Pell and Pell-Lucas Numbers Associated with Brocard-Ramanujan Equation

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ABSTRACT. In this paper, the diophantine equations of the form $A_{n_1}A_{n_2}\cdots A_{n_k} \pm 1 = B^2_m$ where $(A_n)_{n \geq 0}$ and $(B_m)_{m \geq 0}$ are either the Pell sequence or Pell-Lucas sequence are solved by applying the Primitive Divisor Theorem. This is another version of Brocard-Ramanujan equation.


Keywords: Pell number, Pell-Lucas number, Brocard-Ramanujan equation, Diophantine equation.

1. Introduction

The problem of finding all solutions to

$$n! + 1 = m^2$$

is known as Brocard-Ramanujan problem. Some authors [1,3,4] have been worked on this problem. Let $(F_n)_{n \geq 0}$ be the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The variant of this problem, the diophantine equation

$$F_nF_{n+1}\cdots F_{n+k-1} + 1 = F^2_m$$

was investigated by Marques [5]. Also, Szalay [7] and Pongsriiam [6] worked on another version of this diophantine equation.

In this article we will give a new version of Brocard-Ramanujan equation in terms of Pell and Pell-Lucas sequence.

Let $(P_n)_{n \geq 0}$ be the Pell sequence given by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$ and let $(Q_n)_{n \geq 0}$ be the Pell-Lucas sequence given by the same recurrence relation as the Pell sequence with the initial values $Q_0 = Q_1 = 2$.

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2. Preliminaries and Lemmas

Before giving the Primitive Divisor Theorem, we first give some remarks about it. Let $\alpha$ and $\beta$ be algebraic numbers such that $\alpha + \beta$ and $\alpha \beta$ are nonzero coprime integers and $\alpha \beta^{-1}$ is not a root of unity. Let $(u_n)_{n \geq 0}$ be the sequence given by

$$u_0 = 0, u_1 = 1, \text{ and } u_n = (\alpha + \beta)u_{n-1} - (\alpha \beta)u_{n-2} \text{ for } n \geq 2.$$ 

Then we have Binet’s formula for $u_n$ given by

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ for } n \geq 0.$$ 

If $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ then $u_n$ is the Pell sequence.

A prime $p$ is said to be a primitive divisor of $u_n$ if $p \mid u_n$ but $p$ does not divide $u_1 u_2 \cdots u_{n-1}$.

**Theorem 2.1** (Primitive Divisor Theorem [2]). Suppose $\alpha$ and $\beta$ are real numbers such that $\alpha + \beta$ and $\alpha \beta$ are nonzero coprime integers and $\alpha \beta^{-1}$ is not a root of unity. If $n \neq 1, 2, 6$, then $u_n$ has a primitive divisor except when $n = 12$, $\alpha + \beta = 1$ and $\alpha \beta = -1$.

**Lemma 2.2.** For every $m \geq 1$, we have

$$P_{m-1}P_{m+1} = \begin{cases} P_m^2 - 1, & \text{if } m \text{ is odd;} \\ P_m^2 + 1, & \text{if } m \text{ is even.} \end{cases}$$

**Proof.** Let $m$ be an even integer. We know that the roots of quadratic equation of Pell numbers are $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$. So by the help of Binet’s formula it can be proved as follows:

$$P_{m-1}P_{m+1} = \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} = \frac{\alpha^2m + \beta^2m - \alpha^{m-1}\beta^{m+1} - \alpha^{m+1}\beta^{m-1}}{(\alpha - \beta)^2} = \frac{\alpha^2m + \beta^2m - 6(\alpha \beta)^{m-1}(\alpha^2 + \beta^2)}{(\alpha - \beta)^2} = \frac{\alpha^2m + \beta^2m + 6(\alpha \beta)^m}{(\alpha - \beta)^2} = \frac{\alpha^2m + \beta^2m - 2(\alpha \beta)^m + 8}{(\alpha - \beta)^2} = \frac{\alpha^2m + \beta^2m - 2(\alpha \beta)^m + (\alpha - \beta)^2}{(\alpha - \beta)^2} = \frac{\alpha^2m + \beta^2m - 2(\alpha \beta)^m}{(\alpha - \beta)^2} + 1 = P_m^2 + 1.$$ 

Similarly, let $m$ be an odd integer. It can be proved as follows:
\[ P_{m-1}P_{m+1} = \frac{\alpha^{m-1} - \beta^{m-1}}{\alpha - \beta} \frac{\alpha^{m+1} - \beta^{m+1}}{\alpha - \beta} = \frac{\alpha^{2m} + \beta^{2m} - \alpha^{m-1}\beta^{m+1} - \alpha^{m+1}\beta^{m-1}}{(\alpha - \beta)^2} \]
\[ = \frac{\alpha^{2m} + \beta^{2m} - (\alpha\beta)^{m-1}(\alpha^2 + \beta^2)}{(\alpha - \beta)^2} \]
\[ = \frac{\alpha^{2m} + \beta^{2m} - 6(\alpha\beta)^{m-1}}{(\alpha - \beta)^2} \]
\[ = \frac{\alpha^{2m} + \beta^{2m} + 6(\alpha\beta)^{m}}{(\alpha - \beta)^2} \]
\[ = \frac{\alpha^{2m} + \beta^{2m} - 2(\alpha\beta)^{m} - 8}{(\alpha - \beta)^2} \]
\[ = \frac{\alpha^{2m} + \beta^{2m} - 2(\alpha\beta)^{m} - (\alpha - \beta)^2}{(\alpha - \beta)^2} \]
\[ = \frac{\alpha^{2m} + \beta^{2m} - 2(\alpha\beta)^{m}}{(\alpha - \beta)^2} - 1 \]
\[ = p^2_m - 1. \]

\[ \Box \]

