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Approximation by Composition of q-Szász-Mirakyan and q-Durrmeyer-Chlodowsky Operators

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Abstract. In the present paper, q-Szász Mirakyan operators by taking the weight function of q-Chlodowsky-Durrmeyer operators on $C[0,\infty)$ are introduced and their approximation properties are investigated. Weighted approximation theorem is given and some theorems on the degree of approximation are investigated.

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1. Introduction

For a function f defined on the interval $[0,\infty)$, A. İzgi [5] defined the following operators, which are a composition of Szász-Miraykan operators by taking the weight function of Chlodowsky-Durrmeyer operators, as

$$F_n(f;x) = \frac{n+1}{b_n}\sum_{k=0}^{\infty} p_{n,k}\left(\frac{x}{b_n}\right)\int_0^{b_n}\varphi_{n,k}\left(\frac{t}{b_n}\right)f(t)dt,$$

where $p_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}$, $\varphi_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ and b_n is a sequence of positive real numbers satisfy $lim_{n\to\infty}b_n = \infty$, $lim_{n\to\infty}\frac{b_n}{n} = 0$. The q-analogue of Bernstein polynomials were introduced by Phillips [8]. Recently, Gupta and Heping [3] introduced and studied the q-analogues of usual and discretly defined Durrmeyer operators. Mahmudov and Kaffaoğlu [7] studied the local approximation and the Voronovskaja-type theorem. İspir [4] studied Baskakov operators on weighted spaces. Dalmanoglu and Kırcı Serenbay [2] established approximation properties for Chlodowsky type q-Jakimovski-Leviatan operators. Büyükyazıcı, Tanberkan, Kırcı Serenbay and Atakut [1] introduced a Kantorovich type generalization of Jakimovski-Leviatan operators constructed by A. Jakimovski and D. Leviatan and the theorems on convergence and the degrree of convergence are established.Chlodowsky type Jakimovskiy Leviatan operators. For the q-Szász-Mirakjan operators, the weighted of Korovkin-type theorem and the weighted approximation were given. Before introducing the operators, we mention the standart notations of q-calculus. For $n \in \mathbb{N}$ and q be a positive real number q-integer and q-factorial are defined by

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$$[n] = \begin{cases} \frac{1-q^n}{1-q} & \text{if} \quad q \neq 1 \\ n & \text{if} \quad q = 1 \end{cases} \quad \text{and} \quad [0] = 0$$

$$[n]! = [1][2]...[n] \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad [0]! = 1.$$

For integers $0 \leq k \leq n$ q-binomial is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}.$$

There are two q-analagues of the exponential function e^z, see [6]

$$e_q(z) = \sum_{k=0}^{\infty} \frac{z^k}{[k]!} = \frac{1}{(1-(1-q)z)_q^{\infty}} \qquad |z| < \frac{1}{1-q}, \quad |q| < 1,$$

$$E_q(z) = \prod_{j=0}^{\infty}(1+(1-q)q^j z) = \sum_{k=0}^{\infty} q^{k(k-1)/2} \frac{z^k}{[k]!} = (1+(1-q)z)_q^{\infty}, \quad |q| < 1$$

where

$$(1-q)_q^{\infty} = \prod_{j=0}^{\infty}(1-q^j x).$$

It is easily observed that

$$e_q(z)E_q(-z) = e_q(-z)E_q(z) = 1. \tag{1.1}$$

We set

$$p_{n,k}(q;x) = \frac{1}{E_q([n]x)} q^{\frac{k(k-1)}{2}} \frac{([n]x)^k}{[k]!} \quad n = 1, 2, \ldots .$$

It is clear that $p_{n,k}(q;x) \geq 0$ for all $q \in (0,1)$ and $x \in [0,\infty)$ and moreover

$$\sum_{k=0}^{\infty} p_{n,k}(q;x) = \frac{1}{E_q([n]x)} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{([n]x)^k}{[k]!} = 1.$$

