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Application of the Cubic Exponential B-spline Collocation Based on the Lie-Trotter and Strang Splitting Operator Splitting Method for Burgers Equation

Burgers Denklemi için Lie-Trotter ve Strang Splitting Operatör Parçalama Yöntemine Dayalı Kübik Üstel B-spline Kollokasyon Uygulaması

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Abstract
In this study, the cubic exponential B-spline collocation method has been proposed for the numerical solutions of the Burgers equation with the operator splitting. To apply the operator splitting method, the Burgers' equation has decomposed into two sub-equations based on the time term: the linear part (diffusion) and the nonlinear part (convection). Subsequently, for each sub-equation, Crank-Nicolson finite difference schemes in the temporal direction and cubic exponential B-spline functions and their derivatives have been applied at the x_m nodal points in the spatial direction. The algebraic equation systems obtained have been solved numerically using the Lie-Trotter and Strang splitting schemes to get the solutions of the main equation. Some advantages of the splitting methods include preserving the physical characteristics of the solution, yielding more convergent results over long time intervals, enabling simpler algorithms, and facilitating the storage of solution vectors on computer. To assess the accuracy of the computed numerical results the L_2 and L_∞ error norms have been used. Additionally, the obtained results have been compared with some studies in the literature. The stability analysis of the applied method has been investigated using the von Neumann Fourier series method.

Keywords: *Burgers Equation; Cubic Exponential B-spline; Collocation Method; Lie-Trotter Splitting; Strang Splitting.*

Öz
Bu çalışmada, Burgers denkleminin nümerik çözümleri için kübik üstel B-spline kollokasyon ile birlikte operatör parçalama yöntemi önerildi. Operatör parçalama yöntemini uygulamak için Burgers denklemi zaman terimine göre lineer kısım (difüzyon) ve lineer olamayan kısım (konveksiyon) olarak iki alt denkleme parçalandı. Daha sonra her bir alt denkleme zaman yönünde Crank-Nicolson sonlu fark yaklaşımları, konum yönünde ise kübik üstel B-spline fonksiyonlarının ve türevlerinin x_m düğüm noktalarındaki değerleri uygulandı. Elde edilen cebirsel denklem sistemleri Lie-Trotter ve Strang parçalama şemaları kullanılarak ana denklemin nümerik çözümleri bulundu. Parçalama yöntemlerinin bazı avantajları çözümün fiziksel özelliklerini koruması, uzun zaman aralıklarında daha yakınsak sonuçlar vermesi, daha basit algoritmalara olanak sağlaması, çözüm vektörlerinin bilgisayarda depolanması olarak sayılabilir. Hesaplanan sayısal sonuçların doğruluğunu ölçmek için literatürde sıkça kullanılan L_2, L_∞ hata normları kullanıldı. Ayrıca elde edilen sonuçlar literatürdeki bazı çalışmalarla karşılaştırıldı. Uygulanan yöntemin kararlılık analizi Von Neumann Fourier seri yöntemiyle incelendi.

Anahtar Kelimeler: *Burgers Denklemi; Kübik Üstel B-spline; Kollokasyon Yöntemi; Lie-Trotter Parçalama; Strang Parçalama.*

1. Introduction

The nonlinear Burgers' equation is

$$U_t + UU_x - vU_{xx} = 0, a \leq x \leq b, t \geq 0 \quad (1)$$

with the initial condition

$$U(x,0) = \vartheta(x)$$

and the boundary conditions

$$U(a,t) = f_1(t), \ \ U(b,t) = f_2(t),$$

where $U(x,t)$ is a function sufficiently differentiable in the x direction, t is time, and v is the positive kinematic viscosity coefficient. The Burgers' equation was first introduced by (Bateman 1915) in the context of fluid mechanics research. This equation is acknowledged as the most elementary mathematical model delineating the equilibrium between convection and diffusion. The equation (1) was later studied by Burgers (Burgers, 1948), and after this study, it was named the Burgers equation. If $v = 1$ is substituted into equation (1), the inviscid Burgers' equation is obtained, which models shock waves and finds numerous applications in physics (Brezis and Felix 1997). The reasons behind equation (1) attracting the attention of many researchers include its incorporation of the simplest form of nonlinearity, represented by the convective term UU_x, its inclusion of the term vU_{xx} modeling physical wave phenomena, and the possibility of comparison with the exact solution obtained by (Cole 1951). Due to the increasing interest in

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nonlinear phenomena in recent years, the Burgers equation, which models a variety of phenomena such as gas dynamics, heat conduction, traffic flow, and shock wave events, has attracted considerable attention from researchers (Gao *et al*. 2013). In recent years, with the emergence of more powerful computers, efforts have been made to compute solutions of the Burgers equation using numerous numerical methods and techniques. Some of these studies, (Dag *et al*. 2005) applied cubic B-splines bases using a linearization technique and the collocation finite element method. (Saka and Dag 2007) utilized time and space splitting techniques to obtain approximate solutions of the Burgers' equation via quintic B-spline collocation procedures. (Kutluay and Esen 2004) solved the Burgers' equation using a lumped Galerkin method with quadratic B-spline finite elements. (Dag *et al*., 2017) employed cubic Trigonometric B-spline (CTB) functions to establish a collocation method for finding solutions of the Burgers' equation. (Ucar *et al*. 2020) solved the Burgers' equation using the operator splitting cubic B-spline collocation method. (Dag *et al*. 2004) solved the Burgers' equation using both time and space splitting with the quadratic B-spline collocation method. (Mittal and Jain 2012) computed the numerical solutions of the Burgers' equation via collocation-modified cubic B-splines using SSP-RK43 and SSP-RK54. (Ersoy *et al*. 2018) determined the numerical solutions of the Burgers equation using the exponential B-spline collocation method. (Celikkaya and Guzel 2023) constructed numerical solutions of the Burgers equation by employing four numerical schemes based on operator splitting.

