials

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Dedicated to my dear friend Ali Korcan Tan (1986-2023)

Keywords
Positive linear operators, Generating functions, Euler type polynomials, Moment Functions, Korovkin theorem

Abstract – Positive linear operators play a significant role in many domains, particularly numerical and mathematical analysis. Specifically, they are commonly found in a wide variety of methods to resolve optimization and differential equation issues. Basic properties of positive linear operators are linearity, positivity, positive linear and being restrictive. There are various ways to examine the significance of positive linear operators in Approximation Theory. The Convergence Analysis is the most significant of these. In many situations involving numerical analysis and convergence analysis, positive linear operators are essential. Positive linear operators must be able to converge in iterations towards a specific goal, especially in various approximation techniques or iterative solution algorithms. This can be used to solve optimization issues more effectively or to increase the precision of numerical answers. In approximation theory, generating functions are essential. They are specifically used to build algorithms that facilitate the proper approximation to a goal and to examine the approximation in question. The speed at which an approximation converges to a target can also be ascertained via generating functions. An essential tool for evaluating and enhancing the rate of convergence of iterative algorithms is offered by these functions. The aim of this study is to construct a generalized Kantorovich type Szász operators including the generating functions of Euler polynomials with order (-1). Moreover, we derive the moment and central moment functions for these operators. Finally, we show uniformly convergence of operators by using Korovkin theorem.

1. Introduction

An essential component of Approximation Theory is positive linear operators. Finding an approximate solution to a problem is the goal of Approximation Theory as opposed to an exact one. This method makes extensive use of positive linear operators, which are crucial for resolving a wide range of mathematical issues. Positive linear operators are useful tools for addressing mathematical problems and cover a wide range of topics in Approximation Theory. These operators are employed in many engineering and other applied fields, including optimization problems and iterative approaches, because of their properties that guarantee the efficiency and correctness of approximation solutions.

A useful tool in number theory, combinatorics, and other branches of mathematics are generating functions. They encode information about a series of numbers or objects and are simply formal power series. By generating functions, we can simplify issues requiring sequences into problems involving functions, which facilitates their analysis and solution using algebraic and calculus methods (Kilar and Simsek, 2021), (Simsek, 2018) and (Srivastava et. al, 2020).

The generating function of a sequence a_n is defined to be as follows:

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$$A_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n,$$

where the a_n are coefficients of generating function for $i = 1,2, ...$ (McBride, 2012).

Concepts of generating functions and positive linear operators are utilized in several branches of mathematical analysis and have some commonalities. Mathematical issues are solved or approximated using both, albeit they are not utilized interchangeably. Nonetheless, generating functions can be used to express positive linear operators in specific situations. Recently, positive linear operators involving generating functions of some special polynomials have been intensively studied by many researchers.

Atakut and Büyükyazıcı constructed a generalization of Kantorovich type operators involving Brenke type polynomials as:

$$L_n^{\alpha_n,\beta_n}(f;x) = \frac{\beta_n}{A(1)B(\alpha_n x)} \sum_{k=0}^{\infty} p_k(\alpha_n x) \int_{k/\beta_n}^{(k+1)/\beta_n} f(t)dt,$$

for α_n, β_n and $p_k(x)$ see reference (Atakut and Büyükyazıcı, 2016).

İçöz *et al.* introduced a new type Szász operators including Appell polynomials at the following equation:

$$M_n(f;x) = \frac{1}{A\big(g(1)\big)B(nxg(1))} \sum_{k=0}^{\infty} p_k(\alpha_n x)\, f\left(\frac{k}{n}\right),$$

for $A, B, g(x)$ and $p_k(x)$ see reference (İçöz et al., 2016).

Menekşe Yılmaz established generalized Kantorovich type operators involving generating functions of Adjoint Bernoulli polynomials as follows:

$$\tilde{A}_n(f;x) = n\frac{e^{-nx}}{e-1} \sum_{k=0}^{n} \frac{\tilde{\beta}_n(nx)}{k!} \int_{k/n}^{(k+1)/n} f(t)dt$$

where $\tilde{\beta}_n(x)$ are called adjoint Bernoulli polynomials (Yılmaz, 2022). For more information on positive linear operators obtained using generating functions of special polynomials, see (Gezer and Yılmaz, 2023), (Özarslan et. al, 2008), (Sofyalıoğlu and Kanat, 2022), (Taşdelen et. al, 2012) and (Yılmaz, 2023).

