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Edge Co-Even Domination

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Highlights:
- MECEDS(G)
- Domination in graphs
- Graph operations

Keywords:
- Domination
- Edge co-even dominating set
- Edge co-even domination number

ABSTRACT:

Domination is one of those important parameters in graph theory which has a very wide range of applications. There are various types of domination depending on the structure of dominating sets. In this study, a new domination parameter called edge co-even domination number is introduced and denoted by $\gamma'_{coe}(G)$. Some basic graphs such as path, cycle, complete, complete bipartite, star, regular, wheel and their complement graphs are examined of this definition. In addition, some results of this parameter are found under graph operations, such as corona and cartesian product.

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INTRODUCTION

In graph theory, domination has become a more significant measure with a variety of applications in real life. As a result, many researchers are currently studying on earlier and new types of dominating sets in detail. There are various types of domination such as double domination, total domination, weak domination, restrained domination etc.

Let $G = (V, E)$ be a simple graph. A set D subset of V is called a dominating set such that for all $v \in V - D$ is adjacent to at least one vertex in D (Hedetniemi et al., 1998). The domination number is the number of vertices in a smallest dominating set for G and denoted by $\gamma(G)$.

Similarly, a subset S of E is called an edge dominating set, if $e \in E - S$ is adjacent to some edge in S and the minimum cardinality of S called edge domination number $\gamma'(G)$ (Mitchell et al., 1977).

If $deg\ (v)$ is even number for all $v \in V - D$, then D is called co-even dominating set (*CEDS)* and the co-even domination number denoted by $\gamma_{coe}(G)$ is the cardinality of minimum co-even dominating set D (Omran et al., 2020). The co-even domination number of graphs such as thorn graphs, banana tree, coconut tree and binomial tree are examined (Demirpolat et al., 2021). Also (Shalaan et al., 2020 November), (Omran et al., 2021) and (Imran et al., 2022) have obtained new results on co-even domination number.

In this paper, a new type of domination number called edge co-even domination number is defined and denoted by $\gamma'_{coe}(G)$. The edge co-even domination numbers of some graph types such as friendship and fan graphs are examined. Some results are obtained in basic graph structures. This new parameter is also studied under some graph operations such as corona and cartesian product.

MATERIALS AND METHODS

Edge Co-Even Domination Number

In this section, we define a new dominating set called edge co-even dominating set and a new type of domination number depending on this set. This new domination number is called as edge co-even domination number. The results of this new number on path, cycle, star, complete, complete bipartite, wheel, friendship and fan graphs are examined.

Definition 1. *Let G be a graph and D is an edge dominating set, the set D is called edge co-even dominating set if, $deg(e)$ is even number for all $e \in E - D$.*

Definition 2. *Consider G be a graph that has no isolated vertex and D is an edge co-even dominating set, then D is called a minimal edge co-even dominating set if has no proper subset $\dot{D} \subseteq D$ is an edge co-even dominating of G. $MECEDS(G)$ refers to all minimal edge co-even dominating sets of a graph G.*

Definition 3. *The set $|D|$ is called the edge co-even domination number if $|D| = min\{|D_i|, D_i \in MECEDS(G)\}$ and is denoted by $\gamma'_{coe}(G)$* (1)

Proposition 1. *Let G be a graph and D is an edge co-even dominating set, then*

1. *All edges of odd or zero degrees belong to every edge co-even dominating set.*
2. *$deg(e) \geq 2$, for all $e \in E - D$.*
3. *Let G be r-regular graph then $\gamma'_{coe}(G) = \gamma'(G)$.*
4. *$\gamma'(G) \leq \gamma'_{coe}(G)$* (2)

Proof of Proposition 2.1. All four cases are examined as follows.