2. Operator Splitting Method

Operator splitting methods are highly effective techniques used to decompose complex mathematical and engineering problems into simpler ones to obtain their solutions. In splitting methods, the main equation is decomposed into sub equations, and the solutions of each sub equation are obtained independently of the main equation over the time interval $[t_n, t_{n+1}]$. If the equation is decomposed in a way to model different physical problems, the method is called "operator splitting" (Hundsdorfer 2000).

Let split the Burgers equation as follows:

$$U_t = vU_{xx}, \tag{2}$$

$$U_t = -UU_x. \tag{3}$$

The strategy to obtain numerical solutions of the Burgers equation using the operator splitting method involves solving equations (2) and (3) numerically or analytically. Let $\varphi^A_{\Delta t}$ and $\varphi^B_{\Delta t}$ denote the analytical or numerically acceptable solutions of equations (2) and (3) with a time step of Δt, respectively. Then, the numerical solutions of equation (1) can be obtained as $U(x,\Delta t) \cong \omega_{\Delta t}\theta(x)$ with an appropriate Δt, where

$$\omega_{\Delta t} = \varphi^A_{\Delta t a_1} \circ \varphi^B_{\Delta t b_1} \circ \cdots \varphi^A_{\Delta t a_n} \circ \varphi^B_{\Delta t b_m} \circ \varphi^A_{\Delta t a_{m+1}}$$

or

$$\omega_{\Delta t} = \varphi^B_{\Delta t b_1} \circ \varphi^A_{\Delta t a_1} \circ \cdots \varphi^B_{\Delta t b_n} \circ \varphi^A_{\Delta t a_m} \circ \varphi^B_{\Delta t b_{m+1}}.$$

Coefficients a_i and b_i can be obtained using the Baker-Campel-Hausdorff formula to achieve solutions of the desired order (Creutz and Gocksch 1989, Suziki 1990, Yoshida 1990, Sari *et al*. 2019). The simplest splitting method known as Lie-Trotter (Trotter 1959) splitting is defined as follows:

$L_{\Delta t} = \varphi^A_{\Delta t} \circ \varphi^B_{\Delta t}$ or $L^*_{\Delta t} = \varphi^B_{\Delta t} \circ \varphi^A_{\Delta t}$.

The solution algorithm for Lie-Trotter $L^*_{\Delta t}$ can be written in the following form

$$\frac{dU^*(t)}{dt} = AU^*(t), U^*(t_n) = U^0(t_n), t \in [t_n, t_{n+1}],$$

$$\frac{dU^{**}(t)}{dt} = BU^{**}(t), U^{**}(t_n) = U^*(t_{n+1}), t \in [t_n, t_{n+1}].$$

In this scheme, equation (2) is solved with the original initial condition given by the problem, and the obtained results are used as the initial condition in equation (3). Thus, the numerical solutions of the main problem are found as $U(t_{n+1}) = U^{**}(t_{n+1})$. If the order of operators A and B is changed, a similar algorithm can be written for $L_{\Delta t}$. Strang splitting (Strang 1968), one of the commonly used methods in the literature, is defined as follows:

$S_{\Delta t} = \varphi^A_{\frac{\Delta t}{2}} \circ \varphi^B_{\Delta t} \circ \varphi^A_{\frac{\Delta t}{2}}$ or $S^*_{\Delta t} = \varphi^B_{\frac{\Delta t}{2}} \circ \varphi^A_{\Delta t} \circ \varphi^B_{\frac{\Delta t}{2}}$.

The numerical algorithm for $A \circ B \circ A$ scheme of $S_{\Delta t}$ is given as follows:

$$\frac{dU^*(t)}{dt} = AU^*(t), U^*(t_n) = U^0(t_n), t \in \left[t_n, t_{n+\frac{1}{2}}\right],$$

$$\frac{dU^{**}(t)}{dt} = BU^{**}(t), U^{**}(t_n) = U^*\left(t_{n+\frac{1}{2}}\right), t \in [t_n, t_{n+1}].$$

$$\frac{dU^{***}(t)}{dt} = AU^{***}(t), U^{***}\left(t_{n+\frac{1}{2}}\right) = U^{**}(t_{n+1}), t \in \left[t_{n+\frac{1}{2}}, t_{n+1}\right],$$

where, $t_{n+\frac{1}{2}} = t_n + \frac{\Delta t}{2}$ and desired numerical solutions are obtained as $U(t_{n+1}) = U^{***}(t_{n+1})$. Similarly, the $B \circ A \circ B$ scheme can also be formulated in a similar manner as provided above.

In this study, numerical solutions of the Burgers equation were computed using cubic exponential B-spline collocation finite element method with AB, BA, ABA and BAB numerical patterns.

