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Research Article

Geometry of φ-Unit Tangent Bundle with Vertical Rescaled Berger Deformation Metric

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Abstract
In this paper, we talk about the vertical rescaled Berger deformation metric on the φ-unit tangent bundle over an anti-paraKähler manifold (M^{2m}, φ, g). Firstly, we investigate the Levi-Civita connection in this metric. Secondly, we calculate all forms of the Riemannian curvature tensors. Finally, we study the geodesics on the φ-unit tangent bundle concerning the vertical rescaled Berger deformation metric.

Keywords and 2020 Mathematics Subject Classification
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1. Introduction

One can define natural Riemannian metrics on the tangent bundle of a Riemannian manifold. Their construction makes use of the Levi-Civita parallelization. Among them, the so-called Sasaki metric [1] is of particular interest. That is why the geometry of a tangent bundle equipped with the Sasaki metric has been studied by many authors, such as Yano and Ishihara [2], Dombrowski [3], Salimov, Gezer, and Akbulut [4], etc. The Sasaki metric's rigidity has led some researchers to construct and study other metrics on tangent bundles. This is why they have attempted to search for different metrics on the tangent bundle, which are different deformations of the Sasaki metric. Musso and Tricerri have introduced the notion of the Cheeger-Gromoll metric [5]; this metric has also been studied by many authors (see [6, 7]). In this direction, Yampolsky [8] proposes a Berger-type deformed Sasaki metric on a tangent bundle over a Kählerian manifold, which Altunbaş and collaborators studied in [9]. The study of the Berger-type deformed Sasaki metric on the tangent bundle or the cotangent is not limited to those mentioned above. We also refer to new studies by Zagane, among which we [10, 11].

The main idea of this paper is to study the vertical rescaled Berger deformation metric [12] on the φ-unit tangent bundle over an anti-paraKähler manifold (M^{2m}, φ, g). Firstly, we introduce the φ-unit tangent bundle equipped with vertical rescaled Berger deformation metric. Secondly, we investigate the formulas relating to the Levi-Civita connection of this metric (Theorem 7), and we establish all formulas of the Riemannian curvature tensors (Theorem 8). In the last section, we study the geodesics on the φ-unit tangent bundle and also present the necessary and sufficient conditions under which a curve can be a geodesic concerning the vertical rescaled Berger deformation metric (Theorem 11). As well as when the natural lift and the horizontal lift are geodesic (Corollary 12 and Corollary 13). Finally, we also mention some special cases (Theorem 18 and Theorem 20).

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, u^i)_{i=\overline{1,m}}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g. Let $C^\infty(M)$ be the ring of real-valued C^∞ functions on M and $\Im_0^1(M)$ be the module over

$C^\infty(M)$ of C^∞ vector fields on M. The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,u)}TM = V_{(x,u)}TM \oplus H_{(x,u)}TM \tag{1}$$

of the tangent bundle to TM at any $(x,u) \in TM$ into the vertical subspace

$$V_{(x,u)}TM = Ker(d\pi_{(x,u)}) = \{\xi^i \frac{\partial}{\partial u^i}|_{(x,u)}, \xi^i \in \mathbb{R}\}, \tag{2}$$

and the horizontal subspace

$$H_{(x,u)}TM = \{\xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}|_{(x,u)}, \xi^i \in \mathbb{R}\}. \tag{3}$$

Note that the map $\xi \to {}^H\xi = \xi^i \frac{\partial}{\partial x^i}|_{(x,u)} - \xi^i u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}|_{(x,u)}$ is an isomorphism between the vector spaces T_xM and $H_{(x,u)}TM$. Similarly, the map $\xi \to {}^V\xi = \xi^i \frac{\partial}{\partial u^i}|_{(x,u)}$ is an isomorphism between the vector spaces T_xM and $V_{(x,u)}TM$. Obviously, every tangent vector $W \in T_{(x,u)}TM$ can be written in the form $W = {}^HX + {}^VY$, where $X, Y \in T_xM$ are uniquely determined vectors.

Let $Z = Z^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and horizontal lifts of Z are defined by

$$^VZ = Z^i \frac{\partial}{\partial u^i}, \tag{4}$$

$$^HZ = Z^i(\frac{\partial}{\partial x^i} - u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}). \tag{5}$$

We have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial u^i}$, then $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on TM.

In particular, we have the vertical spray ${}^Vu$ and the horizontal spray ${}^Hu$ on TM defined by

$$^Vu = u^i \frac{\partial}{\partial u^i}, \quad {}^Hu = u^i(\frac{\partial}{\partial x^i} - u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}).$$

Here ${}^Vu$ is also called the canonical or Liouville vector field on TM.

The bracket operation of the vertical and horizontal vector fields is given by the formulas [2, 3]:

$$\begin{cases} \left[{}^HX, {}^HY\right] = {}^H[X,Y] - {}^V(R(X,Y)u), \\ \left[{}^HX, {}^VY\right] = {}^V(\nabla_X Y), \\ \left[{}^VX, {}^VY\right] = 0, \end{cases} \tag{6}$$

for all vector fields X and Y on M, where R is the Riemannian curvature of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

3. Vertical rescaled Berger deformation metric

Let M be a $2m$-dimensional Riemannian manifold with a Riemannian metric g. An almost paracomplex manifold is an almost product manifold (M^{2m}, φ), $\varphi^2 = id$, such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank.

A Riemannian metric g is called an anti-paraHermitian metric if

$$g(\varphi X, \varphi Y) = g(X,Y), \tag{7}$$

or equivalently (purity condition), (B-metric)

$$g(\varphi X, Y) = g(X, \varphi Y) \tag{8}$$

for any vector fields X and Y on M [13].

The triple (M^{2m}, φ, g) is called an almost anti-paraHermitian manifold (an almost B-manifold) if (M^{2m}, φ) is an almost paracomplex manifold with an anti-paraHermitian metric g [13]. Furthermore, if φ is parallel to the Levi-Civita connection ∇ of g, that is, $\nabla\varphi = 0$, then (M^{2m}, φ, g) is considered anti-paraKähler manifold (B-manifold) [13].

It is well known that the anti-paraKähler condition $\nabla\varphi = 0$ is equivalent to the paraholomorphicity of the anti-paraHermitian metric g, i.e. $\phi_\varphi g = 0$, where ϕ_φ is the Tachibana operator [14].

If (M^{2m}, φ, g) is an anti-paraKähler manifold, the Riemannian curvature tensor is pure (S.I.E.). Also, we have

$$\begin{cases} R(\varphi Y, Z) &= R(Y, \varphi Z) = R(Y,Z)\varphi = \varphi R(Y,Z), \\ R(\varphi Y, \varphi Z) &= R(Y,Z), \end{cases} \tag{9}$$

for any vector fields Y and Z on M.