The two q-Gamma functions are defined as

$$\Gamma_q(x) = \int_0^{1/1-q} t^{x-1} E_q(-qt) d_q t, \quad \gamma_q^A(x) = \int_0^{\infty/A(1-q)} t^{x-1} e_q(-t) d_q t.$$

For every $A, X > 0$, $\Gamma_q(x) = K(A,x)\gamma_q^A(x)$, where

$$K(A,x) = \frac{1}{1+A} A^x \left(1 + \frac{1}{A}\right)_q^x (1+A)_q^{1-x}.$$

In particular for any positive integer n, $K(A,n) = q^{\frac{n(n-1)}{2}}$ and $\Gamma_q(n) = q^{\frac{n(n-1)}{2}} \gamma_q^A(n)$.

The q-analogue of Beta functions as follows;

$$\beta_q(t,s) = K(A,t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}}.$$

Now we introduce the following q-Szász-Miraykan operator by taking the weight function of q-Chlodowsky-Durrmeyer operators as

$$F_{n,q}(f;x) = \frac{[n+1]}{b_n} \sum_{k=0}^{\infty} q^{-(1+k)(1+n-k)} p_{n,k}\left(q; \frac{x}{b_n}\right) \int_0^{b_n} \varphi_{n,k}\left(q; \frac{t}{b_n}\right) f(t) d_q t, \tag{1.2}$$

where

$$p_{n,k}(q; \frac{x}{b_n}) = \frac{1}{E\left([n]\frac{x}{b_n}\right)} q^{\frac{k(k-1)}{2}} \frac{\left([n]\frac{x}{b_n}\right)^k}{[k]!},$$

$$\varphi_{n,k}(q;\frac{x}{b_n}) = \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)_q^{n-k} = \begin{bmatrix} n \\ k \end{bmatrix} \left(\frac{x}{b_n}\right)^k \prod_{s=0}^{n-k-1} \left(1-q^s\frac{x}{b_n}\right).$$

2. Estimation Moments

In this section our aim is to obtain $F_{n,q}(t^i;x)$,$i=0,1,...$ For $s=0,1,...$, let use u as $u=\frac{t}{b_n}$ and by the defination of q-Beta function [6], we have

$$\begin{aligned} \int_0^{b_n} t^s\varphi_{n,k}(q;\frac{t}{b_n})d_qt =&(b_n)^{s+1}\begin{bmatrix} n \\ k \end{bmatrix}\int_0^1 u^{s+k}(1-u)_q^{n-k}d_qu \\ =&(b_n)^{s+1}\begin{bmatrix} n \\ k \end{bmatrix}B_q(s+k+1,n-k+1) \\ =&(b_n)^{s+1}\begin{bmatrix} n \\ k \end{bmatrix}\frac{\Gamma_q(s+k+1)\Gamma_q(n-k+1)}{\Gamma_q(s+n+2)} \\ =&(b_n)^{s+1}\frac{[n]![k+s]!}{[k]![n+s+1]!}q^{(1+k+s)(1+n-k)}. \end{aligned} \tag{2.1}$$

Lemma 2.1. *For all $n\in\mathbb{N}$ and $0<q\leq 1$, we have*

$$F_{n,q}(1;x)=1,$$

$$F_{n,q}(t;x)=q^n\left(x+\frac{qb_n-2x}{[n+2]}\right),$$

$$F_{n,q}(t^2;x)=\frac{q^{2n-2}x^2[n]^2+q^{2n}([3]+q)x[n]b_n+q^{2n+2}[2]b_n^2}{[n+2][n+3]},$$

$$\begin{aligned} F_{n,q}(t^3;x)=&\frac{q^{3n-6}[n]^3x^3}{[n+2][n+3][n+4]}+\frac{([5]+q[3]+q^2)q^{3n-3}[n]^2x^2b_n}{[n+2][n+3][n+4]} \\ &+\frac{([6]+2q[4]+2q^2[2])q^{3n}[n]xb_n^2}{[n+2][n+3][n+4]}+\frac{q^{3n+3}[2][3]b_n^3}{[n+2][n+3][n+4]}, \end{aligned}$$