(1) By the definion of edge co-even domination, in order for every remaining edge to be $deg(e) \geq 2$, every edge that has $deg(e) < 2$ must be included in the $MECEDS$.
(2) By definition, all edges with degree odd or zero are included in the set $MECEDS$. Therefore, all remaining edges must have degree $deg(e) \geq 2$.
(3) If G is a r-regular graph, then all vertices or edges in the graph have the same degree. Therefore, $\gamma'_{coe}(G) = \gamma'(G)$.
(4) Let G be a regular graph or a graph with all edge degrees even. In this case, $\gamma'_{coe}(G) = \gamma'(G)$ becomes. From another perspective, let G be a graph with all edge degrees odd. Thus, all edges are included in the edge co-even dominating set by definition. Therefore, $\gamma'_{coe}(G) \geq \gamma'(G)$.

Theorem 1. Let G be a path graph with $n \geq 5$, then $\gamma'_{coe}(P_n) = 2 + \left\lceil \frac{n-5}{3} \right\rceil$ (3)

Proof. Let $\{e_1, e_2, \dots, e_{n-1}\}$ be the edge set of the path P_n. By Proposition 2.1(1), the two pendant edges e_1 *and* e_{n-1} lie in each $MECEDS(G)$. These edges dominate the adjacent edges e_2 *and* e_{n-2}. Now, let $D = \{e_{4+3k}, k = 0,1, \dots, \left\lceil \frac{(n-1)-4}{3} \right\rceil - 1\}$. It is clear that D is a $MECEDS(G)$ to the induced subgraph $\langle e_3, e_4, \dots, e_{n-3} \rangle$. Thus, $\gamma'_{coe}(G) = 2 + \left\lceil \frac{n-5}{3} \right\rceil$.

Theorem 2. Let G be a cycle graph where $n \geq 3$, then $\gamma'_{coe}(C_n) = \left\lceil \frac{n}{3} \right\rceil$. (4)

Proof. Let D be a $MECEDS(G)$. Since the degree of all edges is even, we will use the same procedure used to have an edge dominating set. By using Proposition 2.1.(3), the result is obtained. Therefore, $\gamma'_{coe}(G) = \left\lceil \frac{n}{3} \right\rceil$.

Theorem 3. Let G be a star graph where $n \geq 2$, then

$$\gamma'_{coe}(S_n) = \begin{cases} 1, & if\ n\ is\ even \\ n-1, & if\ n\ is\ odd \end{cases}. \quad (5)$$

Proof. Let D be a $MECEDS(G)$. There are two cases to be examined depending on whether n is even or odd.

Case 1. If n is even, it is clear that we can dominate all edges by an edge in the graph as well, because of all other edges with an even degree. Thus, $\gamma'_{coe}(G) = 1$.

Case 2. If n is odd, by Proposition 2.1.(2), every edge in the star belongs to set D. since the degree of all these edges is odd. Thus, $\gamma'_{coe}(G) = n - 1$.

Theorem 4. If G be a complete graph where $n \geq 3$, then $\gamma'_{coe}(K_n) = \left\lfloor \frac{n}{2} \right\rfloor$. (6)

Proof. Let D be a $MECEDS(G)$. Since the degree of all the edges is even in a complete graph, we will use the same procedure used to have an edge dominating set. Since K_n is a r-regular graph where $r = n - 1$ by using Proposition 2.1(3), the result is obtained. Therefore, $\gamma'_{coe}(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

Theorem 5. Let G be a complete bipartite graph, where $m \leq n$, then

$$\gamma'_{coe}(K_{m,n}) = \begin{cases} m, & if\ n\ and\ m\ are\ both\ odd\ or\ even \\ mn, & if\ m\ is\ odd\ and\ n\ is\ even\ or\ m\ is\ even\ and\ n\ is\ odd \end{cases} \quad (7)$$

Proof. Suppose that V_1 and V_2 are the bipartite sets of the graph G of order n and m, respectively and D is a $MECEDS(G)$. Then, there are two cases to be examined depending on n and m as follows.

Case 1. If n and m are both odd or even, then the degree of all edges is even. Then, we will use the same procedure used to have an edge dominating set. It is sufficient to choose as many edges as the number of edges that go from one vertex of n vertices to m vertices. Therefore, $\gamma'_{coe}(G) = m$.

