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Integral Circulant Graphs and So's Conjecture

Ercan Altınışık [1,*], Sümeyye Büşra Aydın[2]

[1] *Department of Mathematics, Faculty of Sciences, Gazi University, 06500, Ankara, Turkey.*
[2] *Graduate School of Natural and Applied Sciences, Gazi University, 06500, Ankara, Turkey*

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Abstract. An integral circulant graph is a circulant graph whose adjacency matrix has only integer eigenvalues. It was conjectured by W. So that there are exactly $2^{\tau(n)-1}$ non-isospectral integral circulant graphs of order n, where $\tau(n)$ is the number of divisors of n [5]. However, the conjecture remains unproven. In this paper, we present the fundamental concepts and results on the conjecture. We obtain the relation between two characterizations of integral circulant graphs given by W. So [5] and by W. Klotz and T. Sander [2]. Finally, we calculate the eigenvalues of the integral circulant graph G if $S(G) = G_n(d)$ for any $d \in D$. Here $G_n(d)$ is the set of all integers less than n that have the same greatest common divisor d with n.

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1. Introduction and Preliminaries

A simple graph is a graph without loop, multi-edge, and orientation. All the graphs throughout the present paper are simple. Let G be a graph with the vertex set V and the edge set E, that is $G = G(V, E)$. The order of G is defined as the number of the elements in V. For the graph $G = G(V, E)$, let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 1$ if $(i, j) \in E$ or $a_{ij} = 0$ otherwise for $i, j \in V$. We call such a matrix A the adjacency matrix of G and denote it by $A(G)$. The eigenvalues of $A(G)$ are defined as the eigenvalues of G. We call the multiset whose elements are all the eigenvalues of G taking in to account with their multiplicities the spectrum of G and denote it by $sp(G)$. It is obvious that $sp(G)$ will be a classical set when every eigenvalue is of multiplicity 1.

The graph having a circulant adjacency matrix is called a circulant graph. More precisely, a graph of order n is called circulant if it is a circulant adjacency matrix, which is an $n \times n$ matrix commuting with the matrix

$$Z = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix},$$

where I_{n-1} is the $(n-1) \times (n-1)$ identity matrix. An integral circulant graph is a circulant graph whose eigenvalues all are integers. There is an alternative definition for a circulant graph in the literature. Let $J \subseteq \left\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. The circulant graph $C_n(J) = (V, E)$ is then defined as the simple graph on the vertex set $V = \mathbb{Z}_n$ whose edge set is $E = \{\{x, x + d\} : x \in \mathbb{Z}_n,\ d \in J\}$. In this context, J is known as the jump set.

Let the vertices of a circulant graph G be labeled $\{0, 1, 2, \ldots, n-1\}$. The set

*Corresponding Author
Email addresses: ealtinisik@gazi.edu.tr (E. Altınışık), sumeyyebusra.aydin@gazi.edu.tr (S.B. Aydın)

$$S(G) = \{k : a_{0,k} = 1\} \subset \{1, 2, \cdots, n-1\}$$

is called the symbol set of G. More precisely, the elements of the symbol set consist of the indices of the vertices adjacent to vertex 0 of the graph G. For example, if $A(G) = J - I$, where J is a matrix with all elements 1 and I_{nxn} is the $n \times n$ identity matrix, $S(G) = \{1, 2, \ldots, n-1\}$ and thus the graph G will be a complete graph with n vertices. In addition, since the matrix $A(G)$ is symmetric, if and only if for $k \in S(G)$ is that it is $n - k \in S(G)$. Indeed, all subset of the set $\{1, 2, \ldots, n-1\}$ that has this property is the symbol of a circulant graph with n vertices. However, there are circulant graphs that are isomorphic to two different sets of symbols. For example, for $n = 5$, the symbol sets $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$ correspond to 5−vertex cycle graphs. So, it is natural to ask the question "how many non-isomorphic circulant graphs are there for a given n"? It has been observed that for $n = 1, 2, 3, \ldots, 8$, the number of non-isomorphic circulant graphs is $1, 2, 2, 4, 3, 8, 4, 12$, respectively. However, there is no general formula. Therefore, since there are $2^{\lfloor n/2 \rfloor}$ different symbols for a graph with n vertices, there are at most $2^{\lfloor n/2 \rfloor}$ non-isomorphic circulant graphs with n vertices.