Definition 1. *Let (M^{2m}, φ, g) be an almost anti-paraHermitian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function on M. Define a fiber-wise vertical rescaled Berger deformation metric noted $\tilde{g}$ on TM, by*

$$\begin{cases} \tilde{g}({}^H X, {}^H Y) &= g(X,Y), \\ \tilde{g}({}^H X, {}^V Y) &= 0, \\ \tilde{g}({}^V X, {}^V Y) &= f\big(g(X,Y) + \delta^2 g(X, \varphi u) g(Y, \varphi u)\big), \end{cases}$$

for any vector fields X and Y on M, where δ is some constant and f is called twisting function [12].

In the following, we consider $\lambda = 1 + \delta^2\alpha$ and $\alpha = g(u,u) = \|u\|^2$, where $\|.\|$ denotes the norm with respect to (M^{2m}, φ, g).

Lemma 2. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold. Then we have the following:*

i. ${}^H X(g(u, \varphi u)) = 0$,

ii. ${}^V X(g(u, \varphi u)) = 2g(X, \varphi u)$,

iii. ${}^H X(g(Y, \varphi u)) = g(\nabla_X Y, \varphi u)$,

iv. ${}^V X(g(Y, \varphi u)) = g(X, \varphi Y)$,

v. ${}^V(\varphi u)(g(Y, \varphi u)) = g(Y, u)$,

for any vector fields X and Y on M.

Proof. The results follow directly from the equations (4) and (5). ∎

The Levi-Civita connection $\widetilde{\nabla}$ of the vertical rescaled Berger deformation metric $\tilde{g}$ on TM is given by the following theorem.

Theorem 3. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then the Levi-Civita connection $\widetilde{\nabla}$ satisfies the following properties:*

i. $\widetilde{\nabla}_{{}^H X}{}^H Y = {}^H(\nabla_X Y) - \frac{1}{2}{}^V(R(X,Y)u)$,

ii. $\widetilde{\nabla}_{{}^H X}{}^V Y = {}^V(\nabla_X Y) + \dfrac{1}{2f}X(f){}^V Y + \dfrac{f}{2}{}^H(R(u,Y)X)$,

iii. $\widetilde{\nabla}_{{}^V X}{}^H Y = \dfrac{1}{2f}Y(f){}^V X + \dfrac{f}{2}{}^H(R(u,X)Y)$,

iv. $\widetilde{\nabla}_{{}^V X}{}^V Y = \dfrac{-1}{2f}\tilde{g}({}^V X, {}^V Y){}^H(grad f) + \dfrac{\delta^2}{\lambda}g(X, \varphi Y){}^V(\varphi u)$,

for any vector fields X and Y on M, where ∇ and R denotes respectively the Levi-Civita connection and the curvature tensor of (M^{2m}, φ, g) [12].

Lemma 4. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then we have*

i. $\widetilde{\nabla}_{{}^H X}{}^V(\varphi u) = \dfrac{1}{2f}X(f){}^V(\varphi u)$,

ii. $\widetilde{\nabla}_{{}^V(\varphi u)}{}^H X = \dfrac{1}{2f}X(f){}^V(\varphi u)$,

ii. $\widetilde{\nabla}_{V_X}{}^V(\varphi u) = \dfrac{-\lambda}{2} g(X,\varphi u)^H(grad\, f) + {}^V(\varphi X) + \dfrac{\delta^2}{\lambda} g(X,u)^V(\varphi u)$,

iv. $\widetilde{\nabla}_{V_{(\varphi u)}}{}^V X = \dfrac{-\lambda}{2} g(X,\varphi u)^H(grad\, f) + \dfrac{\delta^2}{\lambda} g(X,u)^V(\varphi u)$,

v. $\widetilde{\nabla}_{V_{(\varphi u)}}{}^V(\varphi u) = \dfrac{-\lambda\alpha}{2}{}^H(grad\, f) + \dfrac{\delta^2}{\lambda} g(u,\varphi u)^V(\varphi u) + {}^V u$,

for any vector field X on M.

Proof. The results follow directly from Theorem 3. ∎

The Riemannian curvature tensor $\widetilde{R}$ of vertical rescaled Berger deformation metric $\tilde{g}$ on TM is given by the following theorem.

Theorem 5. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(TM,\tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric. Then the Levi-Civita connection $\widetilde{\nabla}$ satisfies the following properties:*

$$\begin{aligned}
\widetilde{R}({}^HX,{}^HY)^HZ &= {}^H(R(X,Y)Z) + \frac{f}{2}{}^H(R(u,R(X,Y)u)Z) + \frac{f}{4}{}^H(R(u,R(X,Z)u)Y) - \frac{f}{4}{}^H(R(u,R(Y,Z)u)X)\\
&\quad + \frac{1}{2}{}^V((\nabla_Z R)(X,Y)u) + \frac{1}{2f}Z(f)^V(R(X,Y)u) + \frac{1}{4f}Y(f)^V(R(X,Z)u) - \frac{1}{4f}X(f)^V(R(Y,Z)u),
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}({}^HX,{}^VY)^HZ &= \frac{1}{2}X(f)^H(R(u,Y)Z) + \frac{1}{4}Z(f)^H(R(u,Y)X) + \frac{f}{2}{}^H((\nabla_X R)(u,Y)Z) - \frac{1}{4}g(R(X,Z)u,Y)^H(grad\, f)\\
&\quad + \frac{1}{2}{}^V(R(X,Z)Y) - \frac{f}{4}{}^V(R(X,R(u,Y)Z)u) + \big(\frac{1}{2f}g(\nabla_X grad\, f,Z) - \frac{1}{4f^2}X(f)Z(f)\big)^V Y\\
&\quad + \frac{\delta^2}{2\lambda}g(R(X,Z)u,\varphi Y)^V(\varphi u),
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}({}^HX,{}^HY)^VZ &= \frac{f}{2}{}^H((\nabla_X R)(u,Z)Y) - \frac{f}{2}{}^H((\nabla_Y R)(u,Z)X) + \frac{1}{4}X(f)^H(R(u,Z)Y) - \frac{1}{4}Y(f)^H(R(u,Z)X)\\
&\quad - \frac{1}{2}g(R(X,Y)u,Z)^H(grad\, f) + {}^V(R(X,Y)Z) - \frac{f}{4}{}^V(R(X,R(u,Z)Y)u) + \frac{f}{4}{}^V(R(Y,R(u,Z)X)u)\\
&\quad + \frac{\delta^2}{\lambda}g(R(X,Y)u,\varphi Z)^V(\varphi u),
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}({}^HX,{}^VY)^VZ &= \frac{1}{4f}X(f)\big(g(Y,Z) + \delta^2 g(Y,\varphi u)g(Z,\varphi u)\big)^H(grad\, f) - \frac{1}{2}\big(g(Y,Z) + \delta^2 g(Y,\varphi u)g(Z,\varphi u)\big)^H(\nabla_X grad\, f)\\
&\quad - \frac{f}{2}{}^H(R(Y,Z)X) - \frac{f^2}{4}{}^H(R(u,Y)R(u,Z)X) + \frac{1}{4}\big(g(Y,Z) + \delta^2 g(Y,\varphi u)g(Z,\varphi u)\big)^V(R(X,grad\, f)u)\\
&\quad - \frac{1}{4}g(R(u,Z)X,grad\, f)^V Y,
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}({}^VX,{}^VY)^HZ &= f^H(R(X,Y)Z) + \frac{f^2}{4}\big({}^H(R(u,X)R(u,Y)Z) - {}^H(R(u,Y)R(u,X)Z)\big)\\
&\quad + \frac{1}{4}\big(g(R(u,Y)Z,grad\, f)^V X - g(R(u,X)Z,grad\, f)^V Y\big),
\end{aligned}$$

