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Inverse Nodal Problem for an Integro-Differential Operator

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Abstract: In this study, we consider an inverse nodal problem of recovering integro-differential operator with the Sturm-Liouville differential part and the integral part of Volterra type. Furthermore, we obtain a reconstruction formula for function M. So, we reconstruct the operator L with a dense subset of nodal points provided that the function q is known. Even if not all nodes are taken as data but a dense subset of nodes, inverse problem is determined.

Keywords: Integro-differential operator, nodal points, reconstruction formula.

1. Introduction

The inverse nodal problem which was first studied by McLaughlin in 1988 is the problem of finding potential function and boundary conditions by using only a dense subset of nodal points of eigenfunctions. She posed and solved this problem for the Sturm-Liouville operator with Dirichlet boundary conditions in addition to showing that knowledge of a dense set of nodal points can alone determine the potential function of the Sturm-Liouville problem up to a constant. Independently, Shen studied the relation between nodes and density function of string equation in 1988 (see [2]). Moreover, nodal data can help to solve inverse problems for certain classes of operators such as integro-differential and string operators (see [3], [4]). The nodes provide more information than spectral data. Such type of problem was studied by many authors (see [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16]).

We consider a perturbation of the Sturm-Liouville operator by a Volterra type integral operator of the form

$$Ly = -y'' + q(x)y + \int_0^x M(x,t)y(t)dt = \lambda y, x \in [0,\pi] \tag{1}$$

$$y(0) = y(\pi) = 0, \tag{2}$$

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where $\lambda = \rho^2$ is a spectral parameter, q and M are real-valued functions, $q \in L_2(0,\pi)$ and $M(x,t)$ is integrable on $D = \{(x,t) : 0 \le t \le x \le \pi\}$ (see [3], [4]).

The inverse problem for this operator consists of the reconstruction of the function M from the spectra by the assumption given q. In [4], Kuryshova and Shieh prove a uniqueness theorem and provide reconstruction formula for the potential function q under the assumption integral perturbation is known. In this study, we obtain some asymptotic formulas for nodal parameters to reconstruct the function M by using a dense set of nodal points and the potential function q. In this respect, our results differ more from the results which are given in [4].

However, classical inverse problem methods are not always applicable for integro-differential operators. Then, there are comparatively few references on inverse problem for L. Nevertheless, some authors have obtained some important results for the problem (1)-(2) (see [17], [18], [19]).

Let $S(x,\lambda)$ be the solution of (1) with the initial conditions

$$S(0,\lambda) = 0, S'(0,\lambda) = 1, \tag{3}$$

of the form [3], [19],

$$S(x,\lambda) = \frac{\sin(\rho x)}{\rho} + \int_0^x \frac{\sin[\rho(x-\tau)]}{\rho} \left(q(\tau)S(\tau,\lambda) + \int_0^\tau M(\tau,s)S(s,\lambda)ds \right) d\tau. \tag{4}$$

Let $0 < x_1^n < ... < x_{n-1}^n < \pi$, $i = 1,2,...,n-1$ be the nodal points of the n-th eigenfunction. The double sequence $\{x_i^n\}$ is called the nodal sequence associated with operator L. Also, let $I_i^n = \left[x_i^n\ ,\ x_{i+1}^n\right]$ be the i-th nodal domain of the n-th eigenfunction and $l_i^n = |I_i^n| = x_{i+1}^n - x_i^n$ be the associated nodal length. We also define the function $j_n(x)$ on $(0,\pi)$ by $j_n(x) = \max\{\, i |\ x_i^n \le x\}$ [1].

2. Main Results

In this section, we obtain some asymptotic results for nodal parameters and a reconstruction formula for the function M which is obtained as a solution of an inverse nodal problem. Since the asymptotic expansions of nodal parameters contain ρ_n, we express the relation of the eigenvalues for the problem (1)-(2) which is given by Yurko in the following lemma (see [19]).