Beside these, by using $S(G)$ the eigenvalues of G are $\lambda_t(G) = \sum_{k \in S(G)} (w^t)^k$ for $0 \le t \le n-1$, where $w = e^{2\pi i/n}$ and thus its spectrum can be written as [1]

$$sp(G) = (\lambda_0(G), \lambda_1(G), \ldots, \lambda_{n-1}(G)).$$

Then, to count all integral circulant graphs, it will be sufficient to find the symbol sets S of the set $\{1, 2, \ldots, n-1\}$, where $\sum_{k \in S} x^k$ is an integer where $x^n = 1$. In this frame, W. So [5] proved Theorem 1.1.

Theorem 1.1 ([5]). *Let G be a circulant graph with n vertices and symbol set $S(G)$. Then, G is integral if and only if $S(G)$ is a combination of $G_n(d)$'s. Here $G_n(d)$ is the set of all integers less than n that have the same greatest common divisor d with n.*

As a consequence of Theorem 1.1, W. So, [5] gave the following result regarding the number of integral circulant graphs with n vertices:

Corollary 1.2 ([5]). *Let $\tau(n)$ denote the number of positive integer divisors of the positive integer n. Then, the number of integral circulant graphs with n vertices is at most $2^{(\tau(n)-1)}$.*

In addition, W. So, identified all possible symbol sets for graphs with fewer than 100 vertices and came up with the following conjecture after observing that there are no two integral circulant graphs with the same spectrum.

Conjecture 1.3 ([5]). *Let G_1 and G_2 be two integral circulant graphs with symbol sets $S(G_1)$ and $S(G_2)$. If $S(G_1) \neq S(G_2)$, then $sp(G_1) \neq sp(G_2)$ and thus G_1 and G_2 are not isomorphic.*

If Conjecture 1.3 is true, the number of non-isomorphic integral circulant graphs with n vertices is exactly $2^{\tau(n)-1}$. It is easy to show that the conjecture is true for some special positive integers n, but it is still unproven. It is obvious that the number of elements of $S(G)$ is the largest eigenvalue of the integral circulant graph. Therefore, if $|S(G_1)| \neq |S(G_2)|$, $sp(G_1) \neq sp(G_2)$. If p is a prime and r is a positive integer, the positive divisors of $n = p^r$ will be $1, p, p^2, \ldots, p^{r-1}$, so the number of elements of $G_n(d)$ sets will be $d|n$, respectively,

$$(p-1)p^{r-1}, (p-1)p^{r-2}, \ldots, (p-1)p, (p-1).$$

As a result, since the symbol sets of $n = p^r$ vertex integral circulant graphs will contain different numbers of elements, Conjecture 1.3 will be correct in the case of $n = p^r$ [5]. Beside this, recently, Conjecture 1.3 was proved in cases where $n = p^k, pq^k, p^2q$, where $2 \le p < q$ are primes and $k \ge 1$ are integers, and $n = pqr$, where $2 \le p < q < r$ are primes. Its accuracy has been shown [5].

In addition to W. So's characterization with the symbol set, W. Klotz and T. Sander [2] introduced gcd-graphs by generalizing unitary Cayley graphs, a special class of circulant graphs, and showed that all eigenvalues of gcd-graphs are integers. Let $\mathbb{Z}_n$ be the additive group of the ring $\mathbb{Z}_n$, of integers with respect to module n. Let $U_n = \{a \in \mathbb{Z}_n : (a, n) = 1\}$ be the subset of the units of the ring $\mathbb{Z}_n$. Here, (a, n) is the greatest common divisor of the integers a and n. The Cayley graph $X_n = Cay(\mathbb{Z}_n, U_n)$ whose vertex set $V(X_n) = \mathbb{Z}_n$ and edge set

$$E(X_n) = \{\{a, b\} : a, b \in \mathbb{Z}_n,\ (a - b) \in U_n\} = \{\{a, b\} : a, b \in \mathbb{Z}_n,\ (a - b, n) = 1\}$$

is called the unitary Cayley graph. W. Klotz and T. Sander [2] showed that the eigenvalues of the Cayley graph X_n are

$$\lambda_r(X_n) = \sum_{\substack{1\le j\le n \\ (j,n)=1}} w^{rj}$$

for $0 \le r \le n-1$. Indeed,

$$\lambda_t(X_n) = c(r,n) = \mu\left(\frac{n}{(r,n)}\right)\frac{\varphi(n)}{\varphi\left(\frac{n}{(r,n)}\right)}$$

can be written where $c(r,n)$ is the Ramanujan sum, φ is Euler's phi-function, and μ is the Möbius function. Here, the eigenvalues of the unitary Cayley graph X_n are integers and since X_n is also circulant, X_n is an integral circulant graph.