$$\begin{aligned}
\widetilde{R}({}^VX,{}^VY)^VZ &= \frac{\delta^2}{2}\big(g(X,\varphi Z)g(Y,\varphi u) - g(Y,\varphi Z)g(X,\varphi u)\big)^H(grad\, f) - \frac{1}{4}\tilde{g}({}^VY,{}^VZ)^H(R(u,X)grad\, f)\\
&\quad + \frac{1}{4}\tilde{g}({}^VX,{}^VZ)^H(R(u,Y)grad\, f)) - \frac{1}{4f^2}\|grad\, f\|^2\big(\tilde{g}({}^VY,{}^VZ)^V X - \tilde{g}({}^VX,{}^VZ)^V Y\big)\\
&\quad + \frac{\delta^2}{\lambda}\big(g(Y,\varphi Z)^V(\varphi X) - g(X,\varphi Z)^V(\varphi Y)\big) + \frac{\delta^4}{\lambda^2}\big(g(Y,u)g(X,\varphi Z) - g(X,u)g(Y,\varphi Z)\big)^V(\varphi u),
\end{aligned}$$

for any vector fields X,Y and Z on M, where ∇ and R denote respectively the Levi-Civita connection and the curvature tensor of (M^{2m},φ,g) [12].

4. Vertical rescaled Berger deformation metric on φ-unit tangent bundle $T_1^{\varphi}M$

The φ-unit tangent sphere bundle over an anti-paraKähler manifold (M^{2m},φ,g), is the hypersurface [11, 15]

$$T_1^{\varphi}M \quad = \quad \big\{(x,u)\in TM,\, g(u,\varphi u)=1\big\}. \tag{10}$$

If we set

$$\begin{array}{rcl} F: TM & \to & \mathbb{R} \\ (x,u) & \mapsto & F(x,u)=g(u,\varphi u)-1, \end{array}$$

the hypersurface $T_1^{\varphi}M$ is given by

$$T_1^{\varphi}M=\{(x,u)\in TM, \quad F(x,u)=0\},$$

the gradient of F with respect to $\tilde{g}$, $\widetilde{grad}F$ is a normal vector field to $T_1^{\varphi}M$. From the Lemma 2, for any vector field X on M, we get

$$\begin{array}{rcl} \tilde{g}({}^{H}X,\widetilde{grad}F) & = & {}^{H}X(F)={}^{H}X(g(u,\varphi u)-1)=0, \\ \tilde{g}({}^{V}X,\widetilde{grad}F) & = & {}^{V}X(F)={}^{V}X(g(u,\varphi u)-1)=2g(X,\varphi u)=\dfrac{2}{f\lambda}\tilde{g}({}^{V}X,{}^{V}(\varphi u)). \end{array}$$

So, we have

$$\widetilde{grad}F=\frac{2}{f\lambda}{}^{V}(\varphi u).$$

Then the unit normal vector field to $T_1^{\varphi}M$ is given by

$$\mathcal{N}=\frac{\widetilde{grad}F}{\sqrt{\tilde{g}(\widetilde{grad}F,\widetilde{grad}F)}}=\frac{{}^{V}(\varphi u)}{\sqrt{\tilde{g}({}^{V}(\varphi u),{}^{V}(\varphi u))}}=\sqrt{\frac{1}{f\lambda\alpha}}\,{}^{V}(\varphi u),$$

where $\lambda=1+\delta^2\alpha$ and $\alpha=g(u,u)$.

The tangential lift ${}^{T}X$ with respect to $\tilde{g}$ of a vector $X\in T_xM$ to $(x,u)\in T_1^{\varphi}M$ as the tangential projection of the vertical lift of X to (x,u) with respect to $\mathcal{N}$, that is

$${}^{T}X={}^{V}X-\tilde{g}_{(x,u)}({}^{V}X,\mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)}={}^{V}X-\frac{1}{\alpha}g_x(X,\varphi u)^{V}(\varphi u)_{(x,u)}.$$

For the sake of notation clarity, we will use $\bar{X}=X-\dfrac{1}{\alpha}g(X,\varphi u)\varphi u$, then ${}^{T}X={}^{V}\bar{X}$. From the above, we get the direct sum decomposition

$$T_{(x,u)}TM=T_{(x,u)}T_1^{\varphi}M\oplus span\{\mathcal{N}_{(x,u)}\}, \tag{11}$$

where $(x,u)\in T_1^{\varphi}M$. Indeed, if $W\in T_{(x,u)}TM$, then they exist $X,Y\in T_xM$, such that

$$\begin{array}{rcl} W & = & {}^{H}X+{}^{V}Y \\ & = & {}^{H}X+{}^{T}Y+\tilde{g}_{(x,u)}({}^{V}Y,\mathcal{N}_{(x,u)})\mathcal{N}_{(x,u)} \\ & = & {}^{H}X+{}^{T}Y+\dfrac{1}{\alpha}g_x(Y,\varphi u)^{V}(\varphi u)_{(x,u)}. \end{array} \tag{12}$$

From the equation (12) we can say that the tangent space $T_{(x,u)}T_1^{\varphi}M$ of $T_1^{\varphi}M$ at (x,u) is given by

$$T_{(x,u)}T_1^{\varphi}M=\{{}^{H}X+{}^{T}Y\,/\,X,Y\in T_xM, Y\in(\varphi u)^{\perp}\},$$

where $(\varphi u)^{\perp}=\big\{Y\in T_xM,\, g(Y,\varphi u)=0\big\}$. Hence $T_{(x,u)}T_1^{\varphi}M$ is spanned by vectors of the form ${}^{H}X$ and ${}^{T}Y$.

