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SOME VARIABLE EXPONENT BOUNDEDNESS AND COMMUTATORS ESTIMATES FOR FRACTIONAL ROUGH HARDY OPERATORS ON CENTRAL MORREY SPACE

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Abstract. In this article, we study the boundedness of the fractional Rough Hardy operator and its adjoint operators on the central Morrey space with a variable exponent. We also establish the same boundedness for their commutators when the symbol functions are on the λ-central BMO space with a variable exponent.

1. Introduction

The Hardy operator is a key operator in mathematical analysis and has been extensively used in recent times. In 1920, Hardy [1] defined an operator for a locally integrable $f \in \mathbb{R}^n$.

$$Hf(x) = \frac{1}{x}\int_0^x f(t)dt, \quad x > 0. \tag{1}$$

He also established a sharp inequality for it. Later, Faris (as seen in [2]) formulated the n-dimensional form of (1). Grafakos and Christ [3] determined the exact norm value on the Lebesgue space for the n-dimensional Hardy operator. Additionally, the Hardy integral inequality has garnered significant attention. Alternate proofs, variants, applications, and generalizations of this inequality were explored in various articles. Some of these inequalities are discussed in [3, 4]. Furthermore, in [5], the

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authors introduced the n-dimensional fractional Hardy operator as follows:

$$H_\beta f(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|\le|z|} f(t)dt, \quad H_\beta^* f(z) = \int_{|t|>|z|} \frac{f(t)}{|t|^{n-\beta}}dt, \quad z \in \mathbb{R}^n \setminus \{0\}, \quad (2)$$

where $|z| = \sqrt{\sum_{i=1}^n z_i^2}$. Moreover, commutators of these operators are defined as follows:

$$[b, H]f = bHf - H(bf), \qquad [b, H^*]f = bH^*f - H^*(bf), \qquad (3)$$

where b is a locally integrable function. Fu, Liu, and Wang [5] established boundedness for the commutator of the n-dimensional fractional Hardy operator. Firstly, Ren and Tao [6] provided the definition of the n-dimensional rough Hardy operator and its adjoint operator as follows:

$$H_{\Omega,\beta} f(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|\le|z|} \Omega(z-t) f(t)dt,$$
$$H_{\Omega,\beta}^* f(z) = \int_{|t|>|z|} \Omega(z-t)\frac{f(t)}{|t|^{n-\beta}}dt,$$

where $\Omega \in L^s(S^{n-1})$, $1 < s \le \infty$, and is homogeneous of degree zero. Commutators of rough hardy operators are defined as:

$$H_{\Omega,\beta}^b f(z) = \frac{1}{|z|^{n-\beta}} \int_{|t|\le|z|} (b(z)-b(t))\Omega(z-t) f(t)dt,$$
$$H_{\Omega,\beta}^{*,b} f(z) = \int_{|t|>|z|} (b(z)-b(t))\Omega(z-t)\frac{f(t)}{|t|^{n-\beta}}dt,$$

which were used by Wei, Zhen, and Wang [7] to develop estimates for the commutator on the Herz space.

It's important to highlight that the function space featuring varying exponents plays a pivotal role in both harmonic analysis and applied mathematics. Orlicz [8] initiated the theory of variable exponent Lebesgue space for the first time. Musieliak Orlicz spaces are defined in [9]. Sobolev and Lebesgue spaces with integrability exponents have been thoroughly examined, as seen in [10–12] and the references therein. Following that, work on variable Lebesgue spaces began, along with the exploration of the boundedness of numerous operators, including the maximum operator on Lebesgue spaces $L^{p(\cdot)}$ [13]. At the same time, the central bounded mean oscillation space, λ-central Morrey space, and similar function spaces offer compelling practical uses through the exploration of operator estimates in tandem with singular integral operators, as detailed in [14, 26]. The analysis of Morrey space can be traced back to Morrey's [15] work on the regularity of solutions of partial differential equations. In [14], the authors defined λ-central Morrey space and central bounded mean oscillation (BMO) space, which are generalized based on bounded central mean oscillation. λ-central Morrey space and central BMO space have impressive applications in analyzing the boundedness of many operators; see, for example, [16,17]. Furthermore, with substantial applications in image processing [18], electrorheological fluid [19], and partial differential equations [20], variable exponent functions have garnered significant attention. Following Kováčik's [21] seminal work, such theories have made significant advances. For the first time, the idea of non-homogeneous variable exponent central Morrey spaces was formulated by Mizuta [22]. In the recent past, Wang et al. defined variable exponent central BMO and established the boundedness of some operators in [23], which was later extended by Zunwei Fu [24] to variable exponent λ-central Morrey space and central BMO space.

In [25, 27], the authors obtained results for the boundedness of several integral operators on function spaces with variable exponents. Additionally, some authors proved results for the boundedness of multilinear integral operators and their commutators as well, as seen in [28, 29].

Motivated by [24, 30, 31], we are going to examine the boundedness of fractional Rough Hardy operators, as well as the boundedness of commutators, on the variable exponent λ-central Morrey space.

