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$AW(k)$ -Type Curves in Modified Orthogonal Frame

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Abstract: The goal of this article is to examine $AW(k)$ -type curves in Euclidean 3-space according to a modified orthogonal frame with non-zero curvature. Firstly, the relations between the curvatures κ,τ for $AW(k)$ -type curves in the modified orthogonal frame are given. Also, the harmonic curvatures of $AW(k)$ -type curves according to this frame are obtained. The results are illustrated in examples. Finally, slant helices are analyzed for the modified orthogonal frame and some relations are obtained for the curvatures of the curve to be of $AW(1)$ and $AW(2)$ type in case the curve is a slant helix.

Modifiye Ortogonal Çatıda $AW(k)$ -Tipinde Eğriler

Anahtar Kelimeler
AW (k)-tipinde eğriler,
Modifiye ortogonal çatı,
Harmonik eğrilikler,
Slant helisler

Öz: Bu çalışmanın amacı 3-boyutlu Öklid uzayı E^3 de $AW(k)$ -tipinde eğrileri, eğrilik ile modifiye edilmiş ortogonal çatıya göre analiz etmektir. İlk olarak modifiye ortogonal çatıya göre $AW(k)$ -tipinde eğriler için κ,τ eğrilikleri arasındaki bağıntılar verilmiştir. Ayrıca bu çatıya göre $AW(k)$ -tipinde eğrilerin harmonik eğrilikleri elde edilmiştir. Bulunan sonuçlar örnekler üzerinde gösterilmiştir. Son olarak, slant helisler modifiye ortogonal çatıya göre incelenmiş ve eğrinin slant helis olması durumunda eğrinin eğriliklerinin $AW(1)$ ve $AW(2)$ tipinde olması için bazı bağıntılar elde edilmiştir.

1. INTRODUCTION

Arslan et al. [1] explored the concept of $AW(k)$ -type submanifolds, and many subsequent studies have focused on $AW(k)$ -type curves. For example, the authors in [2, 3] provided detailed characteristics and curvature criteria for these curves in E^m. In [4], $AW(k)(k=1,2\ or\ 3)$ -type curves and surfaces were considered. $AW(k)$ -type curves in Euclidean, Lorentz, and Galilean spaces have yielded several intriguing findings in [5-7]. In Euclidean 3-space, a moving frame at a particular point on any regular curve is known as a Frenet frame. The tangent, normal, and binormal vectors of the curve, which are the curve's Serret-Frenet vectors, combine to form the orthonormal Frenet frame of the curve. With this frame, the curve's curvature and torsion functions can be determined. In this regard, there are numerous sources of research on the Frenet frame of the regular curve [8–10].

A regular curve is defined as having functions (torsion and curvature) that can be differentiated at every point on the curve, in accordance with the basic theorem of regular curves [11]. On the analytical curves, the curvature function might, nevertheless, be zero at certain locations. It is evident that the curvature function κ isn't always differentiable since the principal normal and binormal vectors of the curves are typically discontinuous at the curvature's zero point. In this instance as well, an analytical curve's Frenet derivative equations lead to uncertainty at the point where the curvature disappears. Various substitute frames have been constructed to eliminate these uncertain situations [12, 13]. Many researchers have examined curves and surfaces using these different frameworks [14–16]. After considering this issue, Hord [17] and Sasai [18] engaged in a different frame that is effective on these issues. Sasai [19] presented an orthogonal frame for unit speed analytical curves in a straightforward but practical method. Even though the curvature function κ multiplies each Frenet vector to get these modified orthogonal frame vectors, they enable the application of a new formula that corresponds to the Frenet derivative equations for the aforementioned situation. With its help, analytical curves with singular points can be explored efficiently.

Subsequently, Bükçü et al. [20] expanded Sasai's [21] investigation and got the newly modified frame by using the Frenet vectors and the torsion. The modified orthogonal frame of a curve in Lorentzian or Euclidean 3-space has been the area of several searches [22-27].

