MAUN Fen Bil. Dergi., 12:1, 31-38 Araştırma Makalesi/ Research Article
MAUN J. of Sci., 12:1, 31-38 DOI: 10.18586/msufbd.1462229

Application and Reversibility of Three Dimensional Cellular Automata

Ferhat ŞAH
Faculty of Economics and Administrative Sciences, Adiyaman University, Adiyaman, Turkey

✉: fsah@adiyaman.edu.tr 0000-0003-4847-9180

Received (Geliş): 31.03.2024 Revision (Düzeltme):02.05.2024 Accepted (Kabul): 08.05.2024

ABSTRACT

In this study, we obtain the characteristic matrices of three-dimensional cellular automata under the null boundary condition. We examine the inverse of characteristic matrices. We obtain a recurrence equation to determine under what conditions the matrix is invertible. Thanks to this equation, we can calculate the inverse of large-dimensional matrices. Finally, we give some applications of cellular automata. We find the minimal polynomial of the characteristic matrix. We find the cycle length and transition length of the characteristic matrix with the help of minimal polynomials. We also find the attractive points of the characteristic matrix. Finally, we draw the State-Transition diagram with the results we obtained.

Keywords: Cellular Automata, Characteristic Matrices, Reversibility

Üç Boyutlu Hücresel Dönüşümlerin Terslenebilirliği ve Uygulaması

ÖZ

Bu çalışmada üç boyutlu hücresel dönüşümlerin karakteristik matrislerini sıfır sınır şartı altında elde ediyoruz. Karakteristik matrislerin tersini inceliyoruz. Matrisin hangi şartlarda tersinin olduğunu belirlemek için rekürans denklem elde ediyoruz. Bu denklem sayesinde büyük boyutlu matrislerin tersini hesaplayabiliriz. Son olarak hücresel dönüşümlerin bazı uygulamalarını veriyoruz. Karakteristik matrisin minimal polinomunu buluyoruz. Minimal polinomlar yardımıyla karakteristik matrisin devir uzunluğu ve geçiş uzunluğunu buluyoruz. Ayrıca karakteristik matrisin çekici noktalarını buluyoruz. Son olarak elde ettiğimiz sonuçlar ile Durum–Geçiş diyagramını çiziyoruz.

Anahtar Kelimeler: Hücresel Dönüşümler, Karakteristik Matrisler, Terslenebilirlik

INTRODUCTION

Cellular Automata (CA for short) was first used to obtain models in the fields of physics, biology and computer science. CA theory was first studied by Ulam and Von Neumann [1]. Later, many researchers became interested in studying CA to model the behavior of a complex system. Hedlund used CA systematically from a purely mathematical perspective [2]. Wolfram with the help of polynomial algebras [3], Pries to explain group properties based on a similar type of polynomial algebras [4] and Inokuchi et al. studied to observe the behavior of one-dimensional CA produced by the 156 rule [5]. Since two dimensional CAs (2D CAs) have widespread applications in physics, biology, mathematics and other sciences, the study of these CAs has accelerated in many branches of science in the last twenty years. On the other hand, Packard and Wolfram started their studies on two dimensional CA (2D CA) by making some observations on two-dimensional CA based on 5-neighborhood CA [6]. Das et al. extended the characterization of one-dimensional CA with the help of matrix algebras and introduced a new method for the theoretical analysis of linear CA [7]. They based the analysis of CA on polynomial algebra. At the same time, hybrid CAs were analyzed with this new method. Matrix characterization of CA was formulated to examine the resulting complex dynamic system.

Khan et al. developed a solution to examine 2D CA linear transformations with nearest neighbors over the field $\mathbb{Z}_2$ [8]. They examined the characterization of 2D CAs under periodic boundary conditions with the method and different rules they developed to separate two dimensional linear CAs.

Choudhury et al. gave the most general characterization of a special hybrid transformation of 2D CAs over the field $\mathbb{Z}_2$ [9]. Additionally, in another study, Choudhury and Dihidar achieved the characterization of 2D CAs by extending the one-dimensional CA theory with the help of matrix algebra[10]. Siap et al. examined two-dimensional cellular automata over the field $\mathbb{Z}_3$ with some special rules under periodic and null boundary conditions[11]. Whether the rule (representative) matrices corresponding to the examined CAs are invertible has emerged as an important problem. If the rule matrix of CA has an inverse, then the CA corresponding to this matrix is said to be invertible. However, there have not been many studies on three-dimensional cellular automata. Tsalides et al. conducted a study on three-dimensional cellular automata and their applications[12]. R.W.Gerling classified 3D CAs in his study[13].