Let's elucidate the structure of this paper. In Section 2, we will revisit certain definitions, lemmas, and propositions in the context of variable exponent Lebesgue space. In the third section of this article, we consider the boundedness of the fractional Rough Hardy operator and its adjoint operator on the central Morrey space with a variable exponent, respectively. In Section 4, we consider the boundedness of commutators of the fractional Rough Hardy operator and its adjoint operator on the λ-central BMO space with a variable exponent, respectively..

Additionally, $|B|$ and χ_B represent the Lebesgue measure and the characteristic function of a measurable set $B \subset \mathbb{R}^n$, respectively. When we write $g \approx h$, we are indicating the existence of constants $c_1, c_2 > 0$ such that $c_1 g \leq h \leq c_2 g$. Here, $B_k = B(0, 2^k) = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$ and the characteristics function $\chi_k = \chi_{A_k}$ for $k \in \mathbb{Z}$ (see [35]).

2. Function spaces along variable exponent

First of all, we will provide some basic definitions and notations concerning Lebesgue spaces over variable exponents. Let's consider an open set $E \subseteq \mathbb{R}^n$, and let $q(\cdot) : E \to [1, \infty)$ be a measurable function. We denote the conjugate exponent as $q'(\cdot)$, which is defined as

$$\frac{1}{q'(\cdot)} + \frac{1}{q(\cdot)} = 1$$

The set $P(E)$ consists of all pairs of exponents $(q(\cdot), q'(\cdot))$ that satisfy

$$1 < q^- = ess\inf\{q(x) : x \in E\}$$

$$q^+ = ess\sup\{q(x) : x \in E\} < \infty.$$

We use $L^{q(\cdot)}$ to represent the space of all measurable functions f such that for some $\zeta > 0$,

$$\int_E \left(\frac{|f(x)|}{\zeta}\right)^{q(x)} dx < \infty,$$

This space is a Banach function space equipped with the Luxemburg norm:

$$||f||_{L^{q(\cdot)}(E)} = \inf\left\{\zeta > 0 : \int_E \left(\frac{|f(x)|}{\zeta}\right)^{q(x)} dx \leq 1\right\}.$$

We define $L^{q(\cdot)}_{\text{loc}}(\delta)$ as the set of functions f belonging to $L^{q(\cdot)}(E)$ for any compact subset $E \subset \delta$:

$$L^{q(\cdot)}_{loc}(\delta) = \left\{f : f \in L^{q(\cdot)}(E) \ for\ every\ compact\ subset\ E \subset \delta\right\}.$$

Here, M denotes the Hardy-Littlewood maximal operator acting on a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and is defined by

$$Mf = \sup_{r>0} \frac{1}{|B_r|} \int_{B_r} |f| dy$$

where $B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}$ is the ball centered at x with radius r.

$\mathfrak{B}$ is a set containing $q(\cdot) \in \mathbb{R}^n$ that satisfy the condition that M is bounded on $L^{q(\cdot)}$. Now, we express a few properties of variable exponents associated with the class $\mathfrak{B}(E)$. Neugebauer, Cruz Uribe, and Fiorenza [12], as well as Nakvinda [32], established the inequalities presented in the proposition below.

Proposition 1. *[32] Let E be an open set, and let $q(\cdot) \in P(E)$ satisfy the requirements given below:*

$$|q(y) - q(x)| \leq \frac{-C}{ln(|y - x|)},\ \frac{1}{2} \geq |y - x|, \tag{4}$$

$$|q(y) - q(x)| \leq \frac{C}{ln(|x| + e)},\ |x| \leq |y|, \tag{5}$$

then $q(\cdot) \in \mathfrak{B}(E)$, where C stands for a positive constant independent of y and x.

Lemma 1. *[21] (Generalized Hölder inequality) Let $q(\cdot)$, $q_1(\cdot)$, and $q_2(\cdot)$ be in $P(E)$.*

- *If $h \in L^{q(\cdot)}$ and $f \in L^{q'(\cdot)}$, then*

$$\int_E |h(x)f(x)|dx \leq r_q \|h\|_{L^{q(\cdot)}} \|f\|_{L^{q'(\cdot)}},$$

where $r_q = 1 + \frac{1}{q^-} - \frac{1}{q^+}$.

- *If $h \in L^{q_1(\cdot)}(E)$ and $f \in L^{q_2(\cdot)}(E)$, and $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$, then*

$$\|hf|_{L^{q(\cdot)}} \leq r_{q,q_1} \|h\|_{L^{q_1(\cdot)}} \|f\|_{L^{q_2(\cdot)}},$$

where $r_{q,q_1} = \left(1 + \frac{1}{(q_1)^-} - \frac{1}{(q_1)^+}\right)^{1/q^-}$.