3. Cubic Exponential B-spline Functions and Application of the Method

A uniform partitioning of the solution domain $[a, b]$ in terms of the nodal points x_m, $a = x_0 < x_1 < \cdots < x_N = b$ and $h = x_{m+1} - x_m$. Let $s = \sinh(ph)$ and $c = \cosh(ph)$, the cubic B-spline functions $E_i(x)$ are defined in terms of the nodal points x_m as follows:

$$E_m(x) = \begin{cases} b_2\left((x_{m-2} - x) - \frac{1}{p}\left(\sinh\left(p\left((x_{m-2} - x)\right)\right)\right)\right), & [x_{m-2}, x_{m-1}] \\ a_1 + b_1(x_m - x) + c_1 \exp(p * (x_m - x)) + d_1 \exp(-p * (x_m - x)), & [x_{m-1}, x_m] \\ a_1 + b_1(x - x_m) + c_1 \exp(p * (x - x_m)) + d_1 \exp(-p * (x - x_m)), & [x_m, x_{m+1}] \\ b_2\left((x - x_{m+2}) - \frac{1}{p}\left(\sinh\left(p\left((x - x_{m+2})\right)\right)\right)\right) & [x_{m+1}, x_{m+2}] \\ 0, & Otherwise, \end{cases} \quad (4)$$

where p is positive free parameter and

$$a_1 = \frac{phc}{phc - s}, b_1 = \frac{p}{2}\left[\frac{c(c-1) + s^2}{(phc - s)(1 - c)}\right],$$

$$b_2 = \frac{p}{2(phc - s)},$$

$$c_1 = \frac{1}{4}\left[\frac{\exp(-ph)(1 - c) + s(\exp(-ph) - 1)}{(phc - s)(1 - c)}\right],$$

$$d_1 = \frac{1}{4}\left[\frac{\exp(ph)(c - 1) + s(\exp(ph) - 1)}{(phc - s)(1 - c)}\right].$$

It is clear that the set $\{E_{-1}(x), E_0(x), \cdots E_{N+1}(x)\}$ forms a basis over the interval $[a, b]$, (Mccartin 1991). Hence, a function defined over the interval $[a, b]$, denoted as $E_m(x), m = -1(1)N + 1$, can be expressed as a linear combination of the functions. That is, with $\delta_m(t)$ being time-dependent parameters, the expression for $U_N(x, t)$ can be approximated as follows:

$$U_N(x, t) \cong \sum_{m=-1}^{N+1} \delta_m(t) E_m(x). \quad (5)$$

Using expressions (4) and (5), the function and its first and second-order derivatives can be determined as follows:

$$U(x, t) = U(x_m, t) = \alpha_1 \delta_{m-1} + \alpha_2 \delta_m + \alpha_1 \delta_{m+1},$$

$$U'^{(x,t)} = U'^{(x_m,t)} = \beta_1 \delta_{m-1} - \beta_1 \delta_{m+1}, \quad (6)$$

$$U''^{(x,t)} = U''^{(x_m,t)} = \gamma_1 \delta_{m-1} - \gamma_2 \delta_m + \gamma_1 \delta_{m+1},$$

where

$$\alpha_1 = \frac{s - ph}{2(phc - s)}, \beta_1 = \frac{p(1 - c)}{2(phc - s)}, \gamma_1 = \frac{p^2 s}{2(phc - s)}, \gamma_2 = \frac{p^2 s}{phc - s}.$$

If the approximations in expression (6) are utilized in equations (2) and (3), the following system of ordinary differential equations is obtained as follows:

$$\alpha_1 \dot{\delta}_{m-1} + \alpha_2 \dot{\delta}_m + \alpha_1 \dot{\delta}_{m+1} - v(\gamma_1 \delta_{m-1} - \gamma_2 \delta_m + \gamma_1 \delta_{m+1}) = 0, \quad (7)$$

$$\alpha_1 \dot{\delta}_{m-1} + \alpha_2 \dot{\delta}_m + \alpha_1 \dot{\delta}_{m+1} + z_m(\beta_1 \delta_{m-1} - \beta_1 \delta_{m+1}) = 0. \quad (8)$$

Where, the symbol $\cdot$ denotes the first-order derivative with respect to t, and

$$z_m = \alpha_1 \delta_{m-1} + \alpha_2 \delta_m + \alpha_1 \delta_{m+1}.$$

If Crank-Nicolson for time discretization and the values of exponential B-spline functions at nodal points x_m for space direction in equations (7) and (8) are employed, the following algebraic equation systems are obtained:

$$r_1 \delta_{m-1}^{n+1} + r_2 \delta_m^{n+1} + r_1 \delta_{m+1}^{n+1} = r_3 \delta_{m-1}^n + r_4 \delta_m^n + r_3 \delta_{m+1}^n, \quad (9)$$

$$r_5 \delta_{m-1}^{n+1} + r_6 \delta_m^{n+1} + r_7 \delta_{m+1}^{n+1} = r_7 \delta_{m-1}^n + r_6 \delta_m^n + r_5 \delta_{m+1}^n. \quad (10)$$