Theorem 1.4 ([2]). *Let $n \ge 2$ and X_n be a unitary Cayley graph. Then, the following is provided:*

(i) *Every nonzero eigenvalue of X_n is a divisor of $\varphi(n)$.*

(ii) *Let m be the maximal squarefree divisor of n. Then, $\lambda_{\min} = \mu(m)\frac{\varphi(n)}{\varphi(m)}$ is a nonzero eigenvalue of X_n of minimal absolute value and multiplicity $\varphi(m)$. Every eigenvalue of X_n is a multiple of $\lambda_{\min}$. If n is odd, then $\lambda_{\min}$ is the only nonzero eigenvalue of X_n with minimal absolute value. If n is even, then $-\lambda_{\min}$ is also an eigenvalue of X_n with multiplicity $\varphi(m)$.*

Corollary 1.5 ([2]). *Let $n \ge 2$ and X_n be a unitary Cayley graph. Then, the following is provided:*

(i) *For at least one of ± 1 to be an eigenvalue of X_n, if and only if that n is squarefree.*

(ii) *If n is squarefree, $\mu(n)$ is an eigenvalue of X_n with multiplicity $\varphi(n)$.*

(iii) *For both ± 1 to have eigenvalues with a multiplicity $\varphi(n)$ of X_n it is necessary and sufficient that n is squarefree and even.*

Theorem 1.6 ([2]). *Let m be the maximal squarefree divisor of n and let M be the set of positive divisors of m. Then, the following statements for the unitary Cayley graph X_n, $n \ge 2$, hold.*

(i) *Repeating every term of the sequence $S = \left(\mu(m)\frac{\varphi(n)}{\varphi(m)}\right)_{t\in M}$ $\varphi(t)$−times results in a sequence $\widetilde{S}$ of lenght m which consists of all nonzero eigenvalues of X_n such that the number of appearences of an eigenvalue is its multiplicity.*

(ii) *The multiplicity of zero as an eigenvalue of X_n is $n-m$.*

(iii) *If $\alpha(\lambda)$ is the multiplicity of the eigenvalue λ of X_n, then $\lambda\alpha(\lambda)$ is a multiple of $\varphi(n)$.*

Now, let $D(n)$ denote the set of positive integer divisors of n. Let D be a subset of $D(n)$ that does not contain integer n. A graph $X_n(D)$ with vertex set $V(X_n) = \mathbb{Z}_n$ and edge set

$$E(X_n) = \{\{a,b\} : a,b \in \mathbb{Z}_n,\ (a-b,n) \in D\}$$

is called gcd-graph. If $D = \{1\}$ is specifically taken, $X_n(D)$ will be a unitary Cayley graph. Let the first row of the adjacency matrix of the gcd-graph $X_n(D)$ is $(a_0, a_1, \ldots, a_{n-1})$ and

$$a_j = \begin{cases} 1 & if\ (j,n) \in D, \\ 0 & if\ (j,n) \notin D. \end{cases}$$

If we substitute a_j into $\lambda_r = \sum_{j=0}^{n-1} a_j w^{rj}$ the eigenvalues of the graph can be obtained as

$$\lambda_r = \sum_{d\in D}\sum_{\substack{1\le j<n \\ (j,n)=d}} w^{rj} = \sum_{d\in D} c(r,n/d)$$

for $0 \le r \le n-1$. Thus, the following theorem is obtained.

Theorem 1.7 ([2]). *The eigenvalues of all gcd-graphs are integers.*

For example, for $n = 6$ and $D = \{1,3\}$, $X_n(D)$ is a gcd-graph with vertex set

$$V(X_n(D)) = \{0,1,2,3,4,5\}$$

and edge set

$$E(X_n(D)) = \{\{0,1\},\{0,3\},\{1,2\},\{1,4\},\ \{2,3\},\{2,5\},\{3,4\},\{4,5\},\{5,0\}\}$$

and its spectrum is $sp(X_n(D)) = \{3,-3,0,0,0,0\}$.