Jan Hemmingsson carried out studies on the quasi-periodic behavior of 3D-CAs [14]. S.G.R. Brown and N.B. Bruce tried to model the free development of 3D CAs in their study[15]. E.G. Leubeck and M.C.M. De Gunst worked on the analysis of cellular deterioration using 3D CAs[16]. Alexandra Agapie gave a simple form of the constant distribution for 3D CA in a special case[17].

In this study, we will define the neighborhood states of 3D CAs. We will examine the invertibility of the characteristic matrices obtained under the zero boundary condition. We will provide information about whether the cellular transformations corresponding to these characteristic matrices are reversible or not. We will make some applications of cellular automata.

THREE DIMENSIONAL CELLULAR AUTOMATA

First, the definition of 3D-CA over the field $\mathbb{Z}_P$ will be given. Then, the characteristic matrices will be examined under the null boundary condition with a special rule and a general form will be obtained. Consider three dimensional $\mathbb{Z}^3$lattices and $\sigma: \mathbb{Z}^3 \rightarrow \mathbb{Z}_p$element $\Omega = {\mathbb{Z}_p}^{\mathbb{Z}^3}$configuration space. σ_n is defined by the value of σ at a $n \in \mathbb{Z}^3$ point. Let $u_1, u_2, \ldots, u_s \in \mathbb{Z}^3$ a finite set of different elements and $f: {\mathbb{Z}_p}^s \rightarrow \mathbb{Z}_p$ be given. (Ω, F) is defined as a pair of CA and a local rule f, where $F: \Omega \rightarrow \Omega, (F\sigma)_n = f\left(\sigma_{n+u_1}, \ldots, \sigma_{n+u_s}\right), n \in \mathbb{Z}^3$ is the global transition function.

Mathematically, the next state transition of the (i,j,k) cell can be represented as a function of the present states of the neighbor cells.

$$
\begin{aligned}
x_{(i,j,k)}^{(t+1)} = f(& x_{(i-1,j-1,k-1)}^{(t)}, x_{(i-1,j,k-1)}^{(t)}, x_{(i-1,j,k+1)}^{(t)}, x_{(i-1,j-1,k)}^{(t)}, x_{(i-1,j-1,k+1)}^{(t)}, \\
& x_{(i-1,j,k)}^{(t)}, x_{(i-1,j+1,k)}^{(t)}, x_{(i-1,j+1,k-1)}^{(t)}, x_{(i-1,j+1,k+1)}^{(t)}, x_{(i,j-1,k-1)}^{(t)}, \\
& x^{(t)}_{(i,j,k-1) x_{(i,j,k+1)}^{(t)}, x_{(i,j-1,k)}^{(t)}, x_{(i,j-1,k+1)}^{(t)}, x_{(i,j,k)}^{(t)},} x_{(i,j+1,k)}^{(t)}, x_{(i,j+1,k-1)}^{(t)}, x_{(i,j+1,k+1)}^{(t)}, x_{(i+1,j-1,k-1)}^{(t)} \\
& x_{(i+1,j,k-1)}^{(t)}, x_{(i+1,j,k+1),}^{(t)} x_{(i+1,j-1,k)}^{(t)}, x_{(i+1,j-1,k+1)}^{(t)}, x_{(i+1,j,k)}^{(t)}, x_{(i+1,j+1,k)}^{(t)}, \\
& x_{(i+1,j+1,k-1)}^{(t)}, x_{(i+1,j+1,k+1)}^{(t)}) \\
= & a_0 . x_{(i-1,j-1,k-1)}^{(t+1)} + a_1 . x_{(i-1,j,k-1)}^{(t+1)} + a_2 . x_{(i-1,j,k+1)}^{(t+1)} + a_3 . x_{(i-1,j-1,k)}^{(t+1)} + \\
& a_4 . x_{(i-1,j-1,k+1)}^{(t+1)} + a_5 . x_{(i-1,j,k)}^{(t+1)} + a_6 . x_{(i-1,j+1,k)}^{(t+1)} + a_7 . x_{(i-1,j+1,k-1)}^{(t+1)} + \\
& a_8 . x_{(i-1,j+1,k+1)}^{(t+1)} + a_9 . x_{(i,j-1,k-1)}^{(t+1)} + a_{10} . x_{(i,j,k-1)}^{(t+1)} \\
& + a_{11} . x_{(i,j,k+1)}^{(t+1)} + a_{12} . x_{(i,j-1,k)}^{(t+1)} + a_{13} . x_{(i,j-1,k+1)}^{(t+1)} + a_{14} . x_{(i,j,k)}^{(t+1)} + \\
& a_{15} . x_{(i,j+1,k)}^{(t+1)} + a_{16} . x_{(i,j+1,k-1)}^{(t+1)} + a_{17} . x_{(i,j+1,k+1)}^{(t+1)} + \\
& a_{18} . x_{(i+1,j-1,k-1)}^{(t+1)} + a_{19} . x_{(i+1,j,k-1)}^{(t+1)} + a_{20} . x_{(i+1,j,k+1)}^{(t+1)} + a_{21} . x_{(i+1,j-1,k)}^{(t+1)} + \\
& a_{22} . x_{(i+1,j-1,k+1)}^{(t+1)} + a_{23} . x_{(i+1,j,k)}^{(t+1)} + a_{24} . x_{(i+1,j+1,k)}^{(t+1)} + a_{25} . x_{(i+1,j+1,k-1)}^{(t+1)} + \\
& a_{26} . x_{(i+1,j+1,k+1)}^{(t+1)} \bmod p a_0, a_1, a_2, \ldots, a_{26} \in \mathbb{Z}_p - \{0\} \qquad (1)
\end{aligned}
$$