Case 2. If n is odd and m is even or vice versa, then all edges belong to $MECEDS(G)$ by Proposition 2.1(1). Since there are mn edges in a complete bipartite graph, $\gamma'_{coe}(G) = mn$.

Theorem 6. Let G be a wheel graph where $n \geq 2$, then

$$\gamma'_{coe}(W_n) = \begin{cases} 1 + \left\lceil \frac{n-3}{3} \right\rceil, & if\ n\ is\ even \\ n-1, & if\ n\ is\ odd \end{cases} \tag{8}$$

Proof. Let D be a $MECEDS(G)$. Here are two cases.

Case 1. If n is even, then let e_j be an edge inside the wheel, the edge dominates all inside edges of the wheel and two edges form the edges of induced subgraph isomorphic to the cycle of order n. Therefore, the remained edges not dominated by the edge e_j are the edges in the induced subgraph isomorphic to the cycle of order n not adjacent to the edge e_j. The number of remaining vertices is $n - 1 - 2 = n - 3$, so we can dominate these edges by $\left\lceil \frac{n-3}{3} \right\rceil$ edges. Then, $\gamma'_{coe}(G) = 1 + \left\lceil \frac{n-3}{3} \right\rceil$.

Case 2. If n is odd, then the degree of all the edges inside of the wheel is odd. These edges belong to $MECEDS(G)$ by using Proposition 2.1(1). Consequently, $\gamma'_{coe}(G) = n - 1$. (9)

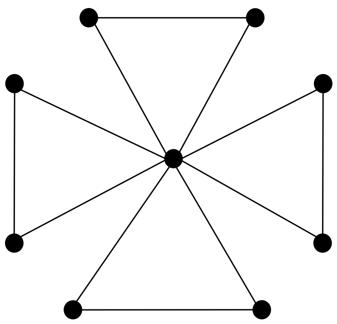

Figure 2.1 The Friendship Graph F_4

Theorem 7. Let G be the Friendship graph, then $\gamma'_{coe}(F_n) = n$. (10)

Proof. Let G be the Friendship graph of order n and D be a $MECEDS(G)$. By the definition of the Friendship graph (Erdös et al., 1996), we know that G is constructed by joining n copies of the cycle graph C_n with a joint vertex. Since the degree of all the edges in G will always be even, choosing one edge for each C_n is sufficient for the edge co-even domination. Therefore, $\gamma'_{coe}(F_n) = n$.

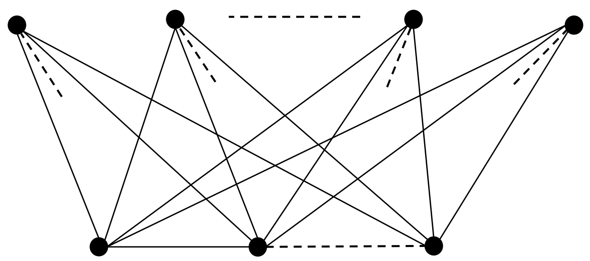

Figure 2.2. $P_n + \overline{K_m}$

Theorem 8. If G is the Fan graph defined by $G \equiv P_n + \overline{K_m}$, then

$$\gamma'_{coe}(G) = \begin{cases} m(n-2) + 2, & if\ m\ is\ odd\ and\ n\ is\ even\ or\ m\ is\ even\ and\ n\ is\ odd \\ 2m + \left\lceil \frac{n-5}{3} \right\rceil + 2, & if\ m, n\ are\ both\ even\ or\ odd \end{cases} \tag{11}$$

Proof. There are two cases depending on n and m as follows.

Case 1. If m is odd and n is even or m is even and n is odd, then the degree of end edges of P_n is odd. These two pendant edges must belong to $MECEDS(G)$ by using Proposition 2.1(1).

Let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of P_n. Except for the edges where the vertices in $\overline{K_m}$ merge with the vertices v_1 and v_n, the degree of the edges where the vertices in $\overline{K_m}$ merge with the remaining $n - 2$ vertices of P_n is odd. So, these m times $n - 2$ edges must belong to $MECEDS(G)$. Therefore, $\gamma'_{coe}(G) = m(n-2) + 2$..(12)

Case 2. If m and n are both even or odd, then the degree of the edges where the vertices in $\overline{K_m}$ merge with the end vertices of P_n is odd. So, these $2m$ edges must belong to $MECEDS(G)$.