$$\begin{aligned} F_{n,q}(t^4;x)=&\frac{q^{4n-12}[n]^4x^4}{[n+2][n+3][n+4][n+5]}+\frac{\left([7]+q[5]+q^2[3]+q^3\right)q^{4n-8}[n]^3x^3b_n}{[n+2][n+3][n+4][n+5]} \\ &+\frac{\left((1+q)[24]+(1+q+q^2)[6]+[2][3]\right)q^{4n-4}[n]^2x^2b_n^2}{[n+2][n+3][n+4][n+5]} \\ &+\frac{\left((1+q^2)[30]+(1+q+q^2+q^3)[4]+[5][4]\right)q^{4n}[n]xb_n^3}{[n+2][n+3][n+4][n+5]}+\frac{[2][3][4]q^{4n+4}b_n^4}{[n+2][n+3][n+4][n+5]}. \end{aligned}$$

Proof. We have to estimate $F_{n,q}(t^s;x)$, $s=0,1,2$. The result can easily be verified for $s=0$. Using the defination (1.2) of $F_{n,q}(f;x)$ for $s=0$, one has

$$\begin{aligned} F_{n,q}(1;x)=&\frac{[n+1]}{b_n}\sum_{k=0}^{\infty}q^{-(1+k)(1+n-k)}p_{n,k}\left(q;\frac{x}{b_n}\right)(b_n)\frac{[n]![k]!}{[k]![n+1]!}q^{(1+k)(1+n-k)} \\ =&\sum_{k=0}^{\infty}p_{n,k}\left(q;\frac{x}{b_n}\right)=1. \end{aligned}$$

When $s = 1$, using $[k+1] = [k] + q^k$, one has

$$\begin{aligned}
F_{n,q}(t;x) =& \frac{[n+1]}{b_n}\sum_{k=0}^{\infty} q^{-(1+k)(1+n-k)} p_{n,k}\left(q;\frac{x}{b_n}\right)\int_0^{b_n}\varphi_{n,k}\left(q;\frac{t}{b_n}\right) t d_q t\\
=& \frac{[n+1]}{b_n}\sum_{k=0}^{\infty} q^{-(1+k)(1+n-k)} p_{n,k}\left(q;\frac{x}{b_n}\right)(b_n)^2\frac{[n]![k+1]!}{[k]![n+2]!}q^{(k+2)(1+n-k)}\\
=& \frac{b_n}{[n+2]}\sum_{k=0}^{\infty} q^{-(1+k)(1+n-k)} p_{n,k}\left(q;\frac{x}{b_n}\right)[k+1]q^{(k+2)(1+n-k)}\\
=& \frac{b_n}{[n+2]}\sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)([k]+q^k)q^{1+n-k}\\
=& \frac{b_n}{[n+2]}\left(\sum_{k=0}^{\infty}[k] p_{n,k}\left(q;\frac{x}{b_n}\right)q^{1+n-k} + \sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)q^{1+n}\right)\\
=& \frac{q^{n+1}b_n}{[n+2]} + \frac{q^n x[n]}{[n+2]} = q^n\left(x + \frac{qb_n - 2x}{[n+2]}\right).
\end{aligned}$$

For $s = 2$, using $[k+1][k+2] = [k]^2 + q^k(1+[2])[k] + q^{2k}[2]$, by the formulas (1.1) and (2.1), we have