In this paper, we characterize the 3D NBCA determined according to local rules. We can use the following local rule to characterize NBCA. First, let's give our specially chosen local rule.

$$
x_{(i,j,k)}^{(t+1)} = g . x_{(i,j,k+1)}^{(t+1)} + r . x_{(i,j+1,k)}^{(t+1)} + u . x_{(i,j-1,k)}^{(t+1)} + w . x_{(i,j,k-1)}^{(t+1)} + y . x_{(i-1,j,k)}^{(t+1)} + z . x_{(i+1,j,k)}^{(t+1)} \bmod p
$$

$$
\left(g, r, u, w, y, z \in \mathbb{Z}_p - \{0\}\right) \qquad (2)
$$

There's the small matter of what the neighborhood of the cells at the other end of the cage should be. In most cases, we take the lattice to be large enough so that these cells are ignored and the lattice can be considered virtually infinite. However, in some cases, the extent of the lattice may be finite and there are various types of boundary conditions, but we will give only one of them. is expressed as follows.

A null boundary CA (NBCA) is one in which the extreme cells are connected to logic 0-state. If we characterize the 3D NBCA with the local rules in Eq. (2) we have obtained the rule matrix as follows:

$$\left(T_{RN}\right)_{mns\times mns}=\begin{pmatrix} K_n & Z_n & O_n & O_n & \cdots & O_n & O_n & O_n \\ B_n & K_n & Z_n & O_n & \cdots & O_n & O_n & O_n \\ O_n & B_n & K_n & Z_n & \cdots & O_n & O_n & O_n \\ O_n & O_n & B_n & K_n & \cdots & O_n & O_n & O_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ O_n & O_n & O_n & O_n & \cdots & K_n & Z_n & O_n \\ O_n & O_n & O_n & O_n & \cdots & B_n & K_n & Z_n \\ O_n & O_n & O_n & O_n & \cdots & O_n & B_n & K_n \end{pmatrix}_{mns\times mns} \quad (3)$$

K_n, Z_n, B_n, O_n are $n\times n$ block matrices where $n = ms$. The submatrices of the rule matrix are as follows:

$$K_n=\begin{pmatrix} S_s(u,r) & w.I_s & O_s & \dots & O_s & O_s \\ g.I_s & S_s(u,r) & w.I_s & \dots & O_s & O_s \\ O_s & g.I_s & S_s(u,r) & \dots & O_s & O_s \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O_s & O_s & O_s & \dots & S_s(u,r) & w.I_s \\ O_s & O_s & O_s & \dots & g.I_s & S_s(u,r) \end{pmatrix}_{ms\times ms}$$

$$Z_n=\begin{pmatrix} y.I_s & O_s & O_s & \dots & O_s & O_s \\ O_s & y.I_s & O_s & \dots & O_s & O_s \\ O_s & O_s & y.I_s & \dots & O_s & O_s \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O_s & O_s & O_s & \dots & y.I_s & O_s \\ O_s & O_s & O_s & \dots & O_s & y.I_s \end{pmatrix}_{ms\times ms}$$