Given a vector field X on M, the tangential lift ${}^{T}X$ of X is given by

$${}^{T}X_{(x,u)}=\big({}^{V}X-\tilde{g}({}^{V}X,\mathcal{N})\mathcal{N}\big)_{(x,u)}={}^{V}X_{(x,u)}-\frac{1}{\alpha}g_x(X_x,\varphi u)^{V}(\varphi u)_{(x,u)}. \tag{13}$$

For any vector field X on M, we have the followings:

i. $\tilde{g}({}^{H}X, \mathscr{N}) = 0$,

ii. $\tilde{g}({}^{T}X, \mathscr{N}) = 0$,

iii. ${}^{T}X = {}^{V}X$ if and only if $g(X, \varphi u) = 0$,

iv. ${}^{T}(\varphi u) = 0$,

v. $g(\bar{X}, \varphi u) = 0$.

Definition 6. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(TM, \tilde{g})$ be its tangent bundle equipped with the vertical rescaled Berger deformation metric. The Riemannian metric $\hat{g}$ on $T_1^{\varphi}M$, induced by $\tilde{g}$, is completely determined by the identities*

i. $\hat{g}({}^{H}X, {}^{H}Y) = g(X, Y)$,

ii. $\hat{g}({}^{T}X, {}^{H}Y) = \hat{g}({}^{H}X, {}^{T}Y) = 0$,

iii. $\hat{g}({}^{T}X, {}^{T}Y) = f\big(g(X, Y) - \frac{1}{\alpha} g(X, \varphi u) g(Y, \varphi u)\big)$.

We shall calculate the Levi-Civita connection $\widehat{\nabla}$ of $T_1^{\varphi}M$ with vertical rescaled Berger deformation metric $\hat{g}$. This connection is characterized by the Gauss's formula:

$$\widehat{\nabla}_{\widehat{X}} \widehat{Y} = \widetilde{\nabla}_{\widehat{X}} \widehat{Y} - \tilde{g}(\widetilde{\nabla}_{\widehat{X}} \widehat{Y}, \mathscr{N}) \mathscr{N}, \tag{14}$$

for all vector fields $\widehat{X}$ and $\widehat{Y}$ on $T_1^{\varphi}M$.

Theorem 7. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(T_1^{\varphi}M, \hat{g})$ be its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric, then we have the following formulas:*

i. $\widehat{\nabla}_{{}^{H}X} {}^{H}Y = {}^{H}(\nabla_X Y) - \frac{1}{2} {}^{T}(R(X, Y)u)$,

ii. $\widehat{\nabla}_{{}^{H}X} {}^{T}Y = {}^{T}(\nabla_X Y) + \frac{1}{2f} X(f) {}^{T}Y + \frac{f}{2} {}^{H}(R(u, Y)X)$,

iii. $\widehat{\nabla}_{{}^{T}X} {}^{H}Y = \frac{1}{2f} Y(f) {}^{T}X + \frac{f}{2} {}^{H}(R(u, X)Y)$,

iv. $\widehat{\nabla}_{{}^{T}X} {}^{T}Y = \frac{-1}{2}\big(g(X, Y) - \frac{1}{\alpha} g(X, \varphi u) g(Y, \varphi u)\big) {}^{H}(grad\, f) - \frac{1}{\alpha} g(Y, \varphi u) {}^{T}(\varphi X) + \frac{1}{\alpha^2} g(X, \varphi u) g(Y, \varphi u) {}^{T}u$,

for any vector fields X and Y on M, where ∇ is the Levi-Civita connection and R is its curvature tensor of (M^{2m}, φ, g).

Proof. In the proof, we will use Theorem 3, Lemma 4 and the formula (14).

i. By direct calculation, we have

$$\begin{aligned}
\widehat{\nabla}_{{}^{H}X} {}^{H}Y &= \widetilde{\nabla}_{{}^{H}X} {}^{H}Y - \tilde{g}(\widetilde{\nabla}_{{}^{H}X} {}^{H}Y, \mathscr{N}) \mathscr{N} \\
&= {}^{H}(\nabla_X Y) - \frac{1}{2} {}^{V}(R(X, Y)u) - \tilde{g}(-\frac{1}{2} {}^{V}(R(X, Y)u), \mathscr{N}) \mathscr{N} \\
&= {}^{H}(\nabla_X Y) - \frac{1}{2} {}^{T}(R(X, Y)u).
\end{aligned}$$

ii. We have $\widehat{\nabla}_{{}^{H}X} {}^{T}Y = \widetilde{\nabla}_{{}^{H}X} {}^{T}Y - \tilde{g}(\widetilde{\nabla}_{{}^{H}X} {}^{T}Y, \mathscr{N}) \mathscr{N}$, by direct calculation, we get

$$\widetilde{\nabla}_{{}^{H}X} {}^{T}Y = {}^{T}(\nabla_X Y) + \frac{1}{2f} X(f) {}^{T}Y + \frac{f}{2} {}^{H}(R(u, Y)X)$$

and

$$\tilde{g}(\widetilde{\nabla}_{{}^{H}X} {}^{T}Y, \mathscr{N}) \mathscr{N} = 0.$$

Hence

$$\widehat{\nabla}_{{}^{H}X} {}^{T}Y = {}^{T}(\nabla_X Y) + \frac{1}{2f} X(f) {}^{T}Y + \frac{f}{2} {}^{H}(R(u, Y)X).$$

iii. Also, we have $\widehat{\nabla}_{^TX}{}^HY = \widetilde{\nabla}_{^TX}{}^HY - \tilde{g}(\widetilde{\nabla}_{^TX}{}^HY, \mathscr{N})\mathscr{N}$, by direct calculation, we get

$$\widetilde{\nabla}_{^TX}{}^HY = \frac{1}{2f}Y(f)^TX + \frac{f}{2}{}^H(R(u,X)Y)$$

and

$$\tilde{g}(\widetilde{\nabla}_{^HX}{}^TY, \mathscr{N})\mathscr{N} = 0.$$

Hence

$$\widehat{\nabla}_{^HX}{}^TY \quad = \quad \frac{1}{2f}Y(f)^TX + \frac{f}{2}{}^H(R(u,X)Y).$$

iv. In the same way above, we have $\widehat{\nabla}_{^TX}{}^TY = \widetilde{\nabla}_{^TX}{}^TY - \tilde{g}(\widetilde{\nabla}_{^TX}{}^TY, \mathscr{N})\mathscr{N}$,

$$\begin{aligned}\widetilde{\nabla}_{^TX}{}^TY \quad = \quad & \frac{-1}{2}\big(g(X,Y) - \frac{1}{\alpha}g(X,\varphi u)g(Y,\varphi u)\big)^H(grad\, f) - \frac{1}{\alpha}g(Y,\varphi u)^V(\varphi X) + \frac{1}{\delta^2\alpha^2}g(X,\varphi u)g(Y,\varphi u)^Tu \\ & + \Big(\frac{\lambda+1}{\lambda\alpha^2}g(X,u)g(Y,\varphi u) + \frac{1}{\lambda\alpha^2}g(X,\varphi u)g(Y,u) - \frac{(\lambda+1)}{\lambda\alpha^3}g(X,\varphi u)g(Y,\varphi u) - \frac{1}{\lambda\alpha}g(X,\varphi Y)\Big)^V(\varphi u)\end{aligned}$$