$$\begin{aligned}
F_{n,q}(t^2;x) =& \frac{[n+1]}{b_n}\sum_{k=0}^{\infty} q^{-(1+k)(1+n-k)} p_{n,k}\left(q;\frac{x}{b_n}\right)\int_0^{b_n}\varphi_{n,k}\left(q;\frac{t}{b_n}\right) t^2 d_q t\\
=& \frac{[n+1]}{b_n}\sum_{k=0}^{\infty} q^{-(1+k)(1+n-k)} p_{n,k}\left(q;\frac{x}{b_n}\right)\frac{[n]![k+2]!}{[k]![n+3]!}q^{(1+n-k)(k+3)}\\
=& \frac{b_n^2}{[n+3][n+2]}\sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)[k+1][k+2]q^{2(1+n-k)}\\
=& \frac{b_n^2}{[n+3][n+2]}\sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)([k]^2 + q^k(1+[2])[k] + q^{2k}[2])q^{2(1+n-k)}\\
=& \frac{b_n^2}{[n+3][n+2]}\left(\sum_{k=0}^{\infty}[k]^2 p_{n,k}\left(q;\frac{x}{b_n}\right)q^{2(1+n-k)} + \sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)q^k(1+[2])[k]q^{2(1+n-k)}\right.\\
&\left. + \sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)q^{2k}[2]q^{2(1+n-k)}\right)\\
=& \frac{b_n^2}{[n+3][n+2]}\left(\frac{[n]^2x^2}{b_n^2}q^{2n-2} + [n]q^{2n-1}([3]+q)\frac{x}{b_n} + [2]q^{2n+2}\right)\\
=& \frac{q^{2n-2}x^2[n]^2 + q^{2n}([3]+q)x[n]b_n + q^{2n+2}[2]b_n^2}{[n+2][n+3]}.
\end{aligned}$$

Also we can write this equation as

$$F_{n,q}(t^2;x) = q^{2n}\left(q^{-2}x^2 + \frac{x\left(([3]+q)[n]b_n - q^n\left(q^{-2}([2]+[3])[n] + q^{n-2}[2][3]\right)x\right)}{[n+2][n+3]} + \frac{q^2[2]b_n^2}{[n+2][n+3]}\right).$$

When $s = 3$ by the formulas (1.1), (2.1) and using

$$[k+3][k+2][k+1] = [k]^3 + q^k(1+[2]+[3])[k]^2 + q^{2k}([2]+[3]+[2][3])[k] + q^{3k}[2][3]$$

we have,

$$\begin{aligned}
F_{n,q}(t^3;x) =& \frac{[n+1]}{b_n}\sum_{k=0}^{\infty} q^{-(1+k)(1+n-k)} p_{n,k}\left(q;\frac{x}{b_n}\right)\int_0^{b_n}\varphi_{n,k}\left(q;\frac{t}{b_n}\right)t^3 d_q t\\
=& \frac{b_n^3}{[n+2][n+3][n+4]}\sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)[k+1][k+2][k+3]q^{3(1+n-k)}\\
=& \frac{b_n^3}{[n+2][n+3][n+4]}\left(\sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)[k]^3 q^{3(1+n-k)} + \sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)q^k(1+[2]+[3])[k]^2 q^{3(1+n-k)}\right.\\
&\left.+\sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)q^{2k}([2]+[3]+[2][3])[k]q^{3(1+n-k)} + \sum_{k=0}^{\infty} p_{n,k}\left(q;\frac{x}{b_n}\right)q^{3k}[2][3]q^{3(1+n-k)}\right)\\
=& \frac{b_n^3}{[n+2][n+3][n+4]}\left\{\frac{q^{3n-6}[n]^3x^3}{b_n^3} + ([5]+q[3]+q^2)\frac{q^{3n-3}[n]^2x^2}{b_n^2}\right.\\
&\left.+([6]+2q[4]+2q^2[2])\frac{q^{3n}[n]x}{b_n} + q^{3n+3}[2][3]\right\}\\
=& \frac{q^{3n-6}[n]^3x^3 + ([5]+q[3]+q^2)q^{3n-3}[n]^2x^2b_n + ([6]+2q[4]+2q^2[2])q^{3n}[n]xb_n^2}{[n+2][n+3][n+4]}\\
&+\frac{q^{3n+3}[2][3]b_n^3}{[n+2][n+3][n+4]}.
\end{aligned}$$

When $s=4$ by the formulas (1.1), (2.1) and using

$$\begin{aligned}
[k+4][k+3][k+2][k+1] =& [k]^4 + q^k(1+[2]+[3]+[4])[k]^3 + q^{2k}([2]+[3]+[2][3]+(1+[2]+[3])[4])[k]^2\\
&+q^{3k}([2][3]+([2]+[3]+[2][3])[4])[k]+[2][3][4]q^{4k}
\end{aligned}$$

we can get the desired result.