The moments and central moments functions in Approximation Theory are engaged to define and analyze approximation methods based on orthogonal polynomials. They construct a framework for obtaining effective and attentive approximations of functions over specified intervals, which have applications in various areas of mathematics, engineering, and science. Moment and central moment functions are defined respectively to be as:

$$e_r(t) = t^r, \qquad r = 0,1,2, ...$$

and

$$L_n((e_1 - e_0 x)^r; x) := L_n((t-x)^r; x),$$

where L_n is a positive linear operator (Gupta and Rassias, 2019).

In Approximation Theory, which deals with approximating complicated functions by simpler ones, Korovkin's theorem offers a key result. It provides universal convergence conditions for function sequences, making approximation techniques easier to research and improve.

Theorem 1 (Korovkin Theorem) Let a sequence of linear positive operators $(L_n)_n$, $L_n: V \to \mathcal{F}[a,b]$ where $F[a,b]$ is space of all real-valued functions in the interval $[a,b]$ and V is a linear subspace of $\mathcal{F}[a,b]$. Suppose that $\varphi_0, \varphi_1, \varphi_2 \in V \cap C[a,b]$ forms a Chebychev system on the interval $[a,b]$, if we have

$$\lim_{n\to\infty} L_n(\varphi_j) = \varphi_j$$

uniformly for $j = 0,1,2$, then

$$\lim_{n\to\infty} L_n(f) = f,$$

uniformly, for any $f \in V \cap C[a,b]$ (Paltanea, 2012).

The theorem of Bohman is the particular version of above theorem when $\varphi_j = e_j, j = 0,1,2$. The monomial functions denoted by e_j are defined to be as moment functions.

2. Materials and Methods

In this section we define a new operator. Then we calculate the moment and central moment functions for our operator. We also give some results involving these functions. Finally, by using the moment functions, the uniform convergence of our operator is obtained with the help of Korovkin's theorem.

2.1. Construction of the operator

Let α_n and β_n be strictly increasing sequences such that

$$\lim_{n\to\infty} \frac{1}{\beta_n} = 0\,, \frac{\alpha_n}{\beta_n} = 1 + O\left(\frac{1}{\beta_n}\right),$$

where the O symbol is called big-O.

The negative order Euler polynomials are defined to be as:

$$\sum_{n=0}^{\infty} E_n^{(-k)}(x)\frac{t^n}{n!} = \left(\frac{e^t+1}{2}\right)^k e^{xt},$$

where $k > 0$ and $n > -k$ (Horadam, 1992).

Substituting $k = 1$ into above equation gives the generating function of the Euler polynomials with order (-1) $E_n^{(-1)}$ as follows:

$$\sum_{n=0}^{\infty} E_n^{(-1)}(x)\frac{t^n}{n!} = \left(\frac{e^t+1}{2}\right) e^{xt}.$$

By using above mathematical tools, we obtain a generalized Kantorovich type operators involving Euler polynomials with order (-1), $E_n^{(-1)}(x)$, at the following equation:

$$T_n^{\alpha_n,\beta_n}(f;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\int_{\frac{k}{\beta_n}}^{\frac{k+1}{\beta_n}} f(t)dt \tag{1}$$

where $T_n^{\alpha_n,\beta_n}: L([0,1]) \to C([0,1])$ and $f \in C([0,1])$.

The derivatives of generating functions of Euler polynomials with order (-1) are obtained to be as follows:

$$\sum_{k=0}^{\infty}\frac{kE_n^{(-1)}(x)}{k!}t^{k-1} = \frac{1}{2}e^{\alpha_n x}(e(\alpha_n x+1)+\alpha_n x) \tag{2}$$

$$\sum_{k=0}^{\infty}\frac{k(k-1)E_n^{(-1)}(x)}{k!}t^{k-2} = \frac{1}{2}e^{\alpha_n x}(e(\alpha_n x+1)^2+(\alpha_n x)^2), \tag{3}$$

$$\sum_{k=0}^{\infty}\frac{k(k-1)(k-2)E_n^{(-1)}(x)}{k!}t^{k-3} = \frac{1}{2}e^{\alpha_n x}(e(\alpha_n x+1)^3+(\alpha_n x)^3), \tag{4}$$

$$\sum_{k=0}^{\infty}\frac{k(k-1)(k-2)(k-3)E_n^{(-1)}(x)}{k!}t^{k-4} = \frac{1}{2}e^{\alpha_n x}(e(\alpha_n x+1)^4+(\alpha_n x)^4). \tag{5}$$

2.2. Moments and Central Moments Functions of Operator

The following lemmas are given to prove the uniform convergence of the operator and to obtain some results of the operator.