Lemma 2. *[34] Assuming that $q(\cdot) \in \mathfrak{B}$ for all measurable subsets I of S, and $I \subset \mathbb{R}^n$, there exists a constant $0 < \delta < 1$ and a constant C such that*

$$\frac{\|\chi_I\|_{L^{q(\cdot)}}}{\|\chi_S\|_{L^{q(\cdot)}}} \leq C\left(\frac{|I|}{|S|}\right)^{\delta}.$$

$$\frac{\|\chi_S\|_{L^{q(\cdot)}}}{\|\chi_I\|_{L^{q(\cdot)}}} \leq C\frac{|S|}{|I|}.$$

Remark 1. *Suppose that $q(\cdot) \in P(\mathbb{R}^n)$ and satisfies conditions (4) and (5) in Proposition 1. Then so does $q^{'}(\cdot)$. Generally, we can see that both $q(\cdot)$ and $q^{'}(\cdot)$ belong to $\mathfrak{B}(\mathbb{R}^n)$ based on Proposition 1. Therefore, by virtue of Lemma 2, we can consider a constant $\delta_1 \in (0, \frac{1}{(q_2)_+})$ such that*

$$\frac{\|\chi_I\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|I|}{|S|}\right)^{\delta_1} \tag{6}$$

which holds for all balls S in $\mathbb{R}^n$ and for $I \subset S$. If $q_1(\cdot) \in P$, using Lemma 2, we can take constant $\delta_3 \in (0, \frac{1}{(q_1^{'})+})$ such that

$$\frac{\|\chi_I\|_{L^{q_1^{'}(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{q_1^{'}(\cdot)}(\mathbb{R}^n)}} \leq C\left(\frac{|I|}{|S|}\right)^{\delta_3}. \tag{7}$$

Lemma 3. *[34] Let $q(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$. The following inequality holds for all balls $B \subset \mathbb{R}^n$ and a positive constant C:*

$$C^{-1} \leq \frac{1}{|B|}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{q^{'}(\cdot)}(\mathbb{R}^n)} \leq C.$$

Definition 1. *[35] Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define*

$$\|b\|_{BMO} = \sup_B \frac{1}{B}\int_B |b(x) - \underset{B}{Avg}\ b|dx,$$

where the function b is considered to have bounded mean oscillation if $\|b\|_{BMO} < \infty$.

Lemma 4. *[36] Assuming that $q(\cdot) \in P(\mathbb{R}^n)$, for $b \in BMO$, and for $j, i \in \mathbb{Z}$ with $j > i$, we have the following inequalities:*

$$C^{-1}\|b\|_{BMO} \leq \sup_{B:Ball} \frac{1}{\|\chi_B\|_{L^{q(\cdot)}}}\|(b - b_B)\chi_B\|_{L^{q(\cdot)}} \leq C\|b\|_{BMO} \tag{8}$$

$$\|(b - b_{B_i})\chi_{B_j}\|_{L^{q(\cdot)}} \leq C(j - i)\|b\|_{BMO}\|\chi_{B_j}\|_{L^{q(\cdot)}} \tag{9}$$

Definition 2. *[24] Let $\lambda \in \mathbb{R}$ and $q(\cdot) \in P(\mathbb{R}^n)$. Then the central Morrey space for the variable exponent $\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)$ is defined as*

$$\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n) = \left\{f \in L^{q(\cdot)}_{Loc}(\mathbb{R}^n) : \|f\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty\right\},$$

where

$$\|f\|_{\dot{B}^{q(\cdot),\lambda}(\mathbb{R}^n)} = \sup_{R>0} \frac{\|f\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}$$

Definition 3. *[24] Let $\lambda < \frac{1}{n}$ and $q(\cdot) \in P(\mathbb{R}^n)$. The variable exponent λ-central BMO space $CBMO^{q(\cdot),\lambda}$ is defined as*

$$CBMO^{q(\cdot),\lambda} = \left\{ f \in L^{q(\cdot)}_{Loc}(\mathbb{R}^n) : \|f\|_{CBMO^{q(\cdot),\lambda}(\mathbb{R}^n)} < \infty \right\}$$

where

$$\|f\|_{CBMO^{q(\cdot),\lambda}} = \sup_{R>0} \frac{\|(f - f_{B(0,R)})\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{|B(0,R)|^{\lambda}\|\chi_{B(0,R)}\|_{L^{q(\cdot)}(\mathbb{R}^n)}}$$

By using the boundedness results of the integral operator I_β, we will demonstrate the boundedness of the fractional rough Hardy operator:

$$I_\beta(f)(t) = \int_{\mathbb{R}^n} \frac{f(z)}{|t-z|^{n-\beta}} dz.$$

Proposition 2. *[37] Let $q_1(\cdot) \in P(\mathbb{R}^n)$, $0 < \beta < \frac{n}{(q_1)_+}$, and define $q_2(\cdot)$ as*

$$\frac{1}{q_2(\cdot)} = \frac{1}{q_1(\cdot)} - \frac{\beta}{n}.$$

Then,

$$\|I_\beta f\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.$$

Lemma 5. *[30]Assuming that β, $q_1(\cdot)$, and $q_2(\cdot)$ are defined similarly to proposition 2, we have*

$$\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C 2^{-j\beta}\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.$$

3. Boundedness Of Fractional Rough Hardy Operators

Theorem 1. *Assume that $\Omega \in L^s(S^{n-1})$, where $\frac{n}{n-1} < s$. Let $q_1(\cdot), p(\cdot), q_2(\cdot) \in P(\mathbb{R}^n)$ satisfy the inequalities (4) and (5) in proposition 1. Define the variable exponent $p(\cdot)$ by*

$$\frac{1}{q_1(\cdot)} = \frac{1}{p(\cdot)} + \frac{\beta}{n}.$$

Let λ_1 satisfy the following condition:

When $\frac{1}{q_2(\cdot)} = \frac{1}{q_1'(\cdot)} - \frac{1}{s}$, there exist $\lambda_1 > -\frac{\beta}{n}$ and $\lambda = \lambda_1 + \frac{\beta}{n}$. If $\delta_3 - \frac{1}{s} + \lambda + \delta_1 > 0$, then the fractional rough Hardy operator is bounded from $\dot{B}^{q_1(\cdot),\lambda_1}$ to $\dot{B}^{p(\cdot),\lambda}$, and the following inequality holds:

$$||H_{\beta,\Omega} f||_{\dot{B}^{p(\cdot),\lambda}} \le C||f||_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

Proof.

$$|H_{\beta,\Omega}f(x)\cdot\chi_k(x)| \le \frac{1}{|x|^{n-\beta}}\int_{B_k}|f(t)||\Omega(x-t)|dt\cdot\chi_k(x)$$
$$\le C2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\chi_k(x).$$

Using $\frac{1}{q_2(\cdot)}+\frac{1}{s}=\frac{1}{q_1'(\cdot)}$

$$|H_{\beta,\Omega}f(x)\cdot\chi_k(x)| \le C2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}}\cdot\chi_k(x).$$

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}\|\chi_j\|_{L^{q_2(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}.$$

Hence we have

$$\|\chi_k\|_{L^{q_2(\cdot)}} \approx |B_k|^{\frac{1}{q_2(\cdot)}} \approx |B_k|^{\frac{1}{q_1'(\cdot)}-\frac{1}{s}} \approx |B_k|^{-\frac{1}{s}}\|\chi_k\|_{L^{q_1'(\cdot)}}$$

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C2^{-k(n-\beta)}\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|_{L^{p(\cdot)}}. \tag{10}$$

Based on Proposition 2, we have

$$I_\beta(\chi_{B_k})(x) \ge C2^{k\beta}\chi_{B_k}(x)$$
$$\chi_{B_k}(x) \le C2^{-k\beta}I_\beta(\chi_{B_k})(x)$$

$$\begin{aligned}\|\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\le C2^{-k\beta}\|I_\beta\chi_{B_k}\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\le C2^{-k\beta}\|\chi_{B_k}\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\\ &\le C2^{k(n-\beta)}\|\chi_{B_k}\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{11}$$

Using inequality (11) in (10), we obtain

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\sum_{j=-\infty}^{k}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}}. \tag{12}$$

Using condition (7), we get

$$\|H_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\sum_{j=-\infty}^{k}2^{(j-k)n\delta_3}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\chi_j\|_{L^{s}(\mathbb{R}^n)}|B_j|^{-\frac{1}{s}}. \tag{13}$$

For $t \in C_j$, $x \in C_k$, and $j \leq k$, we have $0 \leq |x-t| \leq |x| + 2^j \leq 2.2^k$, and

$$\int_{C_j} |\Omega(x-t)|^s dt \leq \int_0^{2^{k+1}} \int_{s^{n-1}} |\Omega(x^{'})|^s d\sigma(x^{'}) r^{n-1} dr \leq C 2^{kn}$$

$$\begin{aligned}\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C \sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_3 - \frac{n}{s})} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \\ &\leq C \sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_3 - \frac{n}{s})} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B_j|^{\lambda_1} ||\chi_j||_{L^{q_1(\cdot)}}.\end{aligned} \tag{14}$$

$$\|\chi_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \approx |B|^{\frac{1}{q_1(\cdot)}} \approx |B|^{\frac{1}{p(\cdot)} + \frac{\beta}{n}} \approx |B|^{\frac{\beta}{n}} \|\chi_j\|_{L^{p(\cdot)}(\mathbb{R}^n)}$$

$$\begin{aligned}\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} &\leq C \sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_3 - \frac{n}{s})} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B_j|^{\lambda_1 + \frac{\beta}{n}} ||\chi_j||_{L^{p(\cdot)}} \\ \leq C \sum_{j=-\infty}^{k} 2^{(j-k)(n\delta_3 - \frac{n}{s})} &\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} \frac{|B_j|^{\lambda}}{|B_k|^{\lambda}} |B_k|^{\lambda} \frac{||\chi_j||_{L^{p(\cdot)}}}{||\chi_k||_{L^{p(\cdot)}}} ||\chi_k||_{L^{p(\cdot)}}.\end{aligned} \tag{15}$$

Using inequality (6), we have

$$\|H_{\beta,\Omega} f \cdot \chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leq C \sum_{j=-\infty}^{k} 2^{n(j-k)(\delta_3 - \frac{1}{s} + \lambda + \delta_1)} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)} |B_k|^{\lambda} ||\chi_k||_{L^{p(\cdot)}}. \tag{16}$$

Since $\delta_3 - \frac{1}{s} + \lambda + \delta_1 > 0$, we obtain

$$\|H_{\beta,\Omega} f\|_{\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)} \leq C \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}. \tag{17}$$