□

Lemma 2.2. *Let $q\in(0,1)$, $x\in[0,\infty)$, we have*

$$F_{n,q}(t-x;x) = (q^n-1)x + q^n\left(\frac{qb_n-2x}{[n+2]}\right),$$

$$F_{n,q}((t-x)^2;x) \le (1+q)\frac{x([n]b_n+[3]x)}{[n+2][n+3]}.$$

Proof. By Lemma 2.1, we have

$$F_{n,q}(t-x;x) = F_{n,q}(t;x) - x = q^n\left(x+\frac{qb_n-2x}{[n+2]}\right) - x = (q^n-1)x + q^n\left(\frac{qb_n-2x}{[n+2]}\right).$$

By Lemma 2.1, we have

$$\begin{aligned}
F_{n,q}((t-x)^2;x) =& F_{n,q}(t^2;x) - 2xF_{n,q}(t;x) + x^2\\
=& \left(\frac{q^{2n-2}[n]^2 + [n+2][n+3] - 2q^n[n][n+3]}{[n+2][n+3]}\right)x^2 + \left(\frac{q^{2n}b_n[n]([3]+q) - 2q^{n+1}b_n[n+3]}{[n+2][n+3]}\right)x\\
&+q^{2n+2}\frac{[2]b_n^2}{[n+2][n+3]}\\
\le& \frac{x\left(\left((1+q)[n]-(1+q+q^2)(1+q)\right)b_n - ([n]-[2][3])\,x\right)}{[n+2][n+3]} + \frac{q^{2n+2}[2]b_n^2}{[n+2][n+3]}\\
\le& (1+q)\frac{x([n]b_n+[3]x)}{[n+2][n+3]}.
\end{aligned}$$

Also by using

$$F_{n,q}((t-x)^4;x) = F_{n,q}(t^4;x) - 4xF_{n,q}(t^3;x) + 6x^2F_{n,q}(t^2;x) - 4x^3F_{n,q}(t;x) + x^4$$

we can get

$$F_{n,q}((t-x)^4;x) \leq \frac{[5][37]^2 b_n^4}{[n+2][n+3]}.$$

□

3. Weighted Approximation in Weighted Spaces

Theorem 3.1. *Let the sequence $q = q_n$ satisfies $q_n \in (0,1)$ and $q_n \to 1$ as $n \to \infty$ and let b_n be a sequence of positive real numbers, increasing and such that*

$$\lim_{n\to\infty} b_n = \infty, \lim_{n\to\infty} \frac{b_n^2}{n} = 0.$$

For $f \in C_\rho[0,\infty]$, we have

$$\lim_{n\to\infty} \left\| F_{n,q_n} f - f \right\|_{\rho,[0,b_n]} = 0.$$

Proof.

$$\lim_{n\to\infty} \left\| F_{n,q_n}(1;x) - 1 \right\|_{\rho,[0,b_n]} = 0,$$

$$\begin{aligned}
\lim_{n\to\infty} \left\| F_{n,q_n}(t;x) - x \right\|_{\rho,[0,b_n]} &= \lim_{n\to\infty} \sup_{x\in[0,b_n]} \frac{\left|(q_n^n-1)x + q_n^n\left(\frac{q_n b_n - 2x}{[n+2]}\right)\right|}{1+x^2} \\
&= \lim_{n\to\infty} \frac{q_n^n - 1}{2} + \frac{q_n^{n+1} b_n + 1}{[n+2]}.
\end{aligned}$$

Since $q_n \in (0,1)$, $\lim_{n\to\infty} q_n = 1$ and we have $\lim_{n\to\infty} \frac{b_n}{[n]} = 0$, so we can obtain $\lim_{n\to\infty} \left\| F_{n,q_n}(t;x) - x \right\|_{\rho,[0,b_n]} = 0$.