and

$$\begin{aligned}\tilde{g}(\widetilde{\nabla}_{^TX}{}^TY, \mathscr{N})\mathscr{N} \quad = \quad & \frac{-1}{\alpha^2}g(X,u)g(Y,\varphi u)^V(\varphi u) + \frac{1}{\alpha^3}g(X,\varphi u)g(Y,\varphi u)^V(\varphi u) \\ & + \Big(\frac{\lambda+1}{\lambda\alpha^2}g(X,u)g(Y,\varphi u) + \frac{1}{\lambda\alpha^2}g(X,\varphi u)g(Y,u) - \frac{\lambda+1}{\lambda\alpha^3}g(X,\varphi u)g(Y,\varphi u) - \frac{1}{\lambda\alpha}g(X,\varphi Y)\Big)^V(\varphi u).\end{aligned}$$

Hence

$$\widehat{\nabla}_{^TX}{}^TY \quad = \quad \frac{-1}{2}\big(g(X,Y) - \frac{1}{\alpha}g(X,\varphi u)g(Y,\varphi u)\big)^H(grad\, f) - \frac{1}{\alpha}g(Y,\varphi u)^T(\varphi X) + \frac{1}{\alpha^2}g(X,\varphi u)g(Y,\varphi u)^Tu.$$

■

Now, we shall calculate the Riemannian curvature tensor of $T_1^{\varphi}M$ with the vertical rescaled Berger deformation metric $\hat{g}$.

Denoting by $\widehat{R}$ the Riemannian curvature tensors of $(T_1^{\varphi}M, \hat{g})$, from the Gauss equation for hypersurfaces we deduce that $\widehat{R}(\widehat{X},\widehat{Y})\widehat{Z}$ satisfies

$$\widehat{R}(\widehat{X},\widehat{Y})\widehat{Z} = {}^t(\widetilde{R}(\widehat{X},\widehat{Y})\widehat{Z}) - B(\widehat{X},\widehat{Z}).A_{\mathscr{N}}\widehat{Y} + B(\widehat{Y},\widehat{Z}).A_{\mathscr{N}}\widehat{X}, \tag{15}$$

for all $\widehat{X}, \widehat{Y}$ and $\widehat{Z}$ vector fields on $T_1^{\varphi}M$, where ${}^t(\widetilde{R}(\widehat{X},\widehat{Y})\widehat{Z})$ is the tangential component of $\widetilde{R}(\widehat{X},\widehat{Y})\widehat{Z}$ with respect to the direct sum decomposition (11), $A_{\mathscr{N}}$ is the shape operator of $T_1^{\varphi}M$ in $(TM, \tilde{g})$ derived from $\mathscr{N}$, and B is the second fundamental form of $T_1^{\varphi}M$ (as a hypersurface immersed in TM), associated to $\mathscr{N}$ on $T_1^{\varphi}M$.

$A_{\mathscr{N}}\widehat{X}$ is the tangential component of $(-\widetilde{\nabla}_{\widehat{X}}\mathscr{N})$ i.e.

$$A_{\mathscr{N}}\widehat{X} = -{}^t(\widetilde{\nabla}_{\widehat{X}}\mathscr{N}),$$

$B(\widehat{X},\widehat{Y})$ is given by Gauss's formula, $\widetilde{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla}_{\widehat{X}}\widehat{Y} + B(\widehat{X},\widehat{Y}).\mathscr{N}$, so

$$B(\widehat{X},\widehat{Y}) = \tilde{g}(\widetilde{\nabla}_{\widehat{X}}\widehat{Y}, \mathscr{N}).$$

Theorem 8. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold and $(T_1^{\varphi}M, \hat{g})$ its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric, then we have the following formulas:*

$$\begin{aligned}\widehat{R}({}^HX, {}^HY){}^HZ \quad = \quad & {}^H(R(X,Y)Z) + \frac{f}{2}{}^H(R(u,R(X,Y)u)Z) + \frac{f}{4}{}^H(R(u,R(X,Z)u)Y) - \frac{f}{4}{}^H(R(u,R(Y,Z)u)X) \\ & + \frac{1}{2}{}^T((\nabla_Z R)(X,Y)u) + \frac{1}{2f}Z(f)^T(R(X,Y)u) + \frac{1}{4f}Y(f)^T(R(X,Z)u) - \frac{1}{4f}X(f)^T(R(Y,Z)u),\end{aligned}$$

$$\widehat{R}(^HX,^TY)^HZ = \frac{1}{2}X(f)^H(R(u,Y)Z)+\frac{1}{4}Z(f)^H(R(u,Y)X)+\frac{f}{2}{}^H((\nabla_X R)(u,Y)Z)-\frac{1}{4}g(R(X,Z)u,Y)^H(grad\,f)$$
$$+\frac{1}{2}{}^T(R(X,Z)\bar{Y})-\frac{f}{4}{}^T(R(X,R(u,Y)Z)u)+\big(\frac{1}{2f}g(\nabla_X grad\,f,Z)-\frac{1}{4f^2}X(f)Z(f)\big)^TY,$$

$$\widehat{R}(^HX,^HY)^TZ = \frac{f}{2}{}^H((\nabla_X R)(u,Z)Y)-\frac{f}{2}{}^H((\nabla_Y R)(u,Z)X)+\frac{1}{4}X(f)^H(R(u,Z)Y)-\frac{1}{4}Y(f)^H(R(u,Z)X)$$
$$-\frac{1}{2}g(R(X,Y)u,Z)^H(grad\,f)+{}^T(R(X,Y)\bar{Z})-\frac{f}{4}{}^T(R(X,R(u,Z)Y)u)+\frac{f}{4}{}^T(R(Y,R(u,Z)X)u),$$

$$\widehat{R}(^HX,^TY)^TZ = \frac{1}{4f}X(f)g(\bar{Y},\bar{Z})^H(grad\,f)-\frac{1}{2}g(\bar{Y},\bar{Z})^H(\nabla_X grad\,f)-\frac{f^2}{4}{}^H(R(u,Y)R(u,Z)X)$$
$$-\frac{f}{2}{}^H(R(\bar{Y},\bar{Z})X)+\frac{1}{4}g(\bar{Y},\bar{Z})^T(R(X,grad\,f)u)-\frac{1}{4}g(R(u,Z)X,grad\,f)^TY,$$