$$B_n=\begin{pmatrix} z.I_s & O_s & O_s & \dots & O_s & O_s \\ O_s & z.I_s & O_s & \dots & O_s & O_s \\ O_s & O_s & z.I_s & \dots & O_s & O_s \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O_s & O_s & O_s & \dots & z.I_s & O_s \\ O_s & O_s & O_s & \dots & O_s & z.I_s \end{pmatrix}_{ms\times ms}$$

$$O_n=\begin{pmatrix} O_s & O_s & O_s & \dots & O_s & O_s \\ O_s & O_s & O_s & \dots & O_s & O_s \\ O_s & O_s & O_s & \dots & O_s & O_s \\ \dots & \dots & \dots & \dots & \dots & \dots \\ O_s & O_s & O_s & \dots & O_s & O_s \\ O_s & O_s & O_s & \dots & O_s & O_s \end{pmatrix}_{ms\times ms}$$

I_s is $s\times s$ an identity matrix. O_s is $s\times s$ zero matrix. $S_s(u,r)$ is as follows:

$$S_s(u,r)=\begin{pmatrix} 0 & r & 0 & 0 & \dots & 0 & 0 & 0 \\ u & 0 & r & 0 & \dots & 0 & 0 & 0 \\ 0 & u & 0 & r & \dots & 0 & 0 & 0 \\ 0 & 0 & u & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & r & 0 \\ 0 & 0 & 0 & 0 & \dots & u & 0 & r \\ 0 & 0 & 0 & 0 & \dots & 0 & u & 0 \end{pmatrix}_{s\times s}$$

Now, we give an example. We take $m=n=s=3$ .we take into account a configuration of size $3\times3\times3$. we study 3D-CA under null boundary conditions. we apply the local rule to all cells of $[X]^t_{3\times3\times3}$. We obtain the characteristic matrix $[T_{RN}]_{27\times27}$ as follows:

$$T_{RN}=\begin{pmatrix} S_3(u,r) & w.I_3 & O_3 & y.I_3 & O_3 & O_3 & O_3 & O_3 & O_3 \\ g.I_3 & S_3(u,r) & w.I_3 & O_3 & y.I_3 & O_3 & O_3 & O_3 & O_3 \\ O_3 & g.I_3 & S_3(u,r) & O_3 & O_3 & y.I_3 & O_3 & O_3 & O_3 \\ z.I_3 & O_3 & O_3 & S_3(u,r) & w.I_3 & O_3 & y.I_3 & O_3 & O_3 \\ O_3 & z.I_3 & O_3 & g.I_3 & S_3(u,r) & w.I_3 & O_3 & y.I_3 & O_3 \\ O_3 & O_3 & z.I_3 & O_3 & g.I_3 & S_3(u,r) & O_3 & O_3 & y.I_3 \\ O_3 & O_3 & O_3 & z.I_3 & O_3 & O_3 & S_3(u,r) & w.I_3 & O_3 \\ O_3 & O_3 & O_3 & O_3 & z.I_3 & O_3 & g.I_3 & S_3(u,r) & w.I_3 \\ O_3 & O_3 & O_3 & O_3 & O_3 & z.I_3 & O_3 & g.I_3 & S_3(u,r) \end{pmatrix}_{27\times27}$$

Now, we write the submatrices of our characteristic matrix.

$$S_3(u,r)=\begin{pmatrix}0 & r & 0\\ u & 0 & r\\ 0 & u & 0\end{pmatrix}_{3\times 3}$$

$$K_3=\begin{pmatrix}S_3(u,r) & w.I_3 & O_3\\ g.I_3 & S_3(u,r) & w.I_3\\ O_3 & g.I_3 & S_3(u,r)\end{pmatrix}$$

$$Z_3=\begin{pmatrix}y.I_3 & O_3 & O_3\\ O_3 & y.I_3 & O_3\\ O_3 & O_3 & y.I_3\end{pmatrix}$$

$$B_3=\begin{pmatrix}f.I_3 & O_3 & O_3\\ O_3 & f.I_3 & O_3\\ O_3 & O_3 & f.I_3\end{pmatrix}$$

Finally, we have obtained our block matrix as follows:

$$T_{RN}=\begin{pmatrix}K_3 & Z.I_3 & O_3\\ B.I_3 & K_3 & Z.I_3\\ O_3 & B.I_3 & K_3\end{pmatrix}_{27\times 27}.$$

REVERSIBILITY

Reversibility of three dimensional cellular automata is a very difficult problem. If our characteristic matrix is invertible, we say that cellular automata is reversible. K. Morita studied the features of reversibility of cellular automata[18]. Z. Çınkır et al. were interested in the problem of reversibility of cellular automata over the field $\mathbf{Z}_m$ under periodic boundary conditions [19]. H. Akın et al. were interested in the problem of reversibility of cellular automata under reflective boundary conditions over the field $\mathbf{Z}_m$[20]. Chang et al. calculated reversibility of multi dimensional cellular automata [21]. Now, we will give a very important algorithm to determine the reversibility of 3D-CA under the null boundary conditions.

