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Rings Whose Certain Modules are Dual Self-CS-Baer

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Abstract
In this work, we characterize some rings in terms of dual self-CS-Baer modules (briefly, ds-CS-Baer modules). We prove that any ring R is a left and right artinian serial ring with $J^2(R) = 0$ iff $R \oplus M$ is ds-CS-Baer for every right R-module M. If R is a commutative ring, then we prove that R is an artinian serial ring iff R is perfect and every R-module is a direct sum of ds-CS-Baer R-modules. Also, we show that R is a right perfect ring iff all countably generated free right R-modules are ds-CS-Baer.

Keywords: *Dual self-CS-Baer module, Harada ring, Lifting module, Perfect ring, QF-ring, Serial ring*

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1. Introduction

Throughout the paper, all rings will have an identity element and all modules will be unitary right modules unless otherwise stated. Let M be a module and N a submodule of M. Then $N \ll M$ means that N is a small submodule of M (namely, M is different from $N + K$ for every proper submodule K of M). $J(R)$ will denote the Jacobson radical of any ring R and $Rad(M)$ will denote the radical of any module M.

A module M is called *lifting* (or *satisfies* (D_1)), if every submodule N of M lies above a direct summand, that is, N contains a direct summand X of M such that $N/X \ll M/X$ (see [1] and [2]). A module M is said to be *dual self-CS-Baer* (briefly, *ds-CS-Baer*) if for every family $(f_i)_{i\in I}$ of homomorphisms $f_i : M \to M$, $\sum_{i\in I} Im(f_i)$ lies above a direct summand of M (see [3]). Clearly, every lifting module is ds-CS-Baer. Moreover, if R is a right Harada ring, then every injective right R-module is ds-CS-Baer. Because, remember that any ring R is called a right *Harada* ring if every injective right R-module is lifting (see [1]). Recall that any right R-module M is called *hollow*, if every proper submodule of M is small in M (see [2, Definition 4.1]) and it is called *local*, if it is hollow and $Rad(M) \neq M$. Note that M is local iff M is cyclic and has a unique maximal submodule (see [4, page 357]). It is not hard to see that every hollow module and so every local module is a lifting module.

In recent years, ds-CS-Baer modules and their related topics have been studied by Crivei, Keskin Tütüncü, Radu and Tribak (see for example [3], [5] and [6]). In this paper, we continue the study of ds-CS-Baer modules.

In section 2, we characterize some rings in terms of ds-CS-Baer modules. Among others, we mainly prove the followings:

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(A) Let R be a ring. Then R is an artinian serial ring with $J^2(R) = 0$ iff for every right R-module M, $R \oplus M$ is ds-CS-Baer (Theorem 2.1).

(B) Let R be a right self-injective ring. Then R is a QF-ring iff every injective right R-module is ds-CS-Baer (Theorem 2.3).

(C) Let R be a ring. Then R is a right perfect ring iff every free right R-module is ds-CS-Baer (Theorem 2.4).

(D) Let R be a commutative ring. Then R is semiperfect iff every cyclic R-module is ds-CS-Baer (Proposition 2.1).

(E) Let R be a commutative ring. Then R is an artinian serial ring iff R is perfect and every 2-f.p. R-module is a finite direct sum of ds-CS-Baer modules (Proposition 2.4).

2. Results

We first give the following easy observation.

Lemma 2.1. *Let R be a ring. Let M be a free right R-module. Then M is lifting iff it is ds-CS-Baer.*

Proof. Let M be a free right R-module. Then we can assume that $M = \oplus_{i \in I} R$. Now the result is obvious by the proof of [3, Proposition 9.4]. □

Let R be ring and M a module. M is called *uniserial* if its submodules are linearly ordered by inclusion and is called *serial* if it is a direct sum of uniserial submodules. The ring R is called *right* (left) *serial* if the right (left) R-module R_R (${}_RR$) is serial. Also R is called *artinian serial* if it is both right and left artinian serial. By [4, Theorem 32.3], we know that if R is an artinian serial ring, then every right R-module and every left R-module is a direct sum of uniserial R-modules.

Now, we characterize artinian serial rings with $J^2(R) = 0$ via ds-CS-Baer modules.