Let $\{e_1, e_2, \dots, e_{n-1}\}$ be the edge set of P_n, then the degree of end edges of P_n is odd. These two pendant edges e_1 and e_{n-1} must belong to $MECEDS(G)$ by using Proposition 2.1(1). These edges dominate the adjacent edges $e_2\ and\ e_{n-2}$. It is clear that for the remaining edges of the induced subgraph $\langle e_3, e_4, \dots, e_{n-3}\rangle$, $D = \{e_{4+3k}, k = 0,1, \dots, \left\lceil\frac{(n-1)-4}{3}\right\rceil - 1\}$ is a $MECEDS(G)$. Thus, $|D|$ set of P_n is $\left\lceil\frac{n-5}{3}\right\rceil + 2$. Therefore, $\gamma'_{coe}(G) = 2m + \left\lceil\frac{n-5}{3}\right\rceil + 2$. ....(13)

RESULTS AND DISCUSSION

MECEDS(G) in the Complement of Graphs

In this section, we study on $MECEDS(G)$ of complement (Bondy et al., 1976) of path, cycle, complete bipartite graphs, and share their proofs.

Theorem 9. Let G be a path graph of order n, then $\gamma'_{coe}(\overline{P_n}) = 2n - 6$. ....(14)

Proof. Let $\{v_1, v_2, \dots, v_n\}$ be the vertex set of P_n. If n is even, the degree of terminal vertices v_1 and v_n are even and the other vertices have odd degrees. Also, if n is odd, then the degree of terminal vertices v_1 and v_n are odd and the other vertices have even degree. Then, all edges that incident on one of the terminal vertex with non-terminal vertex have odd degree. Thus, by using Proposition 2.1(1), all these edges belong to $MECEDS(G)$. It is clear that these edges dominate all edges in P_n and the number of these edges is $(n - 3) + (n - 3) = 2n - 6$. ....(15)

Theorem 10. Let G be a cycle graph where $n \geq 4$, then $\gamma'_{coe}(\overline{C_n}) = \left\lfloor\frac{n}{2}\right\rfloor$. ....(16)

Proof. Let G be a complement of a cycle graph of order n. It is clear that $\overline{C_n}$ is (n - 3) – regular graph. Thus, by Proposition 2.1(3), the result is obtained.

Theorem 11. Let G is a complete bipartite graph of order nm where $n, m \geq 2$, then

$\gamma'_{coe}(\overline{K_{m,n}}) = \left\lfloor\frac{m}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor$. ....(17)

Proof. Let G be a complement of complete bipartite graph of order mn. It is clear that $\overline{K_{m,n}} \equiv K_m \cup K_n$, then by using Theorem 2.4, the result is obtained.

Edge Co-Even Domination Number Under Some Graph Operations

We find results on edge co-even domination number of cartesian product (Klavžar et al., 2008) of P_2 and P_n, corona (Buckley et al., 1990) of C_n, and $\overline{K_p}$ and prove the results.

Theorem 12. Let G be a Cartesian product of P_2 and P_n denoted by $G \equiv P_2 \otimes P_n$, then $\gamma'_{coe}(G) = \left\lceil\frac{2(n-4)}{3}\right\rceil + 4$. ....(18)

Proof. Let $\{e_1^1, e_2^1, \dots, e_{n-1}^1\}$ be the edge set of one of P_n. By Proposition 2.1(1) the two pendant edges $e_1^1\ and\ e_{n-1}^1$ lie in $MECEDS(G)$. These edges dominate the adjacent edges $e_2^1\ and\ e_{n-2}^1$. For the remaining edges, it is clear that $D = \{e_{4+3k}, k = 0,1,\dots,\left\lceil\frac{n-4}{3}\right\rceil - 1\}$ is a $MECEDS(G)$. to the induced subgraph $\langle e_3^1, e_4^1, \dots, e_{n-3}^1\rangle$.