Theorem: For $n,m,s\geq 2$ $(n,m,s\in \mathbf{Z}^+)$, characteristic matrix $(T_{RN})_{mns\times mns}$ be defined as in Eq. (3). The rank of Eq.(3) $(n-1)ms+rank(\lambda_{2n-1})$. where λ_{2n-1} satisfies the following recurrence relation:

$\lambda_1(S)=K$, $\lambda_0(S)=B$

$n\geq 2$,$\lambda_{2n-1}(S)=-Z^{-1}K\lambda_{2n-3}(S)+\lambda_{2n-4}(S)$,

$n\geq 3$,$\lambda_{2n-4}(S)=-Z^{-1}B\ \lambda_{2n-5}(S)$

Proof: we will apply the induction method over $n\geq 2.$ For $n=2$, we have obtained a block matrix as follows.

$$T=\begin{pmatrix}K & ZI\\ BI & K\end{pmatrix}.$$

Now, if we want to calculate the rank of our block matrix, R_1 and R_2 are the rows of the block matrices. Multiply the first row by $-Z^{-1}K$ and add it to the second row. We have obtained the block matrix as follows:

$$\begin{pmatrix}K & ZI\\ -Z^{-1}K^2+B=\lambda_3(S) & O\end{pmatrix}.$$

In this case rank of T depends on $-Z^{-1}K^2+B=\lambda_3(S)$ .For $n=3$, we have obtained the block matrix as follows:

$$T=\begin{pmatrix}K & ZI & O\\ BI & K & ZI\\ O & BI & K\end{pmatrix}.$$

Multiply the second row by $-Z^{-1}K$ and add it to the third row. We have obtained the block matrix as follows:

$$\begin{pmatrix}K & ZI & O\\ FI & K & ZI\\ -Z^{-1}BK=\lambda_2(S) & -Z^{-1}K^2+B=\lambda_3(S) & O\end{pmatrix}$$

If, we multiply the first row of the new matrix by $-Z^{-1}\lambda_2(S)$ and add it to the third row. we have the following block matrix.

$$\begin{pmatrix}K & ZI & O\\ FI & K & ZI\\ -Z^{-1}\lambda_3(S)K+\lambda_2(S)=\lambda_5(S) & O & O\end{pmatrix}.$$

In this case rank of T depends on $-Z^{-1}\lambda_3(S)K+\lambda_2(S)=\lambda_5(S)$.The $(n-1)th$ row of T is found. If we multiply the $(n-1)th$ row by $-Z^{-1}\lambda_1(S)$ and add it to the last row, the last row is obtained as $(0,0,\ldots,\lambda_2(S),\lambda_3(S),0)$. Now, if we multiply the $(n-2)th$ row of the new matrix by $-Z^{-1}\lambda_3(S)$ and add it to the last row.

In this manner, the last row is obtained as $\left(0,0,\ldots,\lambda_4\left(S\right),\lambda_5\left(S\right),0,0\right)$. If we multiply the second row of the new matrix by $-Z^{-1}\lambda_{2n-5}\left(S\right)$ and add it to the last row, the last row is obtained as $\left(\lambda_{2n-4}\left(S\right),\lambda 2n-3\left(S\right),0,0,\ldots,0\right)$. Finally, if we multiply the first row of the new matrix by $-E^{-1}\lambda_{2n-3}\left(S\right)$ and add it to the last row, the last row is obtained as $\left(\lambda_{2n-1}\left(S\right),0,0,0,\ldots,0\right)$. The new matrix is as follows:

$$\left(T_{RN}\right)_{mns\times mns}=\begin{pmatrix} K_n & Z_n & O_n & O_n & \cdots & O_n & O_n & O_n \\ B_n & K_n & Z_n & O_n & \cdots & O_n & O_n & O_n \\ O_n & B_n & K_n & Z_n & \cdots & O_n & O_n & O_n \\ O_n & O_n & B_n & K_n & \cdots & O_n & O_n & O_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ O_n & O_n & O_n & O_n & \cdots & K_n & Z_n & O_n \\ O_n & O_n & O_n & O_n & \cdots & B_n & K_n & Z_n \\ \lambda_{2n-1}(S) & O_n & O_n & O_n & \cdots & O_n & O_n & O_n \end{pmatrix}_{mns\times mns}$$

The proof is completed.