$$\begin{aligned}
&\lim_{n\to\infty} \left\| F_{n,q_n}(t^2;x) - x^2 \right\|_{\rho,[0,b_n]} \\
&= \lim_{n\to\infty} \sup_{x\in[0,b_n]} \left| \left(\frac{x\left(q^{2n}([3]+q)b_n[n] + \left((q^{2n-1}-1)[n]^2 - q^n(5[n]+[2][3]q^n)x\right)\right)}{[n+2][n+3]} + \frac{q^{2n}[2]b_n^2}{[n+2][n+3]} \right) \frac{1}{1+x^2} \right| \\
&= \lim_{n\to\infty} \frac{q^{2n}[2]b_n^2 + q^{2n}([3]+q)b_n[n]/2 + 5[n] + [2][3]}{[n+2][n+3]}.
\end{aligned}$$

Since $q_n \in (0,1)$, $\lim_{n\to\infty} q_n = 1$ and we have $\lim_{n\to\infty} \frac{b_n^2}{[n]} = 0$, so we can obtain $\lim_{n\to\infty} \left\| F_{n,q_n}(t^2;x) - x \right\|_{\rho,[0,b_n]} = 0$.

□

4. A Voronovskaya Type Theorem

In this section, we prove a Voronoskaya type theorem for the operators $F_{n,q}$.

Theorem 4.1. *For every $f \in C_\rho[0,b_n]$ such that $f', f'' \in C_\rho[0,b_n]$, we have*

$$\lim_{n\to\infty} \frac{[n+2]}{b_n} \left\{ F_{n,q}(f,x) - f(x) \right\} = f'(x) + xf''(x)$$

for each fixed $x \in [0,b_n]$.

Proof. Let $f, f', f'' \in C_\rho[0,b_n]$. In order to prove the theorem, by Taylor's theorem we write

$$f(t) = \begin{cases} f(x) + (t-x)f'(x) + 1/2(t-x)^2 f''(x) + (t-x)^2\eta(t-x), & if \quad t \neq x \\ 0, & if \quad t = x \end{cases}$$

where $\eta(h)$ tends to zero as h tends to zero. From Lemma 2.2,

$$\frac{[n+2]}{b_n}\left\{F_{n,q}(f,x)-f(x)\right\}=\frac{[n+2]}{b_n}\left((q^n-1)x+q^n\left(\frac{qb_n-2x}{[n+2]}\right)\right)f'(x)$$
$$+\frac{1}{2}\frac{[n+2]}{b_n}\left((1+q)\frac{x([n]b_n+[3]x)}{[n+2][n+3]}\right)f''(x)+\frac{[n+2]}{b_n}F_{n,q}((t-x)^2\eta(t-x);x).$$

If we apply the Cauchy-Schwarz-Bunyakovsky inequality to

$$F_{n,q}((t-x)^2\eta(t-x);x)$$

we conclude that

$$\frac{[n+2]}{b_n}\left|F_{n,q}((t-x)^2\eta(t-x);x)\right|\leq\sqrt{\frac{[n+2]}{b_n^2}F_{n,q}((t-x)^4;x}\sqrt{[n+2]F_{n,q}((\eta(t-x))^2;x)}.$$

From Lemma 2.1 we can get

$$\sqrt{\frac{[n+2]}{b_n^2}F_{n,q}((t-x)^4;x}\leq\sqrt{\frac{[n+2]}{b_n^2}\left(\frac{[5][37]^2b_n^4}{[n+2][n+3]}\right)}$$

hence

$$\lim_{n\to\infty}\sqrt{\frac{[n+2]}{b_n^2}F_{n,q}((t-x)^4;x}=0.$$

On the other hand, by the assumption

$$\lim_{n\to\infty}\eta(t-x)=0.$$

So it follows that

$$\lim_{n\to\infty}\frac{[n+2]}{b_n}\left|F_{n,q}((t-x)^2\eta(t-x);x)\right|=0.$$

Then we have

$$\frac{[n+2]}{b_n}\left\{F_{n,q}(f,x)-f(x)\right\}=f'(x)-\frac{[2]x}{b_n}f'(x)+xf''(x)-\frac{[2][3]}{[n+3]}xf''(x)-\frac{[9][n+2]x^2}{[n+3]b_n}f''(x).$$

□

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