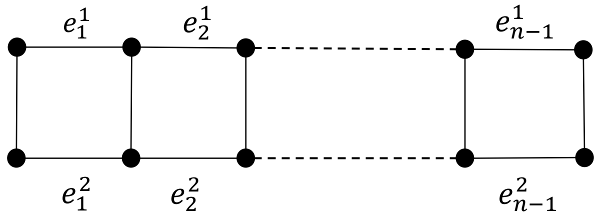

***Figure 4.1**. $P_2 \otimes P_n$*

Same way, let $\{e_1^2, e_2^2, \dots, e_{n-1}^2\}$ be the edge set of other P_n. We determine the edges we will choose, taking care to dominate all P_2. By Proposition 2.1(1) the two pendant edges e_1^2 *and* e_{n-1}^2 lie in $MECEDS(G)$. These edges dominate the adjacent edges e_2^2 *and* e_{n-2}^2 and also dominate the first two and the last two P_2. For the remaining edges, to provide domination of all P_2, we need to choose the edges that indicate one mines of the first path graph edge numbers. It is clear that $D_1 = \{e_{3+3k}, k = 0,1,\dots,\left\lceil\frac{n-4}{3}\right\rceil - 1\}$ is a $MECEDS(G)$ to the induced subgraph $\langle e_3^2, e_4^2, \dots, e_{n-3}^2\rangle$. Therefore, $\gamma'_{coe}(G) = \left\lceil\frac{2(n-4)}{3}\right\rceil + 4$. ....(19)

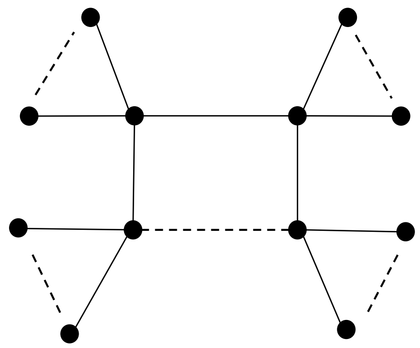

***Figure 4.2**. $C_n \odot \overline{K_p}$*

Theorem 13. *If* $G \equiv C_n \odot \overline{K_p}$, then $\gamma'_{coe}(G) = \begin{cases} np, & if\ p\ is\ even \\ \left\lceil\frac{n}{2}\right\rceil, & if\ p\ is\ odd \end{cases}$. ....(20)

Proof. There are two cases to be examined depending on whether p is even or odd.

Case 1. If p is even, then the degree of edges that connect C_n and $\overline{K_p}$ is odd. Therefore, these edges of G belong to $MECEDS(G)$ and dominate all the edges of the cycle. Thus, domination is provided in graph G. Therefore, $\gamma'_{coe}(G) = np$.

Case 2. If p is odd, then the degree of all the edges is even. Here, providing domination of the cycle graph is sufficient to provide domination of graph G. While determining these edges, it is also important to ensure the domination of edges that connect C_n and $\overline{K_p}$.

Let $\{e_1, e_2, \dots, e_n\}$ be the edges of C_n. Each edge on cycle dominates $2p$ edges except for adjacent edges. It is clear that $D = \{e_{1+2k}, k = 0,1,\dots,\left\lceil\frac{n}{2}\right\rceil - 1\}$ is a $MECEDS(G)$. Thus, $\gamma'_{coe}(G) = \left\lceil\frac{n}{2}\right\rceil$. ....(21)

CONCLUSION

In this paper, we define a new domination consept in the graphs called edge co-even domination number. We obtain results of this number on some certain graph classes as a P_n, C_n, S_n, W_n, K_n, $K_{n,m}$, friendship, fan graph and some graph operations are determined, as cartesian product of P_2 and P_n, and corona of C_n and $\overline{K_p}$.

Conflict of Interest

The article authors declare that there is no conflict of interest between them.

Author's Contributions

The authors declare that they have contributed equally to the article.

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