Theorem 2.1. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is an artinian serial ring with $J^2(R) = 0$.*

(2) *Every right R-module is lifting.*

(3) *For every right R-module M, $R \oplus M$ is lifting.*

(4) *For every right R-module M, $R \oplus M$ is ds-CS-Baer.*

Proof. (1) ⇔ (2): It is satisfied by [1, 29.10].

(3) ⇔ (4): It is proved in [3, Proposition 9.4].

(2) ⇒ (3): It is clear.

(3) ⇒ (2): It is clear since lifting property is preserved by direct summands (see for example [1, Lemma 22.6]). □

The next result is a consequence of Theorem 2.1.

Corollary 2.1. *Let R be a ring. Then R is an artinian serial ring with $J^2(R) = 0$ iff every (finitely generated) right R-module is ds-CS-Baer.*

Proof. This follows from [7, Theorem 3.15], [3, Proposition 9.4] and Theorem 2.1 and the fact that being ds-CS-Baer or lifting is preserved by taking direct summands.

□

Remark 2.1. The left-handed versions of Theorem 2.1 and Corollary 2.1 are equal to being artinian serial ring with $J^2(R) = 0$.

A finitely generated right R-module M is said to be *finitely presented* in case in every exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F finitely generated and free the kernel K is also finitely generated. An exact sequence of right R-modules

$$P_1 \xrightarrow{f} P_0 \longrightarrow M \longrightarrow 0$$

is called a *minimal projective presentation* of M in case P_1 and P_0 are finitely generated projective and $\mathrm{Ker} f \ll P_1$ and $\mathrm{Im} f \ll P_0$. Let M a finitely presented right R-module with no nonzero projective direct summands. Following [4], M is called a *2-f.p. module* if there are primitive idempotents e, e_1 and e_2 of R and there is a minimal projective presentation

$$eR \longrightarrow e_1R \oplus e_2R \longrightarrow M \longrightarrow 0.$$

Therefore a 2-f.p. module is both 2-primitive generated and finitely presented.

Recall from [8] that a module M is called w-local if it has a unique maximal submodule. Clearly, a module M is local if and only if M is a cyclic w-local module.

Next, we can give the following.

Theorem 2.2. *Let R be a ring. Consider the following statements:*

(1) *R is serial and every direct sum of two ds-CS-Baer right R-modules and every direct sum of two ds-CS-Baer left R-modules is ds-CS-Baer.*

(2) *Every finitely presented right R-module and finitely presented left R-module is ds-CS-Baer.*

(3) *Every 2-generated finitely presented right R-module and 2-f.p. left R-module is ds-CS-Baer.*

(4) *R is semiperfect and every 2-f.p. right R-module and 2-f.p. left R-module is ds-CS-Baer.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Proof. (1) $\Rightarrow$ (2): Let M be a finitely presented right R-module and N a finitely presented left R-module. By [9, Corollary 3.4], M and N are finite direct sum of cyclic w-local submodules. In particular, they are finite direct sum of local submodules. Since local modules are lifting, they are also ds-CS-Baer. Therefore M and N are ds-CS-Baer by (1).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): These are clear by definitions and [3, Proposition 5.9]. □

Inspired by Theorem 2.1, we give the following theorem that characterizes QF-rings. First, remember that any ring R is called a *QF-ring*, if R is noetherian and injective as a left (or right) R-module (see for example [4, page 333]).

Theorem 2.3. *Let R be a right self-injective ring. Then the following assertions are equivalent:*

(1) *R is a QF-ring.*

(2) *R is a right Harada ring.*

(3) *For every injective right R-module M, $R \oplus M$ is lifting.*

(4) *For every injective right R-module M, $R \oplus M$ is ds-CS-Baer.*

(5) *Every injective right R-module is ds-CS-Baer.*

Proof. (1) $\Leftrightarrow$ (2): It is clear by [1, 28.10 and 28.16].

(3) $\Leftrightarrow$ (4): It is clear by [3, Proposition 9.4].

(2) $\Rightarrow$ (3): Let M be an injective right R-module. By hypothesis, $R \oplus M$ is an injective right R-module. Since R is right Harada, it follows that $R \oplus M$ is lifting.