Theorem 2. *Let $p(\cdot), q_1(\cdot)$, $q_2(\cdot)$ and β be defined the same as in Theorem 1, and $\Omega \in L^s(S^{n-1})$. If $\lambda = \lambda_1 + \frac{\beta}{n}$ and $\lambda < \frac{1}{s} - \frac{\beta}{n} - 1$, then*

$$\|H^*_{\beta,\Omega} f\|_{\dot{B}^{p(\cdot),\lambda}} \leq C \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

Proof.

$$\begin{aligned}|H^*_{\beta,\Omega} f(x) \cdot \chi_k| &\leq \int_{\mathbb{R}^n \setminus B_k} |f(t)\Omega(x-t)||t|^{\beta-n} dt \cdot \chi_k(x) \\ &\leq C \sum_{j=k+1}^{\infty} 2^{j(\beta-n)} \|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)} \|\Omega(x-t)\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)} \cdot \chi_k(x).\end{aligned}$$

By using $\frac{1}{q_2(\cdot)}+\frac{1}{s}=\frac{1}{q_1'(\cdot)}$

$$\begin{aligned}&\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\le C\sum_{j=k+1}^{\infty}2^{j(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Using inequality (11), we have

$$\begin{aligned}&\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\le C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}\|\chi_j\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

As we know, we obtain

$$\|\chi_j\|_{L^{q_2(\cdot)}}\approx|B_j|^{\frac{1}{q_2(\cdot)}}\approx|B_j|^{\frac{1}{q_1'(\cdot)}-\frac{1}{s}}\approx|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}}$$

$$\begin{aligned}&\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\le C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}|B_j|^{-\frac{1}{s}}\|\chi_j\|_{L^{q_1'(\cdot)}(\mathbb{R}^n)}\|\chi_k\|^{-1}_{L^{q_1'(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Now by using condition (7),

$$\begin{aligned}&\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\le C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-n+n)}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}\|\Omega(x-t)\|_{L^s}|B_j|^{-\frac{1}{s}}.\end{aligned}$$

For further calculations following Theorem 1, we get

$$\begin{aligned}&\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\le C\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-\frac{n}{s})}\|f_j\|_{L^{q_1(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

Hence we have

$$\|H^*_{\beta,\Omega}f\cdot\chi_k\|_{\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)}\le C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}\sum_{j=k+1}^{\infty}2^{(j-k)(\beta-\frac{n}{s}+n\lambda+n)}.$$

By utilizing $\lambda<\frac{1}{S}-\frac{\beta}{n}-1$, we obtain the desired result:

$$\|H^*_{\beta,\Omega}f\|_{\dot{B}^{p(\cdot),\lambda}(\mathbb{R}^n)}\le C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}(\mathbb{R}^n)}.$$

4. Boundedness Commutators Of Fractional Rough Hardy Operators

Theorem 3. *Let $0 < \beta < n, \Omega \in L^s(S^{n-1})$, and $\frac{n}{n-1} < s$. Suppose that $q_1(\cdot), p(\cdot), q(\cdot) \in P(\mathbb{R}^n)$ satisfy conditions (4) and (5) in Proposition 1, and let the variable exponent $q_2(\cdot)$ be define by*

$$\frac{1}{q_1(\cdot)} = \frac{1}{q_2(\cdot)} - \frac{1}{q(\cdot)} + \frac{\beta}{n}.$$

Let λ_1 satisfy the following condition:

When $\frac{1}{q(\cdot)} = \frac{1}{q_1'(\cdot)} - \frac{1}{s}$, there exists $\lambda_1 > -\lambda - \frac{\beta}{n}$ such that $\lambda_2 = \lambda_1 + \lambda + \frac{\beta}{n}$. If $b \in \|b\|_{CBMO^{q(\cdot),\lambda}}$ and $\delta_3 - \frac{1}{s} + \lambda_2 + \delta_1 > 0$, then the following inequality holds:

$$||[b, H_{\beta,\Omega}]f||_{\dot{B}^{q_2(\cdot),\lambda_2}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}||f||_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

Proof.

$$\begin{aligned}
|[b, H_{\beta,\Omega}]f(x) \cdot \chi_B(x)| &\leq \frac{1}{|x|^{n-\beta}} \int_{B(0,|x|)} |f(t)(b(x) - b(t))\Omega(x-t)|dt \cdot \chi_B(x) \\
&\leq \frac{1}{|x|^{n-\beta}} \int_{B(0,|x|)} |f(t)(b(x) - b_B)\Omega(x-t)|dt \cdot \chi_B(x) \\
&+ \frac{1}{|x|^{n-\beta}} \int_{B(0,|x|)} |f(t)(b(t) - b_B)\Omega(x-t)|dt \cdot \chi_B(x) \\
&= A_1 + A_2.
\end{aligned}$$

First, we estimate A_1. Let $\frac{1}{p(x)} = \frac{1}{q_1(x)} - \frac{\beta}{n}$. This implies $\frac{1}{q_2(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}$

$$A_1 = |(b(x) - b_B)\chi_B(x)||H_{\beta,\Omega}f(x)|,$$

$$\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} = \|(b(x) - b_B)\chi_B(x)H_{\beta,\Omega}f\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