$$\widehat{R}(^TX,^TY)^HZ = f^H(R(\bar{X},\bar{Y})Z)+\frac{f^2}{4}\big({}^H(R(u,X)R(u,Y)Z)-{}^H(R(u,Y)R(u,X)Z)\big)$$
$$+\frac{1}{4}\big(g(R(u,Y)Z,grad\,f)^TX-g(R(u,X)Z,grad\,f)^TY\big),$$

$$\widehat{R}(^TX,^TY)^TZ = -\frac{f}{4}\big(g(\bar{Y},\bar{Z})^H(R(u,X)grad\,f)-g(\bar{X},\bar{Z})^H(R(u,Y)grad\,f)\big)-\frac{1}{4f}\|grad\,f\|^2\big(g(\bar{Y},\bar{Z})^TX-g(\bar{X},\bar{Z})^TY\big)$$
$$+\frac{1}{\alpha}\big(g(\bar{Y},\varphi\bar{Z})^T(\varphi X)-g(\bar{X},\varphi\bar{Z})^T(\varphi Y)\big)-\frac{1}{\alpha^2}\big(g(\bar{Y},\varphi\bar{Z})g(X,\varphi u)-g(\bar{X},\varphi\bar{Z})g(Y,\varphi u)\big)^Tu,$$

for all vector fields X,Y and Z on M, where $\bar{X}=X-\frac{1}{\alpha}g(X,\varphi u)\varphi u$.

Proof. Using Theorem 3 and Lemma 4, we obtain

$$A_{\mathcal{N}}{}^HX=0,\; A_{\mathcal{N}}{}^TX=-\sqrt{\frac{1}{f\lambda\alpha}}\Big({}^T(\varphi X)-\frac{1}{\alpha}g(X,\varphi u)^TU\Big), \tag{16}$$

$$B(^HX,^HY)=B(^HX,^TY)=B(^TX,^HY)=0 \tag{17}$$

and

$$B(^TX,^TY) = -\sqrt{\frac{f}{\lambda\alpha}}\,g(\bar{X},\varphi\bar{Y}). \tag{18}$$

It is now sufficient to use Theorem 5 and the equations (15)-(18) to obtain the required formulae for the curvature tensor (see [16]). ■

5. Geodesics of the vertical rescaled Berger deformation metric on the φ-unit tangent bundle

Lemma 9. *Let (M,g) be a Riemannian manifold and $x: I\to M$ be a curve on M, where I is open interval of $\mathbb{R}$. If $C: t\in I\to C(t)=(x(t),u(t))\in TM$ is a curve in TM such $u(t)\in T_{x(t)}M$ (i.e. $u(t)$ is a vector field along $x(t)$), then*

$$\dot{C}={}^H\dot{x}+{}^V(\nabla_{\dot{x}}u), \tag{19}$$

where $\dot{x}=\frac{dx}{dt}$ and $\dot{C}=\frac{dC}{dt}$ [17].

Subsequently we denote $x'_t=\dot{x}$, $x''_t=\nabla_{x'_t}x'_t$, $u'_t=\nabla_{x'_t}u$, $u''_t=\nabla_{x'_t}u'_t$ and $C'_t=\dot{C}$, then $C'_t={}^Hx'_t+{}^Vu'_t$.

Lemma 10. *Let $(T_1^{\varphi}M,\hat{g})$ be a φ-unit tangent bundle over anti-paraKähler manifold (M^{2m},φ,g) endowed with the vertical rescaled Berger deformation metric. If $C(t)=(x(t),u(t))$ is a curve on $T_1^{\varphi}M$, then*

$$C'_t={}^Hx'_t+{}^Tu'_t. \tag{20}$$

Proof. Using the equation (19), we have

$$C'_t \quad = \quad {}^H x'_t + {}^V u'_t = {}^H x'_t + {}^T u'_t + \frac{1}{\alpha} g(u'_t, \varphi u)^V(\varphi u).$$

Since $C(t) = (x(t), u(t)) \in T_1^{\varphi} M$ then $g(u, \varphi u) = 1$, on the other hand

$$0 \quad = \quad x'_t g(u, \varphi u) = 2g(u'_t, \varphi u).$$

Hence, the proof of the lemma is completed. ■

Subsequently, let t be an arc length parameter on C, From equation (20), we have

$$1 = \|x'_t\|^2 + f\|u'_t\|^2. \tag{21}$$

Theorem 11. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(T_1^{\varphi} M, \hat{g})$ be its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric and $C(t) = (x(t), u(t))$ be a curve on $T_1^{\varphi} M$. Then C is a geodesic on $T_1^{\varphi} M$ if and only if*

$$\begin{cases} x''_t &= fR(u'_t, u)x'_t + \dfrac{\kappa^2}{2f^2} grad\, f, \\ u''_t &= \dfrac{-1}{f} x'_t(f) u'_t. \end{cases} \tag{22}$$

Moreover,

$$\begin{cases} \|u'_t\| &= \dfrac{\kappa}{f}, \\ \|x'_t\| &= \sqrt{1 - \dfrac{\kappa^2}{f}}, \end{cases} \tag{23}$$

where $\kappa = constant$ and $0 \leq \kappa \leq 1$.

Proof. Using formula (20) and Theorem 7, we compute the derivative $\widehat{\nabla}_{C'_t} C'_t$.

$$\begin{aligned} \widehat{\nabla}_{C'_t} C'_t &= \widehat{\nabla}_{({}^H x'_t + {}^T u'_t)} ({}^H x'_t + {}^T u'_t) \\ &= \widehat{\nabla}_{{}^H x'_t} {}^H x'_t + \widehat{\nabla}_{{}^H x'_t} {}^T u'_t + \widehat{\nabla}_{{}^T u'_t} {}^H x'_t + \widehat{\nabla}_{{}^T u'_t} {}^T u'_t \\ &= {}^H x''_t - \frac{1}{2} {}^T(R(x'_t, x'_t)u) + {}^T u''_t + \frac{1}{2f} x'_t(f) {}^T u'_t + \frac{f}{2} {}^H(R(u, u'_t)x'_t) + \frac{1}{2f} x'_t(f) {}^T u'_t + \frac{f}{2} {}^H(R(u, u'_t)x'_t) \\ &\quad -\frac{1}{2}\big(\|u'_t\|^2 - \frac{1}{\alpha} g(u'_t, \varphi u)^2\big) grad\, f - \frac{1}{\alpha} g(u'_t, \varphi u)^T(\varphi u'_t) + \frac{1}{\alpha^2} g(u'_t, \varphi u)^{2T} u \\ &= {}^H\big(x''_t - fR(u'_t, u)x'_t - \frac{1}{2}\|u'_t\|^2 grad\, f\big) + {}^T\big(u''_t + \frac{1}{f} x'_t(f) u'_t\big). \end{aligned}$$

If we put $\widehat{\nabla}_{C'_t} C'_t = 0$, we find

$$\begin{cases} x''_t &= fR(u'_t, u)x'_t + \dfrac{1}{2}\|u'_t\|^2 grad\, f, \\ u''_t &= \dfrac{-1}{f} x'_t(f) u'_t. \end{cases}$$

Moreover, $x'_t g(u'_t, u'_t) = 2g(u''_t, u'_t)$ and if we use $u''_t = \dfrac{-1}{f} x'_t(f) u'_t$, we find $x'_t(\ln \|u'_t\|^2) = -2x'_t(\ln f)$. Therefore $\|u'_t\| = \dfrac{\kappa}{f}$ and using the equation (21), we get $\|x'_t\| = \sqrt{1 - \dfrac{\kappa^2}{f}}$, where $\kappa = constant$ and $0 \leq \kappa \leq 1$. ■

If $x(t)$ is a curve on M, then the curve $C(t) = (x(t), x'_t(t))$ is called a natural lift of the curve $x(t)$ [2].