(3) $\Rightarrow$ (2): Let M be an injective right R-module. By (3), $R \oplus M$ is lifting. Therefore, M is lifting. Hence, R is a right Harada ring.

(4) $\Leftrightarrow$ (5): It is clear. □

In the following, we characterize right perfect rings in terms of ds-CS-Baer modules. Firstly, remember that any module M is called $\oplus$-*supplemented*, if for every submodule N of M there exists a direct summand K of M with $M = N + K$ and $N \cap K$ small in K. This notion is a generalization of lifting modules (see [2]).

Theorem 2.4. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is a right perfect ring.*

(2) *$R^{(\mathbb{N})}$ is a ds-CS-Baer right R-module.*

(3) *Every countably generated free right R-module is ds-CS-Baer.*

(4) *Every free right R-module is ds-CS-Baer.*

Proof. (1) $\Rightarrow$ (2): Assume that R is a right perfect ring. Consider the right R-module $M = R^{(\mathbb{N})}$. By [2, Theorem 4.41], M is lifting, and so it is ds-CS-Baer by definitions.

(2) $\Rightarrow$ (1): Assume that the right R-module $R^{(\mathbb{N})}$ is ds-CS-Baer. Since it is free, by Lemma 2.1, it is lifting. Hence it is $\oplus$-supplemented. Therefore, R is a right perfect ring by [7, Theorem 2.10].

(1) $\Rightarrow$ (4): Let M be a free right R-module. Then M is projective. So, M is lifting by [2, Theorem 4.41]. Thus, M is ds-CS-Baer by definitions.

(4) $\Rightarrow$ (1): Assume that every free right R-module is ds-CS-Baer. Then every free right R-module is lifting by Lemma 2.1. By [2, Theorem 4.41], R is a right perfect ring.

(4) $\Rightarrow$ (3) $\Rightarrow$ (2): These are clear. □

Next, we give a characterization of commutative semiperfect rings in terms of cyclic dual self-CS-Baer modules.

Proposition 2.1. *Let R be a commutative ring. Then R is semiperfect iff every cyclic R-module is ds-CS-Baer.*

Proof. Let R be a semiperfect ring. Let M be a cyclic R-module. Assume that $M = xR$, where $x \in M$. We know that $M \cong R/I$, for some ideal I of R. By [1, 4.9 (1)], since I is fully invariant in R, R/I is quasi-projective and hence M is quasi-projective. Then by [2, Theorem 4.41], M is lifting and so M is ds-CS-Baer.

Conversely, assume that every cyclic R-module is ds-CS-Baer. Then R is a ds-CS-Baer R-module. Therefore by [3, Proposition 5.9], R is semiperfect. □

Now, we give a characterization of commutative semiperfect FGC-rings. Let R be a commutative ring. R is called an *FGC-ring*, if every finitely generated R-module is a direct sum of cyclic modules (see [10]).

Proposition 2.2. *Let R be a commutative ring. Then the following assertions are equivalent:*

(1) *Every finitely generated R-module is $\oplus$-supplemented.*

(2) *Every finitely generated R-module is a finite direct sum of ds-CS-Baer modules.*

(3) *R is a semiperfect FGC-ring.*

(4) *R is a direct sum of almost maximal valuation rings.*

Proof. (1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4): These are proved in [7, Proposition 2.8].

(1) $\Rightarrow$ (2): Let M be a finitely generated R-module. By (1), M is $\oplus$-supplemented. By [7, Corollary 2.6], $M = \oplus_{i=1}^{n} x_i R$. Note that each $x_i R$ is quasi-projective since R is commutative. Therefore by [2, Theorem 4.41], each $x_i R$ is lifting and so ds-CS-Baer.

(2) $\Rightarrow$ (1): Let M be a finitely generated R-module. By (2), $M = \oplus_{i=1}^{n} x_i R$, where each $x_i R$ is ds-CS-Baer. By [3, Proposition 5.12], each $x_i R$ is lifting and hence $\oplus$-supplemented. Therefore by [11, Theorem 1.4], M is $\oplus$-supplemented. □

Corollary 2.2. *Let R be a commutative indecomposable ring. Then R is an almost maximal valuation ring iff every finitely generated R-module is a direct sum of cyclic ds-CS-Baer R-modules.*

Next, we characterize commutative serial rings via direct sums of cyclic ds-CS-Baer modules.