Example 1: we take $m=2$, $n=2$ and $s=2$. Let's calculate the rank of the characteristic matrix corresponding to our local rule under the null boundary condition. Firstly, we write our characteristic matrix as follows:

$$T_{RN}=\begin{pmatrix} 0 & r & w & 0 & y & 0 & 0 & 0 \\ u & 0 & 0 & w & 0 & y & 0 & 0 \\ g & 0 & 0 & r & 0 & 0 & y & 0 \\ 0 & g & u & 0 & 0 & 0 & 0 & y \\ z & 0 & 0 & 0 & 0 & r & w & 0 \\ 0 & z & 0 & 0 & u & 0 & 0 & w \\ 0 & 0 & z & 0 & g & 0 & 0 & r \\ 0 & 0 & 0 & z & 0 & g & u & 0 \end{pmatrix}_{8\times 8}$$

$$=\begin{pmatrix} S_2\left(u,r\right) & w.I_2 & y.I_2 & O_2 \\ g.I_2 & S_2\left(u,r\right) & O_2 & y.I_2 \\ z.I_2 & O_2 & S_2\left(u,r\right) & w.I_2 \\ O_2 & z.I_2 & g.I_2 & S_2\left(u,r\right) \end{pmatrix}_{8\times 8}$$

If we want to write the above characteristic matrix as a block matrix,

$$K_2=\begin{pmatrix} S_2\left(u,r\right) & w.I_2 \\ w.I_2 & S_2\left(u,r\right) \end{pmatrix}, Z_2=\begin{pmatrix} y.I_2 & O_2 \\ O_2 & y.I_2 \end{pmatrix}$$

$$B_2=\begin{pmatrix} z.I_2 & O_2 \\ O_2 & z.I_2 \end{pmatrix}.$$

Now let's write our matrix as a block matrix, $T_{RN}=\begin{pmatrix} K_2 & Z_2 \\ B_2 & K_2 \end{pmatrix}$.

İf we take $g=r=u=1$, w $=y=z=2$, $\left(g,r,u,w,y,z\in\mathbf{Z}_3\right)$

we obtained the equation as follows:

$$n\geq 2,\ \lambda_{2n-1}(S)=-Z^{-1}K\lambda_{2n-3}(S)+\lambda_{2n-4}(S),$$

$$\lambda_3(S)=-Z^{-1}K\lambda_1(S)+\lambda_0(S)$$
$$=-Z^{-1}K^2+B$$

$$\lambda_1(S)=K,\ \lambda_0(S)=B$$

$$\left(T_{RN}\right)_{8\times 8}=(n-1)ms+rank(\lambda_{2n-1})$$
$$=(2-1)2.2+rank(\lambda_3)$$
$$=4+4=8$$

The rank of an invertible matrix is equal to the order of the matrix, Thus characteristic matrix is invertible. So CA which corresponding to the characteristic matrix is reversible. If we take $g=r=u=w=y=z=1$,

We obtained the equation as follows:

$$\left(T_{RN}\right)_{8\times 8}=(n-1)ms+rank(\lambda_{2n-1})$$
$$=(2-1)2.2+rank(\lambda_3)$$
$$=4+2=6$$

Characteristic matric hasn't got full rank. So it isn't invertible. Thus CA which corresponding to the characteristic matrix isn't reversible.

APPLICATION

Now, we give an application of cellular automata.we find a minimal polynomial of T transition matrix. We obtain cycle and transient length of T transition matrix with the help of minimal polynomial. We will also determine the attractor points of our T transition matrix. Let's provide some definitions.

Definition 1 Let x be the initial configuration. For $t\in N$, $X_t=T^tX_0$. If there is no $t\in N$ such that $T^tX_0=0$, there will be $T^iX_0=T^jX_0$ for some number i,j since the number of all possible configurations is finite.

$(i,j)<(k,l) \Leftrightarrow i<k$ or $i=k, j<l$, then there is the smallest pair of numbers (t,c) that satisfy the condition $T^t X_0 = T^{t+c} X_0$ .The number $t \in N$ in the expression $T^t X_0 = T^{t+c} X_0$ is called the transition length on the X_0 configuration, and the number $c \in N$ is called the cycle length on the X_0 configuration.

Definition 2 The configuration that returns to itself in a certain time step is called an attractor point. In other words, in a cellular automata configuration tree, it is the configuration that can be seen as the root of the tree.