The arrangement of the paper is as follows. The 2nd section gives some basic concepts on modified orthogonal frames with non-zero curvature over the curves. Curves of type *AW(k)* are given in the 3rd section using the modified orthogonal frame in E^3. In the 4th section, the findings regarding the harmonic curvatures of the $AW(k)$ curves in the modified orthogonal frames and their status as general helix, slant helix, and circular helix are discussed.

2. MATERIAL AND METHOD

Let α be a space curve in Euclidean 3-space according to the arclenght s. At each point $\alpha(s)$ on a curve α, there are vectors the tangent $\boldsymbol{t}$, principal normal $\boldsymbol{n}$, and binormal $\boldsymbol{b}$ respectively. The Serret-Frenet equations are given by

$$\begin{aligned} \boldsymbol{t}' &= \kappa \boldsymbol{n}, \\ \boldsymbol{n}' &= -\kappa \boldsymbol{t} + \tau \boldsymbol{b}, \\ \boldsymbol{b}' &= -\tau \boldsymbol{n}, \end{aligned}$$

where κ, τ represent the first and second curvature of the curve, respectively.

Let $\alpha : I \to E^3$ be a space curve. We assume that the curvature κ of α is not identically zero. As a result, the modified orthogonal frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ with the curvature κ of the curve α can be defined. Now we define the modified orthogonal frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ as follows:

$$\boldsymbol{T} = \frac{d\alpha}{ds}, \quad \boldsymbol{N} = \frac{d\boldsymbol{T}}{ds}, \quad \boldsymbol{B} = \boldsymbol{T} \wedge \boldsymbol{N}.$$

The following represents the relationships between the modified orthogonal frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ and Frenet frame $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ at non-zero positions of κ

$$\begin{aligned} \boldsymbol{T} &= \boldsymbol{t}, \\ \boldsymbol{N} &= \kappa \boldsymbol{n}, \\ \boldsymbol{B} &= \kappa \boldsymbol{b}. \end{aligned} \tag{1}$$

The modified orthogonal frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ satisfies the below relations,

$$\begin{aligned} &< \boldsymbol{T}, \boldsymbol{T} > = 1, \\ &< \boldsymbol{N}, \boldsymbol{N} > = < \boldsymbol{B}, \boldsymbol{B} > = \kappa^2, \\ &< \boldsymbol{N}, \boldsymbol{T} > = < \boldsymbol{B}, \boldsymbol{T} > = < \boldsymbol{B}, \boldsymbol{N} > = 0, \end{aligned} \tag{2}$$

where $\langle , \rangle$ is the inner product.

Due to these equations, the derivative equations of the modified orthogonal frame $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ are given as

$$\begin{aligned} \boldsymbol{T}' &= \boldsymbol{N}, \\ \boldsymbol{N}' &= -\kappa^2 \boldsymbol{T} + \frac{\kappa'}{\kappa} \boldsymbol{N} + \tau \boldsymbol{B}, \\ \boldsymbol{B}' &= -\tau \boldsymbol{N} + \frac{\kappa'}{\kappa} \boldsymbol{B}, \end{aligned} \tag{3}$$

where, $\tau = \dfrac{det(\alpha', \alpha'', \alpha''')}{\kappa^2}$ is the torsion of α. The frame denoted by equations (1) and (3) is called a modified orthogonal frame with curvature [20].

Proposition 2.1 [2] Let γ be a Frenet curve of E^3 osculating order 3. Then we have

$$\begin{aligned} \gamma' &= \boldsymbol{t}, \\ \gamma'' &= \boldsymbol{t}' = \kappa \boldsymbol{n}, \\ \gamma''' &= -\kappa^2 \boldsymbol{t} + \kappa' \boldsymbol{n} + \kappa\tau \boldsymbol{b}, \\ \gamma^{(iv)} &= -3\kappa\kappa' \boldsymbol{t} + (\kappa'' - \kappa^3 - \kappa \tau^2)\boldsymbol{n} + (2\kappa' \tau + \kappa\tau')\boldsymbol{b}, \end{aligned}$$

where, $\kappa = \kappa(s)$, $\tau = \tau(s)$ are the curve's curvature and torsion of the curve γ and $\gamma' = \dfrac{d\gamma}{ds}$.