Corollary 12. *Let (M^{2m}, φ, g) be an anti-paraKähler manifold, $(T_1^{\varphi} M, \hat{g})$ be its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric. The natural lift $C(t) = (x(t), x'_t(t))$ of any geodesic $x(t)$ is a geodesic on $(T_1^{\varphi} M, \hat{g})$.*

A curve $C(t)=(x(t),u(t))$ on TM is horizontal lift of the curve $x(t)$ on M if and only if $u'_t=0$ [2].

Corollary 13. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(T_1^{\varphi}M,\hat{g})$ be its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric. If $C(t)=(x(t),u(t))$ is a horizontal lift of the curve $x(t)$ (i.e. $u'_t=0$) then $C(t)$ is a geodesic on $(T_1^{\varphi}M,\hat{g})$ if and only if $x(t)$ is a geodesic on (M^{2m},φ,g).*

Remark 14. *If $x(t)$ is a geodesic on (M^{2m},φ,g) locally we have*

$$x''_t=0 \quad\Leftrightarrow\quad \frac{d^2x^h}{dt^2}+\sum_{i,j=1}^{2m}\frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^h=0, \quad h=\overline{1,2m}.$$

If $C(t)=(x(t),u(t))$ is a horizontal lift of the curve $x(t)$, locally we have

$$u'_t=0 \quad\Leftrightarrow\quad \frac{du^h}{dt}+\sum_{i,j=1}^{2m}\frac{dx^j}{dt}u^i\Gamma_{ij}^h=0, \quad h=\overline{1,2m}.$$

Remark 15. *Using Remark 14, we can construct an infinity of examples of geodesics on $(T_1^{\varphi}M,\hat{g})$.*

Example 16. *Let $\big(]0,+\infty[^2,g,\varphi\big)$ be an anti-paraKähler manifold such that*

$$g=x^2dx^2+y^2dy^2,$$

and

$$\varphi\partial_x=\frac{x}{y}\partial_y \quad , \quad \varphi\partial_y=\frac{y}{x}\partial_x,$$

where $\partial_x=\dfrac{\partial}{\partial x}$. The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1=\frac{1}{x}\ ,\ \Gamma_{22}^2=\frac{1}{y}.$$

The geodesics $\gamma(t)=(x(t),y(t))$ such that $\gamma(0)=(a,b)$, $\gamma'(0)=(\xi,\eta)\in\mathbb{R}^2$ satisfy the system of equations,

$$\frac{d^2x^h}{dt^2}+\sum_{i,j=1}^{2}\frac{dx^i}{dt}\frac{dx^j}{dt}\Gamma_{ij}^h=0 \quad\Leftrightarrow\quad \begin{cases} \dfrac{d^2x}{dt^2}+\dfrac{(\frac{dx}{dt})^2}{x}=0, \\[2ex] \dfrac{d^2y}{dt^2}+\dfrac{(\frac{dy}{dt})^2}{y}=0, \end{cases}$$

$$\Leftrightarrow\quad \begin{cases} x(t)=\sqrt{2a\xi t+a^2}, \\[1ex] y(t)=\sqrt{2b\eta t+b^2}. \end{cases}$$

Then, $\gamma'(t)=\dfrac{a\xi}{\sqrt{2a\xi t+a^2}}\partial_x+\dfrac{b\eta}{\sqrt{2b\eta t+a^2}}\partial_y$ and $\gamma(t)=(\sqrt{2a\xi t+a^2},\sqrt{2b\eta t+b^2})$.

i. From Corollary 12, the curve $C(t)=(\gamma(t),\gamma'(t))$ is a geodesic on $T_1^{\varphi}]0,+\infty[^2$.

ii. If $C(t)=(\gamma(t),u(t))$ is horizontal lift of the curve $\gamma(t)$ and $u(t)=(v(t),w(t))$ i.e. $u'_t=0$ then,

$$\frac{dy^h}{dt}+\sum_{i,j=1}^{2m}\frac{dx^j}{dt}u^i\Gamma_{ij}^h=0\Leftrightarrow\begin{cases} \dfrac{dv}{dt}+\dfrac{dx}{dt}v=0, \\[2ex] \dfrac{dw}{dt}+\dfrac{dy}{dt}w=0, \end{cases} \Leftrightarrow\begin{cases} v(t)=\dfrac{k_1}{\sqrt{2a\xi t+a^2}}, \\[2ex] w(t)=\dfrac{k_2}{\sqrt{2b\eta t+b^2}}. \end{cases}$$

Hence $u(t)=\dfrac{k_1}{\sqrt{2a\xi t+a^2}}\partial_x+\dfrac{k_2}{\sqrt{2b\eta t+b^2}}\partial_y$, where $k_1,k_2\in\mathbb{R}$. From Corollary 13, the curve $C(t)=(\gamma(t),u(t))$ is a geodesic on $T_1^{\varphi}]0,+\infty[^2$.