Let $\frac{1}{p(\cdot)} = \frac{1}{q_2(\cdot)} - \frac{1}{q(\cdot)}$, and use Hölder inequality ($\frac{1}{q_2(\cdot)} = \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)}$)

$$\begin{aligned}
\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C\|H_{\beta,\Omega}f\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|(b(x) - b_B)\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&= C\|H_{\beta,\Omega}f\|_{\dot{B}^{\mu,p(\cdot)}}|B|^{\mu}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)},
\end{aligned}$$

Given that $\mu = \lambda_1 + \frac{\beta}{n}$, and using the result of Theorem 1

$$\|A_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda_2}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

Next,

$$
\begin{aligned}
A_2 &= \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |f(t)(b(t)-b_B)\Omega(x-t)|dt \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\
&\le \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |f(t)(b(t)-b_{2^jB})\Omega(x-t)|dt \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\
&+ \sum_{k=-\infty}^{0} \frac{1}{|x|^{n-\beta}} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |f(t)(b_B-b_{2^jB})\Omega(x-t)|dt \cdot \chi_{2^kB\backslash 2^{k-1}B}(x) \\
&= A_{21} + A_{22}
\end{aligned}
$$

$$
A_{21} = \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \int_{2^jB\backslash 2^{j-1}B} |f(t)(b(t)-b_{2^jB})\Omega(x-t)|dt \cdot \chi_{2^kB\backslash 2^{k-1}B}(x)
$$

given that $\frac{1}{q(\cdot)} + \frac{1}{q_1(\cdot)} + \frac{1}{s} = 1$, we can now use Holder's inequality

$$
\begin{aligned}
A_{21} &\le C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|(b(t)-b_{2^jB})\chi_{2^jB}\|_{L^{q(\cdot)}} \\
&\quad \times \|f\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s} \\
&= C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=-\infty}^{k} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^jB|^{\lambda} \|\chi_{2^jB}\|_{L^{q(\cdot)}} \\
&\quad \times \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |2^jB|^{\lambda_1} \|\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s} \\
&= C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \\
&\quad \times \sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1} |2^jB|^{\frac{1}{s}+\frac{1}{q_1(\cdot)}+\frac{1}{q(\cdot)}} \\
&= C \sum_{k=-\infty}^{0} |2^kB|^{\frac{\beta}{n}-1} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \\
&\quad \times \sum_{j=-\infty}^{k} |2^j|^{\lambda+\lambda_1+1} |B|^{\lambda+\lambda_1+1} \\
&= C \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^k|^{\frac{\beta}{n}+\lambda+\lambda_1} \chi_{2^kB\backslash 2^{k-1}B}(x) |B|^{\lambda+\lambda_1+\frac{\beta}{n}}
\end{aligned}
$$

$$\begin{aligned}\|A_{21}\|_{L^{q_2(\cdot)}} &\le C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|2^k|^{\frac{\beta}{n}+\lambda+\lambda_1}\|\chi_{2^kB}\|_{L^{q_2(\cdot)}}|B|^{\lambda+\lambda_1+\frac{\beta}{n}}\\ &= C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\sum_{k=-\infty}^{0}|2^k|^{\frac{\beta}{n}+\lambda+\lambda_1}|2^kB|^{\frac{1}{q_2(\cdot)}}|B|^{\lambda+\lambda_1+\frac{\beta}{n}}\\ &= C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}|B|^{\frac{1}{q_2(\cdot)}}|B|^{\lambda_2}\sum_{k=-\infty}^{0}|2|^{k(\lambda_2+\frac{1}{q_2(\cdot)})}\end{aligned}$$

$$\|A_{21}\|_{L^{q_2(\cdot)}} \le C\|b\|_{CBMO^{q(\cdot),\lambda}}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\|\chi_B\|_{L^{q_2(\cdot)}}|B|^{\lambda_2}$$

$$A_{22} = \sum_{k=-\infty}^{0}|2^kB|^{\frac{\beta}{n}-1}\sum_{j=-\infty}^{k}\int_{2^jB\setminus 2^{j-1}B}|(b_B-b_{2^jB})f(t)\Omega(x-t)|dt\cdot\chi_{2^kB\setminus 2^{k-1}B}(x)$$

$$\begin{aligned}|(b_B-b_{2^jB})| &= \sum_{i=j}^{-1}|(b_{2^{i+1}B}-b_{2^iB})\\ &= \sum_{i=j}^{-1}\frac{1}{|2^iB|}\int_{2^iB}|b(t)-b_{2^{i+1}B}|dy\\ &\le C\sum_{i=j}^{-1}\frac{1}{|2^iB|}\|(b-b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{q(\cdot)}}\|\chi_{2^{i+1}B}\|_{L^{q'(\cdot)}}\end{aligned}$$