In this study, we firstly compare two characterizations of integral circulant graphs given by W. So [5] and W. Klotz and T. Sander [2]. Then, we calculate the eigenvalues of the integral circulant graph G if $S(G) = G_n(d)$ for any $d \in D$ and we obtain some properties of the eigenvalues.

2. Eigenvalues for Integral Circulant Graphs of the Form $S(G) = G_n(d)$

W. So [5] carried out the characterization of integral circulant graphs with the aid of the symbol set concept by proving in Theorem 1.1 which states the necessary and sufficient condition for G to be an integral graph is that $S(G)$ is a combination of $G_n(d)$. Besides, J. W. Sander and T. Sander [4] stated that gcd-graphs defined above by W. Klotz and T. Sander [2] with the aid of D divisor sets are integral circulant graphs and that all integral circulant graphs can be characterized with the aid of D divisor sets without proof. Now, we will prove this first.

Theorem 2.1. *Let n be a positive integer, $D \subseteq D(n)$ be a divisor set of n, and $n \notin D$. Then, the gcd-graph $X_n(D)$, is an integral circulant graph with symbol set $S(G) = \bigcup_{d\in D} G_n(d)$.*

Proof. First, we show that $X_n(D)$ is a circulant graph. Let $\{x,y\} \in E(G)$. In this case, from the definition of $E(G)$, there is $d_1 \in D$ such that $(x-y,n) = d_1$. Since

$$(x+k)-(y+k) = x-y$$

for $k \in \mathbb{Z}^+$, $((x+k)-(y+k),n) = d_1$ and for $0 \le x' \le n-1,\ \ 0 \le y' \le n-1$, we have

$$x+k \equiv x' \mod n \ \text{ and } \ y+k \equiv y' \mod n,$$

whenever $\{x',y'\} \in E(G)$. Moreover, since $(x-y,n) = (y-x,n)$ for every $\{x,y\} \in E(G)$, $\{y,x\} \in E(G)$. Thus, the adjacency matrix of G is symmetric. Then, the graph characterized by the divisor set D is a circulant graph.

Now, we show that for the symbol set $S(G)$, $S(G) = \bigcup_{d\in D} G_n(d)$. From Theorem 1.4, we know that $X_n(D)$ is an integral circulant graph. Moreover, according to Theorem 1.1, there is a divisor set $D' \subseteq D(n)$ such that $S(G) = \bigcup_{d'\in D'} G_n(d')$. We will show that $D' = D$. Let $x \in D$. Since $x = (x,n) = (0-x,n)$, we have $x \in S(G)$. Hence, for some $d' \in D'$, $x \in G_n(d')$ and so $(x,n) = d'$. From here $x \in D'$ is obtained. Then, $D \subseteq D'$. Now, let $y \in D'$. Also, since $y \in G_n(y)$, $y \in S(G)$ is obtained. From the definition of the graph $X_n(D)$ it is $y \in D$. Then, $D' \subseteq D$. This completes the proof. □

The graph G, whose symbol set is of the form $S(G) = G_n(1)$, is a unitary Cayley graph. $0 \le r \le n-1$ and $w = \exp\left(\frac{2\pi i}{n}\right)$, the eigenvalues of G are calculated as

$$\lambda_r(G) = \sum_{\substack{1\le k\le n-1\\(k,n)=d}} (w^r)^k = c(r,n) = \mu\left(\frac{n}{(r,n)}\right)\frac{\varphi(n)}{\varphi\left(\frac{n}{(r,n}\right)}$$

with the aid of Ramanujan sums $c(r,n)$. Based on this, if the symbol set of an integral circulant graph G is $S(G) = G_n(d)$, its eigenvalues can be calculated as

$$\begin{aligned}\lambda_r(G) &= \sum_{\substack{1\le k\le n-1\\(k,n)=d}} (w^r)^k \\ &= \sum_{\substack{1\le k/d\le n/d-1\\(k/d,n/d)=1}} (w^r)^{\frac{k}{d}} \\ &= c\left(r,\frac{n}{d}\right) \\ &= \mu\left(\frac{\frac{n}{d}}{\left(t,\frac{n}{d}\right)}\right)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{\frac{n}{d}}{(t,\frac{n}{d})}\right)}\end{aligned} \tag{2.1}$$

for $0 \le r \le n-1$. Here, also $t = \frac{n}{(r,n)}$ and $w = \exp\left(\frac{2\pi i}{n}\right)$ [3]. In addition, the minimum absolute eigenvalue of a unitary Cayley graph is $\lambda_{\min} = \mu(m)\frac{\varphi(n)}{\varphi(m)}$. Here, m is the squarefree maximal divisor of n [2]. Now, we write the formula that gives the minimum absolute values of integral circulant graphs whose symbol set is of the form $G_n(d)$ and prove it in a similar way.