Where

$$r_1 = \alpha_1 - \frac{v\Delta t\gamma_1}{2}, r_2 = \alpha_2 + \frac{v\Delta t\gamma_2}{2}, r_3 = \alpha_1 + \frac{v\Delta t\gamma_1}{2},$$

$$r_4 = \alpha_2 - \frac{v\Delta t\gamma_2}{2}, r_5 = \alpha_1 + \frac{z_m\Delta t\beta_1}{2}, r_6 = \alpha_2,$$

$$r_7 = \alpha_1 - \frac{z_m\Delta t\beta_1}{2}.$$

The algebraic equation systems (9) and (10) consist of (N+1) equations and (N+3) time-dependent parameters $\delta_m(t)$. If boundary conditions $U(a, t) = U(b, t) = 0$ are applied to these systems, the following relations are obtained for parameters δ_{-1} and δ_{N+1} that not in the solution domain:

$$\delta_{-1} = -\frac{\alpha_2}{\alpha_1}\delta_0 - \delta_1 + \frac{f_1}{\alpha_1},$$

$$\delta_{N+1} = -\delta_{N-1} - \frac{\alpha_2}{\alpha_1}\delta_N + + \frac{f_2}{\alpha_1}. \quad (11)$$

By using the equations (11) in the (9) and (10), yielding tri-diagonal band matrices of size $(N + 1) \times (N + 1)$.

3.1 Initial State

To start the solution of systems (9) and (10), the initial condition

$$U_N(x_m,0)=\vartheta(x_m), m=0(1)N$$

and boundary conditions

$$U_{xx}(a,0)=f_1(t), U_{xx}(b,0)=f_2(t)$$

are used. The initial vector δ_m^0 can be obtained as follows:

$$\begin{bmatrix} -\frac{\alpha_1\gamma_2}{\gamma_1}+\alpha_2 & 0 & 0 & & & \\ \alpha_1 & \alpha_2 & \alpha_1 & & & \\ & & \ddots & & & \\ & & \alpha_1 & \alpha_2 & \alpha_1 & \\ & & & & -\frac{\alpha_1\gamma_2}{\gamma_1}+\alpha_2 & \end{bmatrix}\begin{bmatrix} \delta_0^0 \\ \delta_1^0 \\ \vdots \\ \delta_{N-1}^0 \\ \delta_N^0 \end{bmatrix}=\begin{bmatrix} U_0-\frac{\alpha_1 f_1}{\gamma_1} \\ 0 \\ \vdots \\ 0 \\ U_N-\frac{\alpha_1 f_2}{\gamma_1} \end{bmatrix}.$$

3.2 The Von Neumann Stability Analysis

The stability analysis of systems (9) and (10) has been investigated using the Von Neumann Fourier series method (Von Neumann and Richtmyer 1950). In this method, $\delta_m^n=\xi^n e^{i\beta mh}$, where $i=\sqrt{-1}$, ξ is the amplification factor, β is the mod number, and h is the spatial step length. Let ρ_A and ρ_B be the stability parameters associated with systems (9) and (10), respectively. If the expression $\delta_m^n=\xi^n e^{i\beta mh}$ is substituted into systems (9) and (10), and necessary operations are carried out, the following expressions are obtained, where $\theta=\frac{v\Delta t\gamma_1}{2}$:

$$\rho_A\left(\frac{\xi^{n+1}}{\xi^n}\right)=\frac{2\alpha_1 cos\beta h+\alpha_2-2\theta(1-cos\beta h)}{2\alpha_1 cos\beta h+\alpha_2+2\theta(1-cos\beta h)}=\frac{M-N}{M+N},$$

$$\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)=\frac{cos\beta h(r_5+r_7)+r_6-isin\beta h(r_7-r_5)}{cos\beta h(r_5+r_7)+r_6+isin\beta h(r_7-r_5)}=\frac{P-iQ}{P+iQ}.$$

For stability, the conditions $\left|\rho_A\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\le 1$ and $\left|\rho_A\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\le 1$ must be satisfied. To satisfy the condition $\rho_A\le 1$, it is required that $M+N\ge M-N$, since $\theta\ge 0$, the condition $\left|\rho_A\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\le 1$ is satisfied. Similarly, taking the absolute value of a complex number for ρ_B reveals that $\rho_B\le 1$. Thus, the following inequalities are obtained for the Lie-Trotter and Strang schemes, respectively

$$\rho_{L_{\Delta t}}(\xi)=\left|\rho_A\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\le 1,$$

$$\rho_{S_{\Delta t}}(\xi)=\left|\rho_A\left(\frac{\xi^{n+\frac{1}{2}}}{\xi^n}\right)\right|\left|\rho_B\left(\frac{\xi^{n+1}}{\xi^n}\right)\right|\left|\rho_A\left(\frac{\xi^{n+\frac{1}{2}}}{\xi^n}\right)\right|\le 1.$$

Therefore, the numerical algorithms provided above are unconditionally stable.