By virtue of Lemma 3, we have

$$\begin{aligned}|(b_B-b_{2^jB})| &\le C\sum_{i=j}^{-1}\frac{1}{|2^iB|}\|(b-b_{2^{i+1}b})\chi_{2^{i+1}B}\|_{L^{q(\cdot)}}\frac{|2^{i+1}B|}{\|\chi_{2^{i+1}B}\|_{L^{q(\cdot)}}}\\ &\le C\sum_{i=j}^{-1}\|b\|_{CBMO^{q(\cdot),\lambda}}|2^{i+1}B|^{\lambda}\\ &\le C\|b\|_{CBMO^{q(\cdot),\lambda}}\sum_{i=j}^{-1}|2^{i+1}B|^{\lambda}\\ &\le C\|b\|_{CBMO^{q(\cdot),\lambda}}|2^{j+1}B|^{\lambda}|j| \qquad (18)\end{aligned}$$

$$A_{22} \le C \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^{j+1}B|^{\lambda}|j|$$
$$\times \|f\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s} \|\chi_{2^jB}\|_{L^{q(\cdot)}}$$
$$\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1}|j|\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$
$$\times \|\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s} \|\chi_{2^jB}\|_{L^{q(\cdot)}}$$
$$\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1}$$
$$\times \sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1}|j|\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |2^jB|^{\frac{1}{q_1(\cdot)}+\frac{1}{s}+\frac{1}{q(\cdot)}}$$
$$\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1} \sum_{j=-\infty}^{k} |2^jB|^{\lambda+\lambda_1+1}|j|$$
$$\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} \chi_{2^kB\setminus 2^{k-1}B}(x)|2^kB|^{\frac{\beta}{n}-1}|2^kB|^{\lambda+\lambda_1+1}|k|$$
$$\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}} \chi_{2^kB\setminus 2^{k-1}B}(x)$$

$$\|A_{22}\|_{L^{q_2(\cdot)}} \le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}} \|\chi_{2^kB}\|_{L^{q_2(\cdot)}}$$
$$\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k||2^kB|^{\lambda+\lambda_1+\frac{\beta}{n}} |2^kB|^{\frac{1}{q_2(\cdot)}}$$
$$\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k||2^k|^{\lambda_2+\frac{1}{q_2(\cdot)}} |B|^{\lambda_2+\frac{1}{q_2(\cdot)}}$$
$$\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\lambda_2} \|\chi_B\|_{L^{q_2(\cdot)}}$$

Combine all results of A_1, A_2, A_{21}, and A_{22}, we obtain the required result

$$\|[b, H_{\beta,\Omega}]f\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\lambda_2} \|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$
$$\|[b, H_{\beta,\Omega}]f\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

Theorem 4. *Let $p(\cdot), q_1(\cdot)$, $q_2(\cdot)$, and β be defined as in Theorem 2 , and let $\Omega \in L^s(S^{n-1})$. If $b \in \|b\|_{CBMO^{q(\cdot),\lambda}}$, $\lambda_2 = \lambda+\lambda_1+\frac{\beta}{n}$, and $\beta < n(1-\delta_3-\delta_1-\lambda_2+\frac{1}{s})$, then*

$$\|[b, H^*_{\beta,\Omega}]f\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}$$

Proof.

$$\begin{aligned}|[b,H^*_{\beta,\Omega}]f(x)\cdot\chi_B(x)| &\leq \int_{B(0,|x|)^c}\frac{|f(t)(b(x)-b(t))\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_B(x)\\ &\leq \int_{B(0,|x|)^c}\frac{|f(t)(b(x)-b_B)\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_B(x)\\ &+\int_{B(0,|x|)^c}\frac{|f(t)(b(t)-b_B)\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_B(x)\\ &= D_1+D_2.\end{aligned}$$

$$D_1 = |(b(x)-b_B)\chi_B(x)||H^*_{\beta,\Omega}f(x)|,$$

$$\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} = \|(b(x)-b_B)\chi_B(x)H^*_{\beta,\Omega}f(x)\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}.$$

By using Hölder inequality ($\frac{1}{q_2(\cdot)}=\frac{1}{p(\cdot)}+\frac{1}{q(\cdot)}$)

$$\begin{aligned}\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} &\leq C\|(b(x)-b_B)\chi_B(x)\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|H^*_{\beta,\Omega}f(x)\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &= C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}|B|^{\mu}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|H^*_{\beta,\Omega}f\|_{\dot{B}^{\mu,p(\cdot)}},\end{aligned}$$

Given that $\mu=\lambda_1+\frac{\beta}{n}$, and using the result of Theorem 2

$$\|D_1\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}}|B|^{\lambda_2}\|\chi_B\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}},$$

Next,

$$D_2 = \int_{B(0,|x|)^c}\frac{|f(t)(b(t)-b_B)\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_B(x).$$

$$\begin{aligned}D_2 &= \sum_{k=-\infty}^{0}\int_{2^jB\backslash 2^{j-1}B}\frac{|f(t)(b(t)-b_B)\Omega(x-t)|}{|t|^{n-\beta}}dt\cdot\chi_{2^kB\backslash 2^{k-1}B}(x)\\ &\leq \sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\int_{2^jB\backslash 2^{j-1}B}|f(t)(b(t)-b_{2^jB})\Omega(x-t)|dt\\ &+\sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\int_{2^jB\backslash 2^{j-1}B}|f(t)(b_B-b_{2^jB})\Omega(x-t)|dt\\ &= D_{21}+D_{22}\end{aligned}$$

$$D_{21} = \sum_{k=-\infty}^{0}\chi_{2^kB\backslash 2^{k-1}B}(x)\sum_{j=k+1}^{\infty}|2^jB|^{\frac{\beta}{n}-1}\int_{2^jB\backslash 2^{j-1}B}|f(t)(b(t)-b_{2^jB})\Omega(x-t)|dt$$