Theorem 2.2. *For all $n \ge 2$, let G be integral circulant graph, $d|n$ and $S(G) = G_n(d)$. The following statements hold.*

(i) *Every nonzero eigenvalue of G is a divisor of $\varphi\left(\frac{n}{d}\right)$.*
(ii) *Let m be the maximal squarefree divisor of n. Then,*

$$\lambda_{\min} = \mu(m)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi(m)} \tag{2.2}$$

is a non-zero eigenvalue of G with minimum absolute value and multiplicity $\varphi(m)d$. Each eigenvalue of G is a multiple of $\lambda_{\min}$. If $\frac{n}{d}$ is odd, $\lambda_{\min}$ is the unique nonzero eigenvalue of G with minimum absolute value. If $\frac{n}{d}$ is even, $-\lambda_{\min}$ is an eigenvalue of G with multiplicity $\varphi(m)d$.

Proof. (i) By the multiplicative properties of the Euler function $\varphi(a)$ divides $\varphi(n)$, if a is a divisor of n. Therefore, the formula (2.1) implies that the nonzero eigenvalues of G are divisors of $\varphi\left(\frac{n}{d}\right)$.

(ii) For $\lambda_r \neq 0$ and $t_r = \frac{\frac{n}{d}}{(r,\frac{n}{d})}$, since t_r is the divisor of m, $\mu(t_r) \neq 0$. By (2.1), the absolute value of $\lambda_r \neq 0$ is minimal if and only if $\varphi(t_r) = \varphi(m)$. This equation always has the trivial solution $t_r = m$. This means that

$$\lambda_r = \lambda_{\min} = \mu(m)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi(m)}.$$

For $0 \le r \le n-1$, one can obtain

$$\begin{aligned}\lambda_r = \lambda_{\min} \Leftrightarrow t_r &= \frac{\frac{n}{d}}{\left(r,\frac{n}{d}\right)} = m\\ \Leftrightarrow \left(r,\frac{n}{d}\right) &= \frac{\frac{n}{d}}{m}\\ \Leftrightarrow r \in Q &= \left\{x\frac{n}{d.m} : 0 \le x \le m.d,\ (x,m) = 1\right\}.\end{aligned}$$

Thus, $\lambda_{\min}$ has multiplicity $|Q| = \varphi(m)d$. If λ_r is an arbitrary non-zero eigenvalue of G, then t_r is the divisor of m, and thus $\varphi(t_r)$ divides $\varphi(m)$, that is, there is $k \in \mathbb{Z}^+$ such that $\varphi(m) = k\varphi(t_r)$. Thus, λ_r becomes a multiple of $\lambda_{\min}$. From (2.1) and (2.2), we have

$$\lambda_r = \mu(t_r)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi(t_r)} = k.\mu(t_r)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi(t_r)} = \pm k\lambda_{\min}.$$

If $\frac{n}{d}$ is odd and $\lambda_r = t_r = \frac{\frac{n}{d}}{(r,\frac{n}{d})}$ divides m then $\varphi(t_r) = \varphi(m)$ and hence m and t_r are odd. Therefore, $\varphi(t_r) = \varphi(m)$ by the properties of the Euler function. If $\frac{n}{d}$ is even then m is even and hence $\varphi(m) = \varphi\left(\frac{m}{2}\right)$. Thus, we obtain

$$\lambda'_{\min} = \mu\left(\frac{m}{2}\right).\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{m}{2}\right)} = -\mu(m)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi(m)} = -\lambda_{\min}$$

for the eigenvalue λ'_{min}. □

Example 2.3. Let G be an integral circulant graph of order n. Let $n = 6$ and $d = 2$. Let $S(G) = G_6(2)$. Here, the spectrum of G is $\{2, 2, -1, -1, -1, -1\}$. Every non-zero eigenvalue of G is a divisor of $\varphi\left(\frac{6}{2}\right) = \varphi(3) = 2$. Here 3 is the squarefree maximal divisor of $\frac{6}{2}$. Then,

$$\lambda_{\min} = \mu(3)\frac{\varphi\left(\frac{6}{2}\right)}{\varphi(3)} = -1.\frac{2}{2} = -1$$

is a nonzero eigenvalue of G with the minimum absolute value.