Table 1. Comparison of numerical results with some studies of Problem 1 for $h=0.025$, $\Delta t=0.0005$, $v=0.01$, $p=1$.

x	Time	$\mathbf{L}_{\Delta t}$	$\mathbf{L}^*_{\Delta t}$	$\mathbf{S}_{\Delta t}$	$\mathbf{S}^*_{\Delta t}$	(Ersoy et al. 2018)	(Ucar et al. 2020)	(Kutluay and Esen 2004)	Exact
						$\Delta t=0.0001$	$\Delta t=0.001$	$\Delta t=0.00001$	
0.25	0.4	0.34200	0.34191	0.34192	0.34192	0.34192	0.34192	0.34183	0.34191
	0.6	0.26904	0.26896	0.26897	0.26897	0.26897	0.26896	0.26889	0.26896
	0.8	0.22154	0.22148	0.22148	0.22148	0.22148	0.22148	0.22142	0.22148
	1	0.18824	0.18819	0.18819	0.18819	0.18819	0.18819	0.18815	0.18819
	3	0.07513	0.07511	0.07511	0.07511	0.07511	0.07511	0.07511	0.07511
0.5	0.4	0.66084	0.66071	0.66071	0.66071	0.66071	0.66071	0.66066	0.66071
	0.6	0.52955	0.52942	0.52942	0.52942	0.52942	0.52942	0.52938	0.52942
	0.8	0.43925	0.43914	0.43914	0.43914	0.43914	0.43914	0.43910	0.43914
	1	0.37451	0.37442	0.37442	0.37442	0.37442	0.37442	0.37438	0.37442
	3	0.15021	0.15018	0.15018	0.15018	0.15018	0.15018	0.15017	0.15018
0.75	0.4	0.91036	0.91028	0.91029	0.91029	0.91027	0.91027	0.91024	0.91026
	0.6	0.76738	0.76725	0.76725	0.76725	0.76725	0.76725	0.76721	0.76724
	0.8	0.64754	0.64739	0.64740	0.64740	0.64740	0.64740	0.64737	0.64740
	1	0.55618	0.55605	0.55605	0.55605	0.55605	0.55605	0.55603	0.55605
	3	0.22493	0.22490	0.22490	0.22490	0.22483	0.22483	0.22480	0.22481

Table 2. Comparison of numerical results at $t = 0.1$ for Problem 1 with $h = 0.0125$, 0.00625, $\Delta t = 0.0005$, $v = 1$, $p = 1$.

x	h	$L_{\Delta t}$	$L^*_{\Delta t}$	$S_{\Delta t}$	$S^*_{\Delta t}$	(Ersoy et al. 2018)	(Uçar et al. 2020)	(Kutluay and Esen 2004)	Exact
						$\Delta t = 0.0001$	$\Delta t = 0.001$	$\Delta t = 0.00001$	
0.1	0.0125	0.10955	0.10950	0.10953	0.10953	0.10953	0.10953	0.10953	0.10954
0.2		0.20981	0.20973	0.20977	0.20977	0.20977	0.20977	0.20978	0.20979
0.3		0.29191	0.29182	0.29186	0.29186	0.29186	0.29187	0.29187	0.29190
0.4		0.34791	0.34785	0.34788	0.34788	0.34788	0.34788	0.34790	0.34792
0.5		0.37153	0.37153	0.37153	0.37153	0.37153	0.37153	0.37155	0.37158
0.6		0.35897	0.35902	0.35899	0.35899	0.35899	0.35900	0.35901	0.35905
0.7		0.30981	0.30990	0.30985	0.30985	0.30986	0.30986	0.30988	0.30991
0.8		0.22773	0.22783	0.22778	0.22778	0.22778	0.22778	0.22780	0.22782
0.9		0.12064	0.12070	0.12067	0.12066	0.12067	0.12067	0.12068	0.12069
0.1	0.00625	0.10956	0.10951	0.10954	0.10954	0.10954	0.10954	-	0.10954
0.2		0.20983	0.20975	0.20979	0.20979	0.20979	0.20979	-	0.20979
0.3		0.29194	0.29185	0.29189	0.29189	0.29189	0.29189	-	0.29190
0.4		0.34795	0.34789	0.34791	0.34791	0.34792	0.34791	-	0.34792
0.5		0.37157	0.37156	0.37156	0.37156	0.37156	0.37157	-	0.37158
0.6		0.35901	0.35906	0.35903	0.35903	0.35903	0.35903	-	0.35905
0.7		0.30985	0.30994	0.30989	0.30989	0.30989	0.30989	-	0.30991
0.8		0.22776	0.22786	0.22781	0.22781	0.22781	0.22781	-	0.22782
0.9		0.12065	0.12071	0.12068	0.12068	0.12068	0.12068	-	0.12069

4. Numerical Experiment and Results

4.1 Problem 1

In this problem, the Burgers' equation is considered with the following initial and boundary conditions

$U(x,0) = \sin \pi x\,, 0 \le x \le 1,$

$U(0,t) = U(1,t) = 0, t \ge 0.$

The exact solution of this problem was obtained as an infinite series by (Cole 1951) as follows:

$$U(x,t) = 2\pi v \frac{\sum_{j=1}^{\infty} j a_j \sin j\pi x exp(-j^2\pi^2 vt)}{a_0 + \sum_{j=1}^{\infty} a_j \cos j\pi x exp(-j^2\pi^2 vt)},$$

where

$$a_0 = \int_0^1 \mathrm{e}^{\left(-(2\pi v)^{-1}\right)(1-\cos \pi x)}\, dx,$$

$$a_j = 2\int_0^1 \mathrm{e}^{\left(-(2\pi v)^{-1}\right)(1-\cos \pi x)} \cos j\pi x\, dx, j = 1(1) \ldots.$$

Table 1 In Table 1, some nodal values are given for different t values of Problem 1. As observed from the **table 1**, despite the use of smaller time step Δt in some studies, it is clear that the results obtained by our method are more accurate. Although the same type of B-spline is used in the (Ersoy *et al*. 2018), the results obtained with operator splitting are quite close to the exact solution even with larger Δt. In **Table 2,** numerical values at different x at $t = 0.1$ are given. As can be seen from the **Table 2**, the results computed with the Strang scheme are generally more accurate compared to those computed with the Lie-Trotter scheme. Moreover, despite using larger Δt, our numerical results are in good agreement with those in other studies.