Let γ be a Frenet curve of osculating order in E^3, then we have

$$N_1 = \kappa \boldsymbol{n},$$
$$N_2 = \kappa' \boldsymbol{n} + \kappa\tau \boldsymbol{b},$$
$$N_3 = (\kappa'' - \kappa^3 - \kappa\,\tau^2)\boldsymbol{n} + (2\kappa'\tau + \kappa\tau')\boldsymbol{b},$$

where the curve's curvature and torsion are represented by the values of $\kappa = \kappa(s)$, $\tau = \tau(s)$ of the curve γ [2].

Definition 2.1 [2] For the Frenet curves,
i) of type weak $AW(2)$ if they satisfy;

$$N_3 = \langle N_3, N_2^* \rangle N_2^*,$$

ii) of type weak $AW(3)$ if they satisfy;

$$N_3 = \langle N_3, N_1^* \rangle N_1^*,$$

where,

$$N_1^* = \frac{N_1}{\| N_1 \|},\quad N_2^* = \frac{N_2 - \langle N_2, N_1^* \rangle N_1^*}{\| N_2 - \langle N_2, N_1^* \rangle N_1^* \|}.$$

Definition 2.2 [2] Frenet curves are,

i) of type $AW(1)$ if they satisfy;

$$N_3 = 0,$$

ii) of type $AW(2)$ if they satisfy;

$$\| N_2 \|^2 N_3 = \langle N_3, N_2 \rangle N_2,$$

iii) of type $AW(3)$ if they satisfy;

$$\| N_1 \|^2 N_3 = \langle N_3, N_1 \rangle N_1.$$

Definition 2.3 [28] Let γ be a unit speed curve of osculating order d. The functions

$H_i : I \to E, 1 \le j \le d-2$ defined by

$$H_1(s) = \frac{\kappa_1(s)}{\kappa_2(s)},\quad H_k = \{D_{v1}H_{k-1} + H_{k-2}\kappa_k\}\frac{1}{\kappa_{k+1}},$$
$$2 \le j \le d-2,$$

are called the harmonic curvatures of γ, where $\boldsymbol{v}1 = \gamma'$ and $\kappa_1(s), \kappa_2(s), \ldots, \kappa_{d-1}$ Frenet curvatures γ which are not necessarily constant.

Definition 2.4 [29] A curve γ with $\kappa_1 \ne 0$ is called a slant helix if the principal normal lines of the curve γ make a constant angle in a fixed direction.

Theorem 2.1 [23] Let $\gamma : I \to E^3$ be a unit speed curve in E^3. Then γ is a slant helix determined by the modified orthogonal frame if and only if

$$\frac{\tau'}{(\kappa^2 + \tau^2)^{\frac{3}{2}}}$$

is constant.

3. RESULTS

3.1. Curves in the Modified Orthogonal Frame

$AW(k)$-type curves in the modified orthogonal frame in E^3 are introduced in this section, along with some results and examples.