Example 2.4. Let G be an integral circulant graph of order n. Let $n = 12$ and $d = 3$. Let $S(G) = G_{12}(3)$. Here, the spectrum of G is $\{2, 2, -2, -2, 1, 1, 1, 1, -1, -1, -1, -1\}$. Every non-zero eigenvalue of G is a divisor of $\varphi\left(\frac{12}{3}\right) = \varphi(4) = 2$. Here 2 is the square-free maximal divisor of $\frac{12}{3}$. Then,

$$\lambda_{\min} = \mu(2)\frac{\varphi\left(\frac{12}{3}\right)}{\varphi(2)} = -1.\frac{2}{2} = -1$$

is a nonzero eigenvalue of G with the minimum absolute value.

Theorem 2.5. *Let $n \geq 2$ and G be an integral circulant graph with symbol set $S(G) = G_n(d)$. Then, the followings are provided:*

(i) *For at least one of ± 1 to be an eigenvalue of G if and only if $\frac{n}{d}$ is squarefree.*

(ii) *If $\frac{n}{d}$ is squarefree, $\mu\left(\frac{n}{d}\right)$ is the eigenvalue of G whose multiplicity is $\varphi\left(\frac{n}{d}\right).d$.*

(iii) *If $\frac{n}{d}$ is squarefree and even, G has eigenvalues of ± 1.*

Proof. (i) Let at least one of ± 1 be an eigenvalue of G. Then, for $0 \leq r \leq n-1$ there exists an r such that

$$\lambda_r = \mu\left(\frac{\frac{n}{d}}{\left(t, \frac{n}{d}\right)}\right)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{\frac{n}{d}}{\left(t,\frac{n}{d}\right)}\right)} = \pm.1$$

From the definition of the Euler function it must be $\frac{\varphi(n/d)}{\varphi((n/d)/(t,n/d))} = 1$. Thus, $\left(t, \frac{n}{d}\right) = 1$ and $\mu\left((n/d)/(t, n/d)\right) = \mu\left(\frac{n}{d}\right) = \pm 1$. From the definition of the Möbius function, $\frac{n}{d}$ is squarefree.

Conversely, choose r such that $\frac{n}{d}$ is squarefree and $\left(t, \frac{n}{d}\right) = 1$. Then,

$$\lambda_r = \mu\left(\frac{\frac{n}{d}}{\left(t, \frac{n}{d}\right)}\right)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{\frac{n}{d}}{\left(t,\frac{n}{d}\right)}\right)} = \mu\left(\frac{n}{d}\right)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{n}{d}\right)} = \mu\left(\frac{n}{d}\right).$$

Since $\frac{n}{d}$ is squarefree, $\mu\left(\frac{n}{d}\right) = \pm 1$ from the definition of the Möbius function.

(ii) From (i), if $\frac{n}{d}$ is squarefree, $\mu\left(\frac{n}{d}\right)$ is an eigenvalue of G for r values such that $\left(t, \frac{n}{d}\right) = 1$. From the definition of the Euler function, there are t with multiplicity $\varphi\left(\frac{n}{d}\right).d$ such that $\left(t, \frac{n}{d}\right) = 1$. Thus, the multiplicity of the eigenvalue $\mu\left(\frac{n}{d}\right)$ is $\varphi\left(\frac{n}{d}\right).d$.

(iii) Let $\frac{n}{d}$ be squarefree and even. $\frac{n}{d}$ is assumed to have an even number of prime factors.

$$\lambda_r = \mu\left(\frac{\frac{n}{d}}{\left(t, \frac{n}{d}\right)}\right)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{\frac{n}{d}}{\left(t,\frac{n}{d}\right)}\right)} = \mu\left(\frac{n}{d}\right)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{n}{d}\right)} = \mu\left(\frac{n}{d}\right) = 1$$

for r such that $\left(t, \frac{n}{d}\right) = 1$. For r such that $\left(t, \frac{n}{d}\right) = 2$ since $\frac{n}{2d}$ has an odd number of prime factors and $\varphi\left(\frac{n}{d}\right) = \varphi\left(\frac{n}{2d}\right)$, thus,

$$\lambda_r = \mu\left(\frac{\frac{n}{d}}{\left(t, \frac{n}{d}\right)}\right)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{\frac{n}{d}}{\left(t,\frac{n}{d}\right)}\right)} = \mu\left(\frac{n}{2d}\right)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(\frac{n}{2d}\right)} = \mu\left(\frac{n}{2d}\right) = -1.$$