4.2 Problem 2

In this problem, the Burgers' equation is considered with the given initial condition at $t = 1$ and the following boundary conditions (Asaithambi 2010, Mittal and Jain 2012)

$$U(x,1) = \frac{x}{1 + \exp\left(\frac{1}{4v}\left(x^2 - \frac{1}{4}\right)\right)},$$

$U(0,t) = U(1,t) = 0, t \ge 1.$

The exact solution of this problem is given as follows:

$$U(x,t) = \frac{x/t}{1 + \sqrt{\frac{t}{t_0}} \exp\left(\frac{1}{4v}\left(x^2 - \frac{1}{4}\right)\right)}, t \ge 1,$$

where $t_0 = \exp\left(1/(8v)\right)$. The numerical solutions of this problem depict the represent of shock waves as time progresses. It can be seen from the **Figure 2** that as viscosity parameter v decreases, the shock waves are getting steeper. The results are computed with the Lie-

Trotter and Strang splitting schemes are given in **Table 3** for Problem 2.

Table 3. Comparison of numerical results at different times for Problem 2 with $h = 0.005$, $\Delta t = 0.01$, $v = 0.0005$, $p = 1$.

x	t	$\mathbf{L}_{\Delta t}$	$\mathbf{L}^*_{\Delta t}$	$\mathbf{S}_{\Delta t}$	$\mathbf{S}^*_{\Delta t}$	(Ersoy et al. 2018)	(Dağ et al. 2005)	Exact
0.1	1.7	0.05882	0.05882	0.05882	0.05882	0.05882	0.05883	0.05882
0.2		0.11765	0.11765	0.11765	0.11765	0.11765	0.11765	0.11765
0.3		0.17647	0.17647	0.17647	0.17647	0.17647	0.17648	0.17647
0.4		0.23529	0.23529	0.23529	0.23529	0.23529	0.23531	0.23529
0.5		0.29412	0.29412	0.29412	0.29412	0.29412	0.29414	0.29412
0.6		0.35294	0.35294	0.35294	0.35294	0.35294	0.35296	0.35294
0.7		0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.8		0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.9		0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	2.5	0.04000	0.04000	0.04000	0.04000	0.04000	0.04000	0.04000
0.2		0.08000	0.08000	0.08000	0.08000	0.08000	0.08000	0.08000
0.3		0.12000	0.12000	0.12000	0.12000	0.12000	0.12001	0.12000
0.4		0.16000	0.16000	0.16000	0.16000	0.16000	0.16001	0.16000
0.5		0.20000	0.20000	0.20000	0.20000	0.20000	0.20001	0.20000
0.6		0.24000	0.24000	0.24000	0.24000	0.24000	0.24001	0.24000
0.7		0.28000	0.28000	0.28000	0.28000	0.28000	0.28001	0.28000
0.8		0.01121	0.01121	0.01121	0.01121	0.01121	0.00811	0.01121
0.9		0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.1	3.25	0.03077	0.03077	0.03077	0.03077	0.03077	0.03077	0.03077
0.2		0.06154	0.06154	0.06154	0.06154	0.06154	0.06154	0.06154
0.3		0.09231	0.09231	0.09231	0.09231	0.09231	0.09231	0.09231
0.4		0.12308	0.12308	0.12308	0.12308	0.12308	0.12308	0.12308
0.5		0.15385	0.15385	0.15385	0.15385	0.15385	0.15385	0.15385
0.6		0.18461	0.18461	0.18461	0.18462	0.18462	0.18462	0.18462
0.7		0.21538	0.21538	0.21538	0.21538	0.21538	0.21539	0.21538
0.8		0.24615	0.24615	0.24615	0.24615	0.24615	0.24616	0.24615
0.9		0.12539	0.12221	0.12221	0.12461	0.12394	0.12358	0.12435

Furthermore, numerical results are compared with some existing studies for the same parameters found in the literature. As clearly seen from the table, the results obtained with our method are quite close to the exact solution.

4.3 Problem 3

As the final problem, the Burgers' equation is considered with the following exact solution

$U(x,t) = \frac{\alpha+\mu+\mu(\mu-\alpha)e^{\eta}}{1+e^{\eta}}, 0 \leq x \leq 1, t \geq 0,$ where $\eta = \frac{\alpha(x-\mu t-\gamma)}{v}$.

With boundary conditions $U(0,t) = 1, U(1,t) = 0.2$ and the initial condition can be obtained from the exact solution for $t = 0$.

In **Table 4**, the results are compared with the (Celikkaya and Guzel 2023), which used operator splitting with the cubic trigonometric B-spline collocation, for the parameters $h = 1/36$, $\Delta t = 0.01$, $v = 0.01$, $p = 1$. As seen from the table, the nodal values computed with our method are in good agreement with those of in the (Celikkaya and Guzel 2023).