Definition 3 After starting with the initial configuration, the configurations reached until returning to the configuration itself in a certain time step are called the basin of the last obtained configuration, and this diagram obtained during this period is called the State-Transition Diagram.

Example2: If we take $m=2$, $n=2$ and $s=2$. we write our characteristic matrix as follows:

$$T_{RN} = \begin{pmatrix} 0 & r & w & 0 & y & 0 & 0 & 0 \\ u & 0 & 0 & w & 0 & y & 0 & 0 \\ g & 0 & 0 & r & 0 & 0 & y & 0 \\ 0 & g & u & 0 & 0 & 0 & 0 & y \\ z & 0 & 0 & 0 & 0 & r & w & 0 \\ 0 & z & 0 & 0 & u & 0 & 0 & w \\ 0 & 0 & z & 0 & g & 0 & 0 & r \\ 0 & 0 & 0 & z & 0 & g & u & 0 \end{pmatrix}_{8\times 8}$$

If we take $g=w=u=y=z=1$ and $r=0$, $(g,r,u,w,y,z \in \mathbf{Z}_3)$,
we have obtained the characteristic matrix as follows:

$$T_{RN} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}_{8\times 8}$$

Let's try to find the transition length, cycle length and basin of the attractive points, if any, by arbitrarily choosing any of the $2^8 = 256$ vectors over the feld $\mathbf{Z}_2$. Let's choose an arbitrary vector as follows.

$$F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{8\times 1} \rightarrow 1$$

$$T_{RN}.F = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}_{8\times 8} . \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}_{8\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{8\times 1} \rightarrow 20$$

$$T_{RN}.F = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}_{8\times 8} . \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{8\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}_{8\times 1} \rightarrow 0$$

If we continue similarly, let's write some results as follows:

State-Transition Diagram

$$1-21-40-60-64-84-105-125-135-147-174-186-198-210-239-251$$

$$\downarrow$$

$$20$$

$$\downarrow$$

$$0$$

$$2-22-43-63-67-87-106-126-132-144-173-185-197-209-236-248 \downarrow 41 \downarrow 0$$

$$3-23-42-62-66-86-107-127-133-145-172-184-196-208-237-249 \downarrow 61 \downarrow 0$$

$$4-16-45-57-69-81-108-120-130-150-171-191-195-215-234-254 \downarrow 65 \downarrow 0$$

$$5-17-44-56-68-80-109-121-131-151-170-190-194-214-235-255 \downarrow 85 \downarrow 0$$

$$6-18-47-59-71-83-110-122-128-148-169-189-193-213-232-252 \downarrow 104 \downarrow 0$$

$$7-19-46-58-70-82-111-123-129-149-168-188-192-212-233-253 \downarrow 124 \downarrow 0$$

$$8-28-33-53-73-93-96-116-142-154-167-179-207-219-230-242 \downarrow 134 \downarrow 0$$

$$9-29-32-52-72-92-97-117-143-155-166-178-198-206-218-231-243 \downarrow 146 \downarrow 0$$

$$10-30-35-55-75-95-98-118-140-152-165-177-205-217-228-240 \downarrow 175 \downarrow 0$$

$$11-31-34-54-74-91-99-119-141-153-164-176-204-216-229-241 \downarrow 187 \downarrow 0$$

$$12-24-37-49-77-89-100-112-138-158-163-183-203-223-226-246 \downarrow 199 \downarrow 0$$

$$13-25-36-48-76-88-101-113-139-159-162-182-202-222-227-247 \downarrow 211 \downarrow 0$$

$$14-26-39-51-79-91-102-114-136-156-161-181-201-221-224-244-251 \downarrow 238 \downarrow 0$$

$$15-27-38-50-78-90-103-115-137-157-160-180-200-220-225-245 \downarrow 250 \downarrow 0$$

If we continue in this way, each element will go to zero after the second pass.

$$T_{RN} = \begin{pmatrix} 0&0&1&0&1&0&0&0\\ 1&0&0&1&0&1&0&0\\ 1&0&0&0&0&0&1&0\\ 0&1&1&0&0&0&0&1\\ 1&0&0&0&0&0&1&0\\ 0&1&0&0&1&0&0&1\\ 0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&1&0 \end{pmatrix}_{8\times 8}$$

The minimal polynomial is x^8. The transition length is 2 and the cycle length is 0. Also, the attractive points are 0 and all configurations are basins of 0.