Lemma 1 (Moment functions) For all $x \in [0,1]$ and $n \in \mathbb{N}$, the $T_n^{\alpha_n,\beta_n}$ satisfy the following equations:

$$T_n^{\alpha_n,\beta_n}(e_0(x);x) = 1, \tag{6}$$

$$T_n^{\alpha_n,\beta_n}(e_1(x);x) = \frac{\alpha_n}{\beta_n}x + \frac{3e+1}{2\beta_n(e+1)}, \tag{7}$$

$$T_n^{\alpha_n,\beta_n}(e_2(x);x) = \frac{\alpha_n^2}{\beta_n^2}x^2 + \frac{\alpha_n}{\beta_n^2}\frac{4e+2}{e+1}x + \frac{10e+1}{3\beta_n^2(e+1)}, \tag{8}$$

where $e_i(x) = x^i \in C([0,1])$ for $i = 0,1,2$.

Proof Let $f(x) = e_0(x) = 1$. By substituting $f(x) = 1$ in (1), the following equation is obtained:

$$T_n^{\alpha_n,\beta_n}(1;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\int_{\frac{k}{\beta_n}}^{\frac{k+1}{\beta_n}} f(t)dt. \tag{9}$$

Using the definition of generating function of $E_n^{(-1)}(x)$ and taking integral in (9), we have

$$T_n^{\alpha_n,\beta_n}(1;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\left(\frac{e+1}{2}\right)e^{\alpha_n x}\frac{1}{\beta_n} = 1, \tag{10}$$

where $t = 1$ and $x \to \alpha_n x$.

Let $f(x) = e_1(x) = x$. By substituting $f(x) = x$ in (1), we give

$$T_n^{\alpha_n,\beta_n}(x;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\int_{k/\beta_n}^{k+1/\beta_n} t dt. \tag{11}$$

Using the definition of generating function of $E_n^{(-1)}(x)$ and taking integral in (11), we obtain

$$T_n^{\alpha_n,\beta_n}(x;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\frac{2k+1}{2{\beta_n}^2}$$

$$= \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{{\beta_n}^2}\sum_{k=0}^{\infty}\frac{kE_n^{(-1)}(\alpha_n x)}{k!} + \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{2{\beta_n}^2}\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}. \tag{12}$$

Applying (2) into (12), we obtain

$$T_n^{\alpha_n,\beta_n}(x;x) = \frac{\alpha_n}{\beta_n}x + \frac{3e+1}{2\beta_n(e+1)}. \tag{13}$$

Let $f(x) = e_2(x) = x^2$. By substituting $f(x) = x^2$ in (1), we give

$$T_n^{\alpha_n,\beta_n}(x^2;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\int_{k/\beta_n}^{k+1/\beta_n} t^2 dt. \tag{14}$$

Using the definition of generating function of $E_n^{(-1)}(x)$ and taking integral in (14), we obtain

$$T_n^{\alpha_n,\beta_n}(x;x) = \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}\frac{3k^2+3k+1}{3{\beta_n}^3}$$

$$= \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{{\beta_n}^3}\sum_{k=0}^{\infty}\frac{k^2E_n^{(-1)}(\alpha_n x)}{k!} + \beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{3{\beta_n}^3}\sum_{k=0}^{\infty}\frac{kE_n^{(-1)}(\alpha_n x)}{k!} +$$
$$\beta_n e^{-\alpha_n x}\left(\frac{2}{e+1}\right)\frac{1}{3{\beta_n}^3}\sum_{k=0}^{\infty}\frac{E_n^{(-1)}(\alpha_n x)}{k!}. \tag{15}$$

Applying (2) and (3) into (15), we obtain

$$T_n^{\alpha_n,\beta_n}(x^2;x)=\frac{\alpha_n^2}{\beta_n^2}x^2+\frac{\alpha_n}{\beta_n^2}\frac{4e+2}{e+1}x+\frac{10e+1}{3\beta_n^2(e+1)}. \tag{16}$$

From (10), (13) and (16), the desired results are obtained.