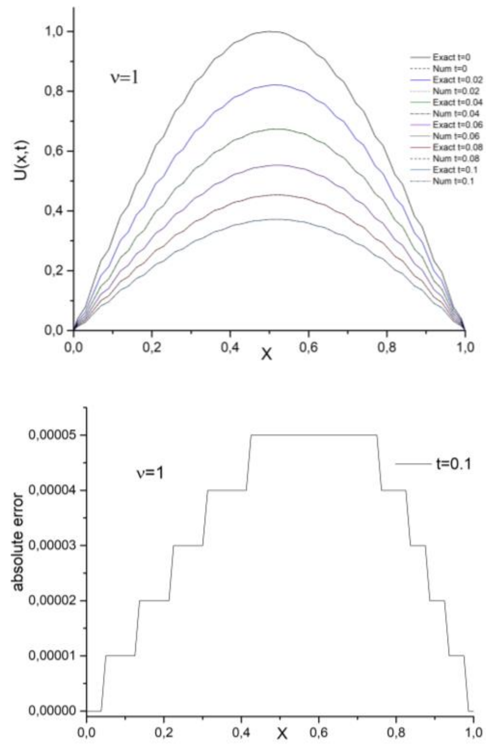
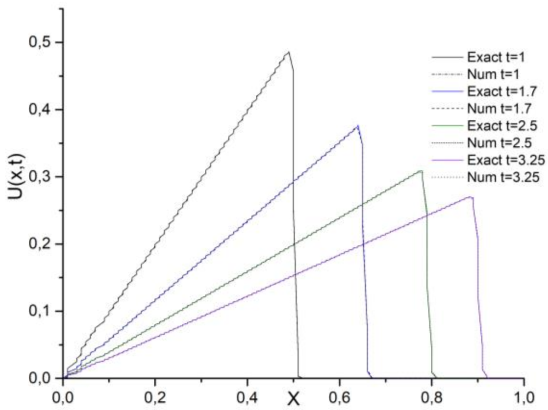


Figure 1. The physical behavior of Problem 1 for $v = 1$, $h = 0.025$, $\Delta t = 0.0005$ (Above) and absolute error at $t = 0.1$(Below).

Table 4. Comparison of numerical results of Problem 3 at $t = 0.5$ for $h = 1/36$, $\Delta t = 0.01$, $v = 0.01$, $p = 1$.

					(Celikkaya and Guzel 2023)				
x	$\mathbf{L}_{\Delta t}$	$\mathbf{L}^*_{\Delta t}$	$\mathbf{S}_{\Delta t}$	$\mathbf{S}^*_{\Delta t}$	$L°N$	$N°L$	$L°N°L$	$N°L°N$	**Exact**
0.000	0.998	0.997	0.998	0.997	1.000	1.000	1.000	1.000	1.000
0.056	1.000	1.000	1.002	1.000	1.000	1.000	1.000	1.000	1.000
0.111	1.000	1.000	1.002	1.000	1.000	1.000	1.000	1.000	1.000
0.167	1.000	1.000	1.002	1.000	1.000	1.000	1.000	1.000	1.000
0.222	1.000	1.000	1.002	1.000	1.000	1.000	1.000	1.000	1.000
0.278	0.999	0.999	1.001	0.999	0.999	0.999	1.001	0.999	0.998
0.333	0.981	0.984	0.985	0.982	0.981	0.984	0.985	0.982	0.980
0.389	0.837	0.848	0.845	0.843	0.835	0.848	0.845	0.843	0.847
0.444	0.461	0.451	0.457	0.457	0.458	0.451	0.457	0.457	0.452
0.500	0.240	0.235	0.237	0.237	0.239	0.235	0.237	0.237	0.238
0.556	0.204	0.203	0.203	0.203	0.203	0.203	0.203	0.203	0.204
0.611	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200
0.667	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200
0.722	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200
0.778	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200
0.833	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200
0.889	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200
0.944	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200
0.100	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200	0.200



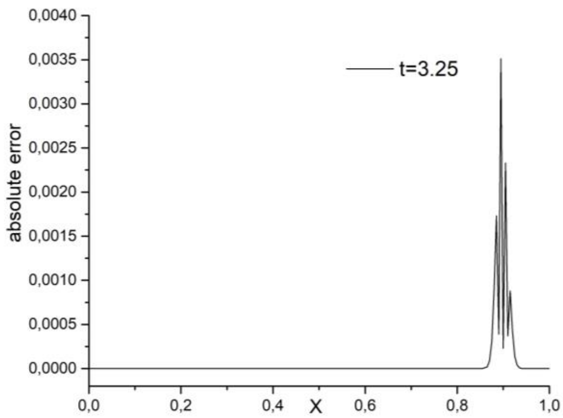


Figure 2. The physical behavior of Problem 2 for $v = 0.0005$, $h = 0.005$, $\Delta t = 0.01$ (Above) and absolute error at $t = 3.25$ (Below).