Proposition 2.3. *Let R be a commutative ring. Then the following assertions are equivalent:*

(1) *R is serial.*

(2) *R is semiperfect and every 2.f.p. R-module is $\oplus$-supplemented.*

(3) *R is semiperfect and every finitely presented R-module is a finite direct sum of ds-CS-Baer modules.*

(4) *R is semiperfect and every 2-generated finitely presented R-module is a finite direct sum of ds-CS-Baer modules.*

(5) *R is semiperfect and every 2-f.p. R-module is a finite direct sum of ds-CS-Baer modules.*

Proof. (1) $\Leftrightarrow$ (2): This follows from [7, Theorem 3.5].

(1) $\Rightarrow$ (3): Clearly, R is semiperfect. Now, let M be a finitely presented R-module. Note that M is finitely generated. By [9, Corollary 3.4], $M = \oplus_{i=1}^{n} M_i$, where each M_i is w-local and cyclic. Note that each M_i $(1 \leq i \leq n)$ is a local module. Hence each M_i is ds-CS-Baer.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5): These are clear.

(5) $\Rightarrow$ (2): Let M be a 2-f.p. R-module. By (5), $M = \oplus_{i=1}^{n} M_i$, where each M_i is a cyclic ds-CS-Baer R-module. By [3, Proposition 5.12], each M_i is lifting and hence $\oplus$-supplemented. Hence M is $\oplus$-supplemented by [11, Theorem 1.4]. □

Finally, we characterize commutative artinian serial rings as follows.

Proposition 2.4. *Let R be a commutative ring. Then the following assertions are equivalent:*

(1) *R is an artinian serial ring.*

(2) *R is perfect and every 2-f.p. R-module is $\oplus$-supplemented.*

(3) *R is perfect and every R-module is a direct sum of ds-CS-Baer modules.*

(4) *R is perfect and every countably generated R-module is a direct sum of ds-CS-Baer modules.*

(5) *R is perfect and every finitely presented R-module is a finite direct sum of ds-CS-Baer modules.*

(6) *R is perfect and every 2-f.p. R-module is a finite direct sum of ds-CS-Baer modules.*

Proof. (1) $\Leftrightarrow$ (2): It is proved in [7, Corollary 3.13].

(1) $\Rightarrow$ (3): By [4, Corollary 28.8], R is a perfect ring. Now, let M be any R-module. By [4, Theorem 32.3], $M = \oplus_{i \in I} M_i$, where each M_i is uniserial. Clearly every uniserial module is hollow. Since R is perfect, then each M_i has small radical (see [4, Remark 28.5]). Therefore, each M_i is local, and so cyclic. Hence M is a direct sum of cyclic ds-CS-Baer modules.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6): These are clear.

(6) $\Rightarrow$ (2): Let M be a 2-f.p. R-module. By (6), $M = \oplus_{i=1}^{n} M_i$, where each M_i is a cyclic ds-CS-Baer R-module. By [3, Proposition 5.12], each M_i is lifting and hence $\oplus$-supplemented. Therefore M is $\oplus$-supplemented by [11, Theorem 1.4]. □

Propositions 2.3 and 2.4 are not true over noncommutative rings as we see in the following example.

Example 2.1. (see [7, Example 3.16]) Let R be a local artinian ring with Jacobson radical $J(R)$ such that $J^2(R) = 0, Q = R/J(R)$ is commutative, $\dim({}_Q J(R)) = 1$ and $\dim(J(R)_Q) = 2$. Then R is left serial but not right serial. Let $J(R) = uR \oplus vR$. $A_1 = R/J(R), A_2 = R/uR$ and $A_3 = R_R$ are the only three isomorphism types of indecomposable right R-modules. Here each A_i is lifting and hence ds-CS-Baer. Note that every right R-module is a direct sum of indecomposable modules, and hence a direct sum of cyclic ds-CS-Baer modules. However, R is not a serial ring.

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