Lemma 2 (Central Moment functions) For all $x\in[0,1]$ and $n\in\mathbb{N}$, the $T_n^{\alpha_n,\beta_n}$ satisfy the following equations:

$$T_n^{\alpha_n,\beta_n}(e_1-x;x)=\left(\frac{\alpha_n}{\beta_n}-1\right)x+\frac{3e+1}{2\beta_n(e+1)}, \tag{17}$$

and

$$T_n^{\alpha_n,\beta_n}((e_1-x)^2;x)=\left(\frac{\alpha_n^2}{\beta_n^2}-\frac{2\alpha_n}{\beta_n}+1\right)x^2+\frac{\alpha_n}{\beta_n}\left(\frac{7e+3}{(e+1)\beta_n}\right)+\frac{10e+1}{3{\beta_n}^2(e+1)} \tag{18}$$

Proof Applying the linear property of $T_n^{\alpha_n,\beta_n}$, we have

$$T_n^{\alpha_n,\beta_n}(e_1-x;x)=T_n^{\alpha_n,\beta_n}(e_1;x)-xT_n^{\alpha_n,\beta_n}(1;x)=\left(\frac{\alpha_n}{\beta_n}-1\right)x+\frac{3e+1}{2\beta_n(e+1)}, \tag{19}$$

and

$$\begin{aligned}T_n^{\alpha_n,\beta_n}((e_1-x)^2;x)&=T_n^{\alpha_n,\beta_n}(e_2;x)-2xT_n^{\alpha_n,\beta_n}(e_1;x)+x^2T_n^{\alpha_n,\beta_n}(1;x)\\&=\left(\frac{\alpha_n^2}{\beta_n^2}-\frac{2\alpha_n}{\beta_n}+1\right)x^2+\frac{\alpha_n}{\beta_n}\left(\frac{7e+3}{(e+1)\beta_n}\right)+\frac{10e+1}{3{\beta_n}^2(e+1)}.\end{aligned} \tag{20}$$

From (19) and (20), the proof is completed.

3. Uniformly Convergence of $T_n^{\alpha_n,\beta_n}$

In this section, we give uniformly convergence property of $T_n^{\alpha_n,\beta_n}$ by using moment functions at the following theorem:

Theorem 2 If $f\in C([0,1])$,

$$\lim_{n\to\infty}T_n^{\alpha_n,\beta_n}(f;x)=f(x) \tag{21}$$

uniformly convergence on $[0,1]$.

Proof We know that,

$$\lim_{n\to\infty}T_n^{\alpha_n,\beta_n}(e_0(x);x)=\lim_{n\to\infty}1=1 \tag{22}$$

$$\lim_{n\to\infty}T_n^{\alpha_n,\beta_n}(x;x)=\lim_{n\to\infty}T_n^{\alpha_n,\beta_n}(e_1(x);x)=\lim_{n\to\infty}\left(\frac{\alpha_n}{\beta_n}x+\frac{3e+1}{2\beta_n(e+1)}\right)=x \tag{23}$$

and

$$\lim_{n\to\infty}T_n^{\alpha_n,\beta_n}(x^2;x)=\lim_{n\to\infty}T_n^{\alpha_n,\beta_n}(e_2(x);x)=\lim_{n\to\infty}\left(\frac{\alpha_n^2}{\beta_n^2}x^2+\frac{\alpha_n}{\beta_n^2}\frac{4e+2}{e+1}x+\frac{10e+1}{3\beta_n^2(e+1)}\right)=x^2\,. \tag{24}$$

Using Theorem 1, the desired result is obtained.

4. Conclusion

In this study, we first introduced a new generalized Kantorovich type Szász operators involving the generating functions of Euler type polynomials with order (-1) and we showed the moment and central moment functions of our operators. The moment and central moment functions are important mathematical tools used to investigate the convergence properties of positive linear operators. With the help of moment functions, we showed that our operator is uniformly convergent.

In future studies, properties of the operator such as convergence speed and convergence error estimation can be investigated.

Ethics Permissions

This paper does not require ethics committee approval.

Conflict of Interest

Author declare that there is no conflict of interest for this paper.

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