The characteristic polynomial is x^8. If, we examine the kernel of our matrix above, we obtained as follows:

$$K = \{(10000110),(00010100),(01101000),(00101001)\}$$

.If, we obtain the elements of the vector space from here, we have obtaned a result as follows.

$$V = \begin{Bmatrix} 00000000, 10000110(134), 00010100(20), 01101000(104), \\ 00101001(41), 10010010(146), 11101110(238), 10101111(175), \\ 01111100(124), 01000001(65), 00111101(61), 11111010(250), \\ 01010101(85), 11000111(199), 10111011(189), 11010011(211) \end{Bmatrix}$$

If we look carefully, we see that each of the elements corresponds to a root.

CONCLUSION

First, the characteristic matrix of three-dimensional cellular automata was obtained under the null boundary condition. Then, the invertibility of our characteristic matrix was examined with the help of a theorem. Thanks to this theorem, we were able to obtain information about the invertibility of very large matrices. We show that if our characteristic matrix is invertible, its corresponding cellular transformations are also reversible. Finally, we gave some applications of three dimensional cellular automata under null boundary conditions.

REFERENCES

[1] Von N.J. The theory of self-reproducing automata, (Edited by A. W. Burks) University of Illinois Press, Urbana, 1966.

[2] Hedlund G.A. Endomorphisms and automorphisms of full shift dynamical system, Mathematical Systems Theory. 3 320- 375, 1969.

[3] Wolfram S. Statistical mechanics of cellular automata, Reviews of Modern Physics. 55 601-644, 1983.

[4] Pries W., Thanaılakıs A., Card H.C. Group properties of cellular automata and Vlsı applications, IEEE Transactions on Computers. 35 1013-1024,1986.

[5] Inokuchı S. On behaviors of cellular automata with rule 156, Bulletin of Informatics and Cybernetics. 30 121-131, 1998.

[6] Wolfram S., Packard N.H. Two dimensional cellular automata, Journal of Statistical Physics. 38 5-6, 1985.

[7] Das A.K., Chaudhurı P.P. Vector space theoretic analysis of additive cellular automata and its applications for pseudo exhaustive test pattern generation, IEEE Transactions on Computers. 42 340-352, 1993.

[8] Khan A.R., Choudhury P.P., Dihidar K., Mitra S., Sarkar P. VLSI architecture of a cellular automata, Computers Mathematics with Applications. 33 79-94, 997.

[9] Chattopadhyay P., Choudhury P.P., Dihidar K. Characte-rization of a particular hybrid transformation of two-dimensional cellular automata, Computers Mathematics with Applications. 38 207-216, 1999.

[10] Dihidar K., Choudhury P. P. Matrix algebraic formulae concerning some exceptional rules of two dimensional cellular automata, Information Sciences. 165 91-101, 2004.

[11] Siap I., Akin H., Sah F. Characterization of two dimensional cellular automata over ternary fields, Journal of the Franklin Institute. 348 1258-1275, 2011.

[12] Tsalides P., Hicks P.J., York T.A. Three dimensional cellular automata and VLSI applications, IEE Proceedıngs. 136 490-495,1989.

[13] Gerling R.W. Classification of three-dimensional cellular automata, Physcia A. 162 187-195, 1989.

[14] Hemmingsson J.A. Totalistic three dimensional cellular automata withquasiperiodic behaviour, Physica A. Statistical Mechanics and its Applications. 183 255-261, 1992.

[15] Brown S.G.R, Bruce N.B. Three-dimensional cellular automaton models of microstructural evolution during solidification, Journal of Materials Science. 30 1144-1150, 1995.

[16] Leubeck E.G., De Gunst, M.C.M. A sterological method for the analysis of cellular lesions in tissue sections using three-dimensional cellular automata, Mathematical and Computer Modelling. 33 1387-1400, 2001.

[17] Agapie A. Simple form of the stationary distribution for 3D cellular automata in a special case, Physica A. 389 2495-2499, 2010.

[18] Morita K. Reversible computing and cellular automata-A survey, Theorical Computer Science. 395 101-131, 2008.

[19] Cinkir Z., Akın H., Siap İ. Reversibility of 1D cellular automata with periodic boundary over finite fields Z_p, Journal of Statistical Physics. 143 807-823, 2011.

[20] Akın H., Siap İ., Sah F. On 1D reversible cellular automata with reflective boundary over the prime field of order p, International Journal of Modern Physics C. 23 1-13, 2012.

[21] Chang C.H., Su J.Y, Akın H., Sah F. Reversibility problem of multidimensional finite cellular automata, Journal of Statistical Physics. 168 208-231, 2017.