Corollary 17. *Let (M^{2m},φ,g) be an anti-paraKähler manifold, $(T_1^{\varphi}M,\hat{g})$ its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric and $C(t)=(x(t),u(t))$ be a curve on $T_1^{\varphi}M$. If f is a constant function, then $C(t)$ is a geodesic if and only if*

$$\begin{cases} x_t'' &= fR(u_t',u)x_t', \\ u_t'' &= 0. \end{cases} \tag{24}$$

Theorem 18. *Let (M^{2m},φ,g) be a locally symmetric anti-paraKähler manifold, $(T_1^{\varphi}M,\hat{g})$ be its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric and $C(t)=(x(t),u(t))$ be a geodesic on $T_1^{\varphi}M$. If f is a constant function, then all geodesic curvatures of the projected curve $\gamma=\pi\circ C$ are constants.*

Proof. The first equation of (24), gives $x_t''=fR(u_t',u)x_t'$. It is easy to see that

$$x_t'\|x_t'\|^2=x_t'g(x_t',x_t')=2g(x_t'',x_t')=2fg(R(u_t',u)x_t',x_t')=0,$$

hence $\|x_t'\|=$ constant (constant along the curve $x(t)$).

$$\begin{aligned} x_t''' &= x_t'(f)R(u_t',u)x_t'+f(\nabla_{x_t'}R)(u_t',u)x_t')+fR(u_t'',u)x_t'+fR(u_t',u_t')x_t'+fR(u_t',u)x_t'' \\ &= fR(u_t',u)x_t''. \end{aligned}$$

Since

$$x_t'\|x_t''\|^2=x_t'g(x_t'',x_t'')=2g(x_t''',x_t'')=2fg(R(u_t',u)x_t'',x_t'')=0,$$

hence $\|x_t''\|=constant$. Continuing the process we obtain

$$x_t^{(p+1)}=fR(u_t',u)x_t^{(p)},\quad p\geq 1$$

and

$$x_t'\|x_t^{(p)}\|^2=x_t'g(x_t^{(p)},x_t^{(p)})=2g(x^{(p+1)},x^{(p)})=2fg(R(u_t',u)x_t^{(p)},x^{(p)})=0.$$

Thus, we get

$$\|x_t^{(p)}\|=constant,\quad p\geq 1. \tag{25}$$

Denote by s an arc length parameter on γ, i.e. $(\|x_s'\|=1)$. Then $x_t'=x_s'\frac{ds}{dt}$, and using the equation (23), we get

$$\frac{ds}{dt}=\sqrt{1-\frac{\kappa^2}{f}}=constant. \tag{26}$$

Let $\nu_1=x_s'$ be the first vector in the Frenet frame $\nu_1,\ldots,\nu_{2m-1}$ along γ and let $k_1,\ldots,k_{2m-1}$ the geodesic curvatures of γ. Then the Frenet formulas verify

$$\begin{cases} (\nu_1)_s' &= k_1\nu_2, \\ (\nu_i)_s' &= -k_{i-1}\nu_{i-1}+k_i\nu_{i+1}, \quad 2\leq i\leq 2m-2 \\ (\nu_{2m-1})_s' &= -k_{2m-2}\nu_{2m-2}. \end{cases} \tag{27}$$

From the equation (26), we have

$$x_t'=x_s'\frac{ds}{dt}=\sqrt{1-\frac{\kappa^2}{f}}\nu_1. \tag{28}$$

Using the equation (28) and the Frenet formulas (27), we obtain

$$x_t''=\sqrt{1-\frac{\kappa^2}{f}}(\nu_1)_t'=\sqrt{1-\frac{\kappa^2}{f}}(\nu_1)_s'\frac{ds}{dt}=(1-\frac{\kappa^2}{f})k_1\nu_2.$$

Now the equation (25) implies $k_1=constant$. Next, in a similar way, we have

$$x_t'''=(1-\frac{\kappa^2}{f})k_1(\nu_2)_t'=(1-\frac{\kappa^2}{f})k_1(\nu_2)_s'\frac{ds}{dt}=(1-\frac{\kappa^2}{f})\sqrt{1-\frac{\kappa^2}{f}}k_1(-k_1\nu_1+k_2\nu_3).$$

Likewise, using the equation (25), we find $k_2=constant$. By continuing the process, we finish the proof. ∎

Lemma 19. *Let (M^{2m},φ,g) be an anti-paraKähler manifold and $(T_1^{\varphi}M,\hat{g})$ be its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric. If $C(t)=(x(t),u(t))$ is a curve on $T_1^{\varphi}M$, then we have*

i. $\Gamma(t)=(x(t),\varphi u(t))$ *is a curve on* $T_1^{\varphi}M$.

ii. $\Gamma(t)$ *is a geodesic on* $T_1^{\varphi}M$ *if and only if* $C(t)$ *is a geodesic on* $T_1^{\varphi}M$.

Proof. i. We put $y(t)=\varphi u(t)$, then $g(y,\varphi y)=g(\varphi u,\varphi(\varphi u))=g(\varphi u,u)$, since $C(t)=(x(t),u(t))\in T_1^{\varphi}M$, we get $g(u,\varphi u)=1$ hence, $g(y,\varphi(\varphi y))=1$, i.e. $\Gamma(t)=(x(t),y(t))\in T_1^{\varphi}M$.

ii. In a similar way the proof of the equation (22), and using $y'_t=\varphi u'_t$ and $y''_t=\varphi u''_t$, we have

$$\begin{aligned}\widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= {}^{H}\Big(x''_t - fR(y'_t,y)x'_t - \frac{1}{2}\|y'_t\|^2 grad\, f\Big) + {}^{T}\Big(y''_t + \frac{1}{f}x'_t(f)y'_t\Big)\\ &= {}^{H}\Big(x''_t - fR(u'_t,u)x'_t - \frac{1}{2}\|u'_t\|^2 grad\, f\Big) + {}^{T}\Big(\varphi u''_t + \frac{1}{f}x'_t(f)\varphi u'_t\Big)\end{aligned}$$

and

$$\begin{aligned}\widehat{\nabla}_{\Gamma'_t}\Gamma'_t = 0 \quad &\Leftrightarrow \quad \begin{cases} x''_t = fR(u'_t,u)x'_t + \frac{1}{2}\|u'_t\|^2 grad\, f\Big) \\ \varphi u''_t = -\frac{1}{f}x'_t(f)\varphi u'_t \end{cases}\\ &\Leftrightarrow \quad \begin{cases} x''_t = fR(u'_t,u)x'_t + \frac{1}{2}\|u'_t\|^2 grad\, f\Big) \\ u''_t = -\frac{1}{f}x'_t(f)u'_t \end{cases}\\ &\Leftrightarrow \quad \widehat{\nabla}_{C'_t}C'_t = 0.\end{aligned}$$

■

From Theorem 18 and Lemma 19, we have the following theorem:

Theorem 20. *Let (M^{2m},φ,g) be a locally symmetric anti-paraKähler manifold, $(T_1^{\varphi}M,\hat{g})$ be its φ-unit tangent bundle equipped with the vertical rescaled Berger deformation metric and $\Gamma(t)=(x(t),\varphi u(t))$ be a geodesic on $T_1^{\varphi}M$. If f is a constant function, then all geodesic curvatures of the projected curve $\gamma=\pi\circ\Gamma$ are constants.*

6. Conclusions

In this work, we studied some geometric properties of a φ-unit tangent bundle with the vertical rescaled Berger deformation metric. We checked the Levi-Civita connection for this metric. We also calculated all forms of the Riemannian curvature tensors. Finally, we study the geodesics on the φ-unit tangent bundle concerning the vertical rescaled Berger deformation metric.

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