□

Theorem 2.6. *Let $n \geq 2$ and m be the squarefree maximal divisor of the integer $\frac{n}{d}$, M be the set of positive divisors of m, and G be an integral circulant graph with symbol set $S(G) = G_n(d)$. Then, the followings are provided:*

(i) *Repeating every term of the sequence $S = \left(\mu(t)\varphi\left(n/d\right)/\varphi(t)\right)_{t\in M}$ $d\varphi(t)-$ times results in a sequence $\widetilde{S}$ of length m which consists of all nonzero eigenvalues of G such that the number of appearences of an eigenvalue is its multiplicity.*

(ii) *The multiplicity of zero as an eigenvalue of G is $n - md$.*

(iii) *If $\alpha(\lambda)$ is the multiplicity of the eigenvalue λ of G, then $\lambda.\alpha(\lambda)$ is a multiple of $\varphi\left(\frac{n}{d}\right)$.*

Proof. (i) The number of terms in the resulting sequence $\widetilde{S}$ is

$$\sum_{t\in M}\varphi(t)=\sum_{t|m}\varphi(t)=md.$$

Equation 2.1 describes the sequence $(\lambda_r)_{0\le r\le n-1}$ of all eigenvalues of G in which each eigenvalue is listed according to its multiplicity. As $\mu(t_r)=0$ for $t_r\in M$, we get the subsequence $\widetilde{T}$ of nonzero eigenvalues for $0\le r\le n-1$, $t_r\in M$

$$\widetilde{T}=\left(\mu(t_r)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi\left(t_r\right)}\right)_{0\le r\le n-1,\ t_r\in M},$$

where $t_r=\frac{n}{d(r,\frac{n}{d})}$. Let t be an arbitrary element of M. For $0\le r\le n-1$, i.e. $r\in\mathbb{Z}_n$, we have $t=t_r$ if and only if $\left(r,\frac{n}{d}\right)=\frac{\frac{n}{d}}{t}$. Elementary number theory shows

$$\begin{aligned}Q_t:=&\left\{r\in\mathbb{Z}_n\ :\left(r,\frac{n}{d}\right)=\frac{\frac{n}{d}}{t}\right\}\\ =&\left\{x\frac{n}{dt}:x\in\mathbb{Z}_t,\ (x,t)=1\right\},\end{aligned}$$

which implies that Q_t has $\varphi(t)$ elements. Therefore, the sequence $\widetilde{T}$ consists of all terms

$$\mu(t)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi(t)},\ t\in M$$

where each of these terms appears $\varphi(t).d-$times. If we take every term only once, then we arrive at the sequence S and see that $\widetilde{S}$ and $\widetilde{T}$ coincide apart possibly from the order of their elements.

(ii) By (i) the length of the sequence $\widetilde{S}$ equals the number of nonzero eigenvalues, each of them counted according to its multiplicity. As $\widetilde{S}$ has length md, the eigenvalue zero has multiplicity $n-md$.

(iii) The statement is trivially true for $\lambda=0$. Let λ be a nonzero eigenvalue of G. Then, there is an integer $t\in M$ such that

$$\lambda=\mu(t)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi(t)}.$$

By (i) λ has at least multiplicity $\varphi(t)$, more precisely

$$\lambda.\alpha(\lambda)=\mu(t)\frac{\varphi\left(\frac{n}{d}\right)}{\varphi(t)}k_t\varphi(t)=\mu(t)k_t\varphi\left(\frac{n}{d}\right).$$

Thus, $\lambda\alpha(\lambda)$ is a multiple of $\varphi\left(\frac{n}{d}\right)$. □

3. Comments and Future Work

In the present paper, we have presented the background of the So conjecture about circulant graphs and we have obtained the relation between the characterizations of integral circulant graphs given by W. So [5] and by W. Klotz and T. Sander [2]. Then, we have calculated the eigenvalues of the integral circulant graph G if $S(G)=G_n(d)$ for any $d\in D$. In this context, we believe that the natural goal of future studies is to attempt to prove the conjecture.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statements

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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