Proposition 3.1 Let γ be a curve with arclenght parameter s and belonging to osculating order 3 in E^3 and $\{T, N, B\}$ be a modified orthogonal frame. We have

$$\gamma' = T_\gamma,$$
$$\gamma'' = T_\gamma' = N_\gamma,$$
$$\gamma''' = -\kappa^2 T_\gamma + \frac{\kappa'}{\kappa} N_\gamma + \tau B_\gamma,$$
$$\gamma^{(iv)} = -3\kappa\kappa' T_\gamma + \left(\frac{\kappa''}{\kappa} - \kappa^2 - \tau^2\right) N_\gamma + \left(\frac{2\tau\kappa'}{\kappa} + \tau'\right) B_\gamma.$$

We will use some notations for defining $AW(k)$-type curves determined by the modified orthogonal frame like as follow

$$A_{\gamma 1} = N_\gamma, \tag{4}$$

$$A_{\gamma 2} = \frac{\kappa'}{\kappa} N_\gamma + \tau B_\gamma, \tag{5}$$

$$A_{\gamma 3} = \left(\frac{\kappa''}{\kappa} - \kappa^2 - \tau^2\right) N_\gamma + \left(\frac{2\tau\kappa'}{\kappa} + \tau'\right) B_\gamma. \tag{6}$$

Corollary 3.1 $\gamma', \gamma'', \gamma'''$ and $\gamma^{(iv)}$ are linearly dependent if and only if $A_{\gamma 1}, A_{\gamma 2}$ and $A_{\gamma 3}$ are linearly dependent.

Definition 3.1 Frenet curves are,
i) of type modified $AW(1)$ if they satisfy;

$$A_{\gamma 3} = 0, \tag{7}$$

ii) of type modified $AW(2)$ if they satisfy;

$$\| A_{\gamma_2} \|^2 A_{\gamma 3} = \langle A_{\gamma_3}, A_{\gamma_2} \rangle A_{\gamma_2}, \tag{8}$$

iii) of type modified $AW(3)$ if they satisfy,

$$\|\widetilde{A_{\gamma_1}}\|^2 \widetilde{A_{\gamma_3}} = \langle \widetilde{A_{\gamma_3}}, \widetilde{A_{\gamma_1}} \rangle \widetilde{A_{\gamma_1}}. \tag{9}$$

Definition 3.2 The unit speed curves of order 3 are

i) of type weak modified $AW(2)$ if they hold;

$$\widetilde{A_{\gamma_3}} = < \widetilde{A_{\gamma_3}}, \widetilde{A_{\gamma_2}}^* > \widetilde{A_{\gamma_2}}^*, \tag{10}$$

ii) of type weak modified $AW(3)$ if they hold**;**

$$\widetilde{A_{\gamma_3}} = < \widetilde{A_{\gamma_3}}, \widetilde{A_{\gamma_1}}^* > \widetilde{A_{\gamma_1}}^*, \tag{11}$$

where,

$$\widetilde{A_{\gamma_1}}^* = \frac{\widetilde{A_{\gamma_1}}}{\|\widetilde{A_{\gamma_1}}\|},$$

$$\widetilde{A_{\gamma_2}}^* = \frac{\widetilde{A_{\gamma_2}} - < \widetilde{A_{\gamma_2}}, \widetilde{A_{\gamma_1}}^* > \widetilde{A_{\gamma_1}}^*}{\|\widetilde{A_{\gamma_2}} - < \widetilde{A_{\gamma_2}}, \widetilde{A_{\gamma_1}}^* > \widetilde{A_{\gamma_1}}^*\|}.$$

Theorem 3.1 Let γ be a curve with arc-length parameter s and belonging to osculating order 3. If γ is of type modified $AW(1)$, then the curvature equations are given by

$$\frac{\kappa''}{\kappa} - \kappa^2 - \tau^2 = 0 \tag{12}$$

and

$$\tau = \frac{c}{\kappa^2}. \ c \in \mathbb{R}, (\kappa \neq 0) \tag{13}$$

Proof. We can easily obtain from equation (6) and (7)
Corollary 3.2 Every plane curve of $AW(1)$ type is also weak $AW(2)$ type.