Lemma 2.3. For every \( m \geq 1 \), we have

(i) \( Q_m^2 - 1 = \begin{cases} 
8P_{m-1}P_{m+1} + 3, & \text{if } m \text{ is odd}; \\
\frac{P_{3m}}{P_m}, & \text{if } m \text{ is even}.
\end{cases} \)

(ii) \( Q_m^2 + 1 = \begin{cases} 
\frac{P_{3m}}{P_m}, & \text{if } m \text{ is odd}; \\
8P_{m-1}P_{m+1} - 3, & \text{if } m \text{ is even}.
\end{cases} \)

Proof. This can be checked easily by using Binet’s formula. \( \Box \)

3. Main Results

Theorem 3.1. The diophantine equation

\[ P_{n_1}P_{n_2}\cdots P_{n_k} + 1 = P_m^2 \] (3.1)

in positive integers \( k, m \) and \( 3 \leq n_1 < n_2 < \cdots < n_k \) has an infinite family of solutions given by

\[ P_{m-1}P_{m+1} + 1 = P_m^2. \]

Proof. Taking a solution of (3.1) by Lemma 2.2 we get

\[ P_{n_1}P_{n_2}\cdots P_{n_k} = P_{m-1}P_{m+1}. \]

Suppose that \( m \geq 14 \). Then \( 13 \leq m - 1 \leq m + 1 \) and therefore, by Primitive Divisor Theorem, \( P_{m+1} \) has a primitive divisor. Then \( n_k = m + 1 \) and hence (3.1) reduces to

\[ P_{n_1}P_{n_2}\cdots P_{n_{k-1}} = P_{m-1}. \] (3.2)

Now \( P_{m-1} > 1 \) and this implies \( k \geq 2 \). Using the same arguments linked to primitive divisors as above (3.2) provides \( n_{k-1} = m - 1 \). As a result, \( k = 2 \), i.e. there are no more terms on the left hand side of (3.2). Thus we get the infinite family of solution \( P_{m-1}P_{m+1} + 1 = P_m^2, m \geq 14. \) \( \Box \)
Theorem 3.2. The diophantine equation

\[ Q_{n_1}Q_{n_2} \cdots Q_{n_k} + 1 = Q_m^2 \]  \hspace{1cm} (3.3)

in positive integer \( k \), even integer \( m \) and in non-negative integers \( n_1 < n_2 < \cdots < n_k \) \((n_i \neq 1)\) has no solution.

Proof. Since we know that \( P_nQ_n = P_{2n} \), (3.3) reduces to

\[ \frac{P_{2n_1}}{P_{n_1}} \frac{P_{2n_2}}{P_{n_2}} \cdots \frac{P_{2n_k}}{P_{n_k}} = \frac{P_{3m}}{P_m}. \]  \hspace{1cm} (3.4)

Suppose that \( m \geq 14 \). Then \( P_{3m} \) has a primitive divisor. Thus, \( 2n_k = 3m \), i.e. \( n_k = \frac{3m}{2} > m \). If \( k = 1 \) (3.4) reduces to \( P_m = P_{n_1} \), and we get a contradiction by \( m = n_1 \). Supposing \( k = 2 \), (3.4) simplifies to

\[ \frac{P_{2n_1}}{P_{n_1}} P_m = P_{n_2}. \]

Since \( n_2 > m \), \( P_{n_2} \) contains a primitive divisor. Thus \( n_2 = 2n_1 \), and \( m = n_1 \) follows. This contradicts to \( n_2 = \frac{3m}{2} \). If \( k \geq 3 \) then observe that \( n_{k-1} < m \) holds, otherwise we could cause a contradiction by \( Q_{n_{k-1}}Q_m > Q_{m^2} \). Thus the equation

\[ \frac{P_{2n_1}}{P_{n_1}} \frac{P_{2n_2}}{P_{n_2}} \cdots \frac{P_{2n_{k-1}}}{P_{n_{k-1}}} P_m = P_{n_{k-1}} \]  \hspace{1cm} (3.5)

has no solution since \( m \geq 14 \), therefore \( P_m \) has a primitive divisor on the left hand side of (3.5), which can not exist on the right hand side.

Theorem 3.3. The diophantine equation

\[ P_{n_1}P_{n_2} \cdots P_{n_k} + 1 = Q_m^2 \]  \hspace{1cm} (3.6)

in positive integer \( k \), even integer \( m \) and in non-negative integers \( n_1 < n_2 < \cdots < n_k \) \((n_i \neq 1)\) has no solution.

Proof. Suppose that \( m > 3 \). We can write (3.6) as:

\[ P_{n_1}P_{n_2} \cdots P_{n_k}P_m = P_{3m}. \]  \hspace{1cm} (3.7)

Then \( P_{3m} \) has a primitive divisor. This implies that \( n_k = 3m \). Then (3.7) reduces to

\[ P_{n_1}P_{n_2} \cdots P_{n_{k-1}}P_m = 1. \]

Thus, \( 1 = P_{n_1}P_{n_2} \cdots P_{n_{k-1}}P_m > P_m > P_3 = 5 \) which is a contradiction. Therefore \( m < 3 \), it means \( m = 0 \) or \( 2 \). In this situation one can check that there is no solution.

References