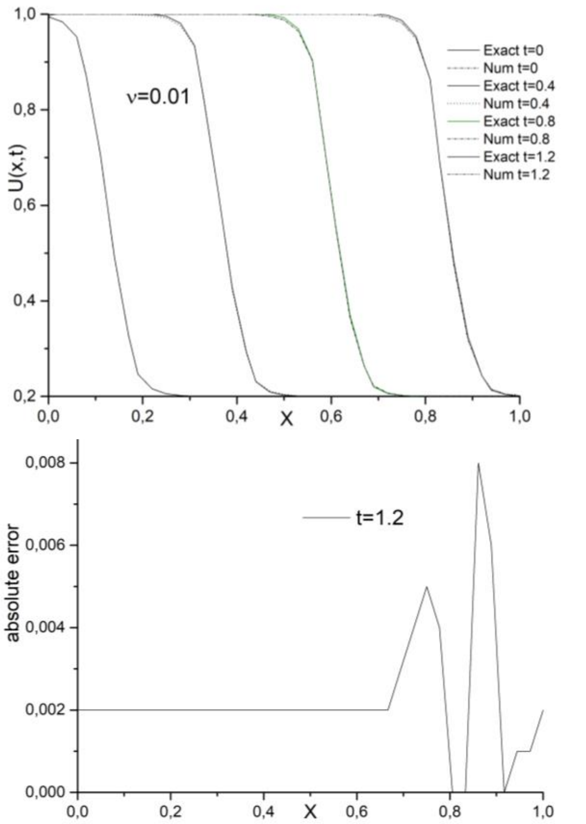


Figure 3. The physical behavior of Problem 3 for $v = 0.01$, $h = 1/36$, $\Delta t = 0.01$ (Above) and absolute error at $t = 1.2$ (Below).

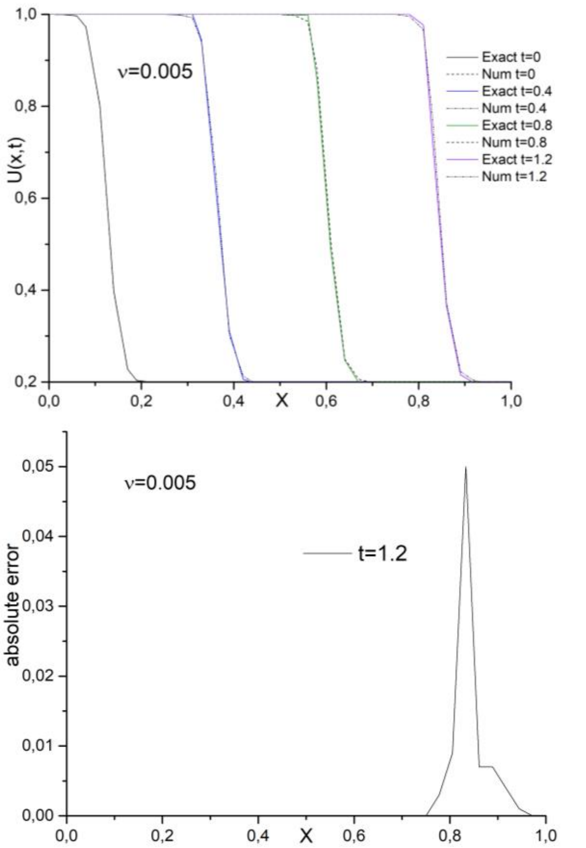


Figure 4. The physical behavior of Problem 3 for $v = 0.005$, $h = 1/36$, $\Delta t = 0.01$ (Above) and absolute error $t = 0.1$ (Below).

Figure 1 shows that the results are computed with the $S_{\Delta t}$ for $v = 1$. As seen from the **Figure 1**, the exact solution closely resembles the numerical solution to the extent that they are practically indistinguishable. Furthermore, it is observed that the error is high in the middle of the solution domain at $t = 0.1$. **Figure 2** illustrates the shock waves generated by Problem 2 for $v = 0.0005$. As seen from the Figure, as time progresses, the height of the shock wave decreases and moving towards the right. Additionally, it is observed that the error concentrates on the right side of the solution domain at $t = 3.25$. **Figures 3** and **4** show the shock waves obtained for the Problem 3 with viscosity values of $v = 0.01$ and $v = 0.005$, respectively. As clearly seen from the Figures, as the viscosity parameter v decreases, the shock waves steeper, and it is observed that the errors reach their highest values at the right boundary at t=3.25.

5. Conclusions

In this study, the numerical solutions of the Burgers' equation are obtained using the exponential B-spline collocation method with operator splitting. The computed results are supported with graphs and tables. Additionally, the obtained numerical results are compared with more computationally intensive methods such as Galerkin method. It has been observed that quite accurate results are obtained for larger Δt values using operator splitting. Furthermore, a comparison has been made with the (Celikkaya and Guzel 2023), which solved the equation using the cubic trigonometric B-spline collocation method with operator splitting.

It is observed that the results obtained with exponential B-spline are in good agreement with those found in (Celikkaya and Guzel 2023). Splitting a given partial differential equation has been demonstrated as an effective method with simpler algorithms, proving to be effective in the numerical solutions of partial differential equations. It is clearly seen from the provided graphs that the operator splitting method preserves the physical structure of the solution. The cubic exponential B-spline collocation operator splitting method will be an effective and suitable method for the numerical solutions of partial differential equations with more complex structures.

Declaration of Ethical Standards
The authors declare that they comply with all ethical standards.

Credit Authorship Contribution Statement
Author 1: Investigation, Methodology / Study design, Writing – original draft, Conceptualization.

Declaration of Competing Interest
The authors have no conflicts of interest to declare regarding the content of this article.

Data Availability
All data generated or analysed during this study are included in this Published paper.

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