Theorem 3.2 Let the curve γ be a unit speed curve of order 3. If the curve γ is of type modified $AW(2)$, then the curvature relations of the curve is given by

$$2\tau\kappa'^2 + \tau'\kappa\kappa' + \tau\kappa^4 - \tau\kappa\kappa'' + \tau^3\kappa^2 = 0. \tag{14}$$

Proof If equation (5) and equation (6) are substituted into equation (8), then we get equation (14).

Theorem 3.3 Let the curve γ be a unit speed curve of order 3. If the curve γ is of type modified $AW(3)$, then there is a relation between curvatures as follows

$$2\tau\kappa' + \tau'\kappa = 0 \tag{15}$$

and

$$\tau = \frac{c}{\kappa^2}, \quad c \in \mathbb{R}, \quad (\kappa \neq 0). \tag{16}$$

Proof If equation (4) and equation (6) substitute into equation (9) we get equations (15) and (16).

Theorem 3.4 Let the curve γ be a unit speed curve of order 3. If the curve γ is of type weak modified $AW(2)$ then the curvature relation is provided by

$$\boldsymbol{-\kappa^2 + \frac{\kappa''}{\kappa} - \tau^2 = 0.} \tag{17}$$

Proof By the below equation

$$\tilde{A}_2^* = \frac{1}{\kappa} B \tag{18}$$

if we equations (6) and (18) substitute into equation (10), then we have equation (17) .

Theorem 3.5 Let the curve γ be a unit speed curve of order 3. If the curve γ is of type weak modified $AW(3)$ then the curvature equations is given by

$$2\tau\kappa' + \tau'\kappa = 0. \tag{19}$$

Proof By the below equation

$$\tilde{A}_1^* = \frac{1}{\kappa} N, \tag{20}$$

if equations (6) and (20) are substituted into equation (11), then we have equation (19).

Example 3.1 Let us consider the curve $c_1(s)$ which is given by

$$c_1(s) = (\cos\frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}).$$

The curve c_1 satisfies modified $AW(2), AW(3)$ - type curves conditions, for $\kappa(s) = \frac{1}{2}$ and $\tau(s) = \frac{1}{2}$.

Example 3.2 Let us consider the curve $c_2(s)$ which is given by

$$c_2 = \begin{pmatrix} s, \frac{s}{6}(2\sinh(2\ln s) - \cosh(2\ln s), \\ \frac{s}{6}(2\cosh(2\ln s) - \sinh(2\ln s) \end{pmatrix}.$$

The curve $c_2(s)$ satisfies modified $AW(3)$ type curve condition, for $\kappa = 1$ and $\tau = -2$, see Figure 1.

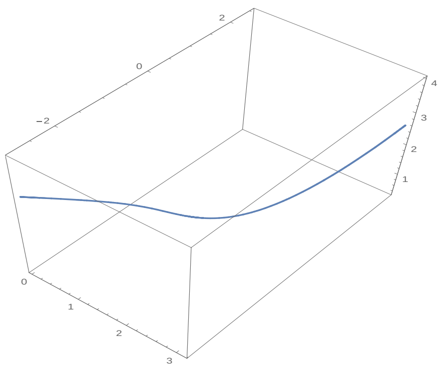

Figure 1. The curve of c_2.

3.2. Curves with Harmonic Curvatures in the Modified Orthogonal Frame

Theorem 3.6 $\gamma'', \gamma''', \gamma^{(iv)}$ are linearly dependent if and only if γ is a general helix.

Proof If γ'', γ''' and $\gamma^{(iv)}$ are linearly dependent, then the following equation is valid:

$$\begin{vmatrix} 0 & 1 & 0 \\ -\kappa^2 & \frac{\kappa'}{\kappa} & \tau \\ -3\kappa\kappa' & \left(\frac{\kappa''}{\kappa} - \kappa^2 - \tau^2\right) & \left(\frac{2\tau\kappa'}{\kappa} + \tau'\right) \end{vmatrix} = 0.$$

We have $\frac{\kappa'}{\tau'} = \frac{\kappa}{\tau}$ and $\frac{d}{ds}(\frac{\kappa(s)}{\tau(s)}) = 0$. Thus $\frac{\kappa}{\tau} =$ constant and γ is a general helix. In reverse γ is a general helix, thus $\frac{\kappa}{\tau} =$ constant and $\gamma'', \gamma''', \gamma^{(iv)}$ are linearly dependent.