Using Holder's inequality ($\frac{1}{q(\cdot)}+\frac{1}{q_1(\cdot)}+\frac{1}{s}=1$).

$$\begin{aligned}
D_{21} \le C &\sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)\\
&\times \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \|(b(t)-b_{2^jB})\chi_{2^jB}\|_{L^{q(\cdot)}} \|f\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s}\\
= C &\sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x)\\
&\times \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^jB|^{\lambda} \|\chi_{2^jB}\|_{L^{q(\cdot)}}\\
&\qquad \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |2^jB|^{\lambda_1} \|\chi_{2^jB}\|_{L^{q_1(\cdot)}} \|\Omega(x-t)\chi_{2^jB}\|_{L^s}\\
= C &\sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}\\
&\times \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} |2^jB|^{\lambda+\lambda_1} |2^jB|^{\frac{1}{s}+\frac{1}{q_1(\cdot)}+\frac{1}{q(\cdot)}}\\
= C &\sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{j=k+1}^{\infty} |2^jB|^{\lambda_2}\\
= C &\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2} \chi_{2^kB\backslash 2^{k-1}B}(x)
\end{aligned}$$

$$\begin{aligned}
\|D_{21}\|_{L^{q_2(\cdot)}} &\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2} \|\chi_{2^kB}\|_{L^{q_2(\cdot)}}\\
&\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |2^{(k+1)}B|^{\lambda_2} |2^kB|^{\frac{1}{q_2(\cdot)}}\\
&\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\lambda_2+\frac{1}{q_2(\cdot)}} \sum_{k=-\infty}^{0} |2^{(k+1)}|^{\lambda_2+\frac{1}{q_2(\cdot)}}\\
&\le C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\lambda_2+\frac{1}{q_2(\cdot)}}
\end{aligned}$$

$$D_{22} = \sum_{k=-\infty}^{0} \chi_{2^kB\backslash 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^jB|^{\frac{\beta}{n}-1} \int_{2^jB\backslash 2^{j-1}B} |(b_B-b_{2^jB})f(t)|dt$$

Here we use inequality (18)

$$\begin{aligned}
D_{22} \leq C \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^j B|^{\frac{\beta}{n}-1} \|b\|_{CBMO^{q(\cdot),\lambda}} |2^{j+1}B|^{\lambda} |j| \|f\chi_{2^j B}\|_{L^{q_1(\cdot)}} \\
\times \|\Omega(x-t)\chi_{2^j B}\|_{L^s} \|\chi_{2^j B}\|_{L^{q(\cdot)}} \\
\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^j B|^{\frac{\beta}{n}-1} |2^j B|^{\lambda+\lambda_1} |j| \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \\
\times \|\Omega(x-t)\chi_{2^j B}\|_{L^s} \|\chi_{2^j B}\|_{L^{q(\cdot)}} \|\chi_{2^j B}\|_{L^{q_1(\cdot)}} \\
\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) \sum_{j=k+1}^{\infty} |2^j B|^{\lambda+\lambda_1+\frac{\beta}{n}} |j| \\
\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} \chi_{2^k B\setminus 2^{k-1}B}(x) |2^{k+1}B|^{\lambda_2} |k+1| \\
\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k+1| |2^{k+1}B|^{\lambda_2} \chi_{2^k B\setminus 2^{k-1}B}(x)
\end{aligned}$$

$$\begin{aligned}
\|D_{22}\|_{L^{q_2(\cdot)}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k+1| |2^{k+1}B|^{\lambda_2} \|\chi_{2^k B}\|_{L^{q_2(\cdot)}} \\
\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k+1| |2^{k+1}B|^{\lambda_2} |2^k B|^{\frac{1}{q_2(\cdot)}} \\
\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} \sum_{k=-\infty}^{0} |k+1| |2^{k+1}|^{\lambda_2+\frac{1}{q_2(\cdot)}} |B|^{\lambda_2+\frac{1}{q_2(\cdot)}} \\
\leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\lambda_2} \|\chi_B\|_{L^{q_2(\cdot)}}
\end{aligned}$$

Combining all the results from D_1, D_2, D_{21}, and D_{22}, we obtain the required results:

$$\|[b, H^*_{\beta,\Omega}] f\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}} |B|^{\lambda_2} \|\chi_B\|_{L^{q_2(\cdot)}(\mathbb{R}^n)}$$

$$\|[b, H^*_{\beta,\Omega}] f\|_{\dot{B}^{q_2(\cdot),\lambda_2}} \leq C\|b\|_{CBMO^{q(\cdot),\lambda}} \|f\|_{\dot{B}^{q_1(\cdot),\lambda_1}}.$$

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