We will use some notations for defining $AW(k)$-type curves determined by the modified orthogonal with harmonic curvatures like the following

$$A_{\gamma 1} = N_\gamma, \tag{21}$$

$$A_{\gamma 2} = \frac{\kappa'}{\kappa} N_\gamma + \frac{\kappa}{H} B_\gamma, , \tag{22}$$

$$\widetilde{A}_{\gamma 3} = \left(\frac{\kappa''}{\kappa} - \kappa^2 (1 + \frac{1}{H^2})\right) N_\gamma + \left(\frac{3H\kappa' - \kappa H'}{H^2}\right) B_\gamma, \tag{23}$$

where κ, τ, H are curvature, torsion, and harmonic curvature of $AW(k)$-type curves respectively. Also

$$\tau = \frac{\kappa}{H} \text{ and } \tau' = \frac{\kappa' H - \kappa H'}{H^2}.$$

Theorem 3.7 If γ is a weak harmonic curve of the modified $AW(2)$ -type, then the equation is provided by

$$\frac{\kappa''}{\kappa} - \kappa^2\left(1 + \frac{1}{H^2}\right) = 0. \tag{24}$$

Proof By the below equation

$$A^*_{\gamma 2} = \frac{1}{\kappa} B_\gamma, \tag{25}$$

if equations (23) and (25) are substituted into equation (10), then we have equation (24).

Theorem 3.8 Let the curve γ be a unit speed curve of order 3. The modified $AW(1)$ -type with harmonic curvature has no general, circular helix.

Proof Let the curve γ be a helix. Thus H is constant, $H' = 0$. Since γ is a modified $AW(1)$ -type with harmonic curvature we get

$$\begin{Bmatrix} \left(\frac{\kappa''}{\kappa} - \kappa^2(1 + \frac{1}{H^2})\right) N \\ + \left(\frac{3H\kappa' - \kappa H'}{H^2}\right) B \end{Bmatrix} = 0, \tag{26}$$

$$-\kappa^2(1 + \frac{1}{H^2}) \neq 0 .$$

Since there are no solutions of the differential equations in equation (26), the proof is completed.

Theorem 3.9 If the curve γ is a weak harmonic curve of the modified $AW(3)$-type, then the equation is given by

$$\frac{3H\kappa' - \kappa H'}{H^2} = 0. \tag{27}$$

Proof By the below equation

$$A_{\gamma_1}^{*} = \frac{1}{\kappa} N_\gamma , \tag{28}$$

if equations (23) and (28) are substituted into equation (11), then we get equation (27) .

Theorem 3.10 If the curve γ is a harmonic curve of the modified $AW(1)$ -type, then the equation is provided by

$$\left(\frac{\kappa''}{\kappa} - \kappa^2 (1 + \frac{1}{H^2}) \right) = 0 , \quad \left(\frac{3H\kappa' - \kappa H'}{H^2} \right) = 0.$$

Proof From equations (7) and (23) , the proof is obvious.

Theorem 3.11 If the curve γ is a harmonic curve of the modified $AW(2)$-type, then the equation is given by

$$\frac{\left\{ \begin{array}{l} 3H\kappa'^2 - \kappa\kappa' H' \\ -H\kappa\kappa'' + \kappa^4 H(1 + \frac{1}{H^2}) \end{array} \right\}}{\kappa H^2} = 0. \tag{29}$$

Proof If equations (22) and (23)substitute into equation (8), then we get equation(29).

Corollary 3.3 If the curve γ is the general helix of the modified $AW(2)$ -type with harmonic curvature, then the equation is given by

$$3\kappa'^2 - \kappa \kappa'' + \kappa^4 (1 + \frac{1}{H^2}) = 0.$$

Theorem 3.12 If the curve γ is a harmonic curve of the modified $AW(3)$ -type, then the equation is given by

$$3H\kappa' - \kappa H' = 0 . \tag{30}$$

Proof If equations (21) and (23) are substituted into equation (9), then we get equation (30).

Corollary 3.4 Let the curve γ be a harmonic curve of the modified $AW(3)$-type curve. If γ is a general helix, then γ is a circular helix.

Proof As the curve γ be a helix, H is constant and $H' = 0$. By equation (30), we have $3H\kappa' = 0$. Since $H \neq 0$, $\kappa' = 0$ and τ = constant, thus γ is a circular helix.

3.3 Slant Helices in the Modified Orthogonal Frame

Corollary 3.5 If the curve γ is a slant helix belonging to the osculating order 3 in E^3 , then the curvature equation is provided by

$$\tau' = c(\kappa^2 + \tau^2)^{\frac{3}{2}}, \quad c \in \mathbb{R}. \tag{31}$$

Theorem 3.13 Let the curve γ be a slant helix belonging to the osculating order 3 in E^3 . If the curve γ is of the $AW(1)$-type, then the formula is provided by

$$\kappa'' = \left(\begin{array}{l} \frac{3}{2}\frac{c_1}{c} \left\{ \frac{\kappa'}{\kappa^4} (\kappa^6 + c^2)^{\frac{3}{2}} - 9\kappa^5 \kappa' (\kappa^6 + c^2)^{\frac{1}{2}} \right\} \\ -\kappa^3 - \frac{c^2}{\kappa^3} = 0. \end{array} \right).$$

Proof From derivated equation (13),

$$\tau' = -\frac{2c\kappa'}{\kappa^3}$$

is obtained. If we substitute equation (13) into equation (14), then we get the proof.

Theorem 3.14 Let γ be a slant helix belonging to the osculating order 3 in E^3. If γ is of the $AW(2)$ - type, then the equation is provided by

$$2\tau\left(\kappa'\right)^2 + c_1\left(\kappa^2 + \tau^2\right)^{3/2} \kappa'\kappa - \kappa^4\tau - \tau\kappa\kappa'' + \tau^3\kappa^2 = 0. \tag{32}$$

Proof Using equation (31) and equation (14) we obtain equation (32).

Theorem 3.15 Let the curve γ be a slant helix belonging to an osculating order 3 in E^3. If γ is of the $AW(3)$ - type, then the formula is provided by

$$2c\kappa^3\kappa' + c_1(\kappa^6 + c^2)^{3/2} = 0.$$

Proof Since the curve γ is a slant helix of $AW(3)$ - type equations (15), (16) and (31) hold.

4. DISCUSSION AND CONCLUSION

In 1999, Arslan et al. [2] reduced the notion of a submanifold of type $AW(k)$ to a curve of type $AW(k)$ and showed the relations between the first and second curvatures of such curves with respect to the Frenet frame.

In this study, we first define $AW(k)$-type curves determined by the modified orthogonal frame in 3-dimensional Euclidean space and obtain some relations between their curvatures that the curve is $AW(k)$ -type curves. Then, the first harmonic curvature of this curve according to the modified orthogonal frame is found and the condition needed that is both sufficient and necessary for the curve to be a general helix is given. In addition, the curve is $AW(k)$-type curves, the first harmonic curvature, and some relations between the curvatures are found and a conclusion is given about the general helix and circular helix.

Finally, slant helices according to the modified orthogonal frame are analyzed and some relations are obtained for the curvatures of the curve to be $AW(1)$ -type curve and $AW(2)$-type curve in case the curve is a slant helix.

This study will shed light on the characterization of $AW(k)$-type curves for different frames for researchers working in this field.

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