Lemma 1 ([19]). The eigenvalues $\{\lambda_n\}_{n\ge1}$ of the boundary value problem (1)-(2) coincide with the zeros of the function $\Delta(\lambda) = S(\pi,\lambda)$ and have the following asymptotic formula, for $n \to \infty$

$$\rho_n = \sqrt{\lambda_n} = n + \frac{A_1}{n} + \frac{K_n}{n},\ \{K_n\} \in l_2,\ A_1 = \frac{1}{2\pi}\int_0^\pi q(t)dt. \tag{5}$$

Theorem 1. Let $q \in L^1(0,\pi)$ and $M(x,t) \in W_1^2(D)$. Then, as $n \to \infty$,

$$x_i^{(n)} = \frac{i\pi}{\rho_n} + \frac{1}{2\rho_n^2} \int_0^{x_i^{(n)}} [1 - \cos(2\rho_n \tau)]\, q(\tau) d\tau - \frac{1}{2\rho_n^3} \int_0^{x_i^{(n)}} M(\tau,\tau) \sin(2\rho_n \tau)\, d\tau + o\left(\frac{1}{\rho_n^3}\right), \quad (6)$$

$$l_i^{(n)} = \frac{\pi}{\rho_n} + \frac{1}{2\rho_n^2} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} [1 - \cos(2\rho_n \tau)] q(\tau) d\tau - \frac{1}{2\rho_n^3} \int_{x_i^{(n)}}^{x_{i+1}^{(n)}} M(\tau,\tau) \sin(2\rho_n \tau)\, d\tau + o\left(\frac{1}{\rho_n^3}\right), \quad (7)$$

for the problem (1)-(2).

Proof: We consider the solution of the integral equation (1)

$$S(x,\lambda) = \frac{\sin(\rho x)}{\rho} + \int_0^x \frac{\sin[\rho(x-\tau)]}{\rho} \left(q(\tau) S(\tau,\lambda) + \int_0^\tau M(\tau,s) S(s,\lambda) ds \right) d\tau.$$

After some algebraic operations, we obtain

$$\begin{aligned} S(x,\lambda) &= \frac{\sin(\rho x)}{\rho} + \frac{\sin(\rho x)}{2\rho^2} \int_0^x \sin(2\rho\tau)\, q(\tau) d\tau - \frac{\cos(\rho x)}{2\rho^2} \int_0^x [1 - \cos(2\rho\tau)]\, q(\tau) d\tau \\ &\quad + \frac{1}{\rho^2} \int_0^x \sin[\rho(x-\tau)] \int_0^\tau M(\tau,s) \sin(\rho s) ds d\tau + o\left(\frac{1}{\rho^3}\right). \end{aligned}$$

By using some trigonometric formulas and a change of variables in the last term, we obtain

$$\begin{aligned} S(x,\lambda) &= \frac{\sin(\rho x)}{\rho} + \frac{\sin(\rho x)}{2\rho^2} \int_0^x \sin(2\rho\tau) q(\tau) d\tau - \frac{\cos(\rho x)}{2\rho^2} \int_0^x [1 - \cos(2\rho\tau)]\, q(\tau) d\tau \\ &\quad - \frac{\sin(\rho x)}{\rho^3} \int_0^x M(\tau,\tau) \cos^2(\rho\tau)\, d\tau + \frac{\cos(\rho x)}{\rho^3} \int_0^x M(\tau,\tau) \cos(\rho\tau) \sin(\rho\tau)\, d\tau + o\left(\frac{1}{\rho^3}\right). \end{aligned}$$

If $S(x,\lambda) = 0$, then as long as $\cos(\rho x)$ is not close to zero, then

$$\begin{aligned} 0 &= \frac{\tan(\rho x)}{\rho} + \frac{\tan(\rho x)}{2\rho^2} \int_0^x \sin(2\rho\tau) q(\tau) d\tau - \frac{1}{2\rho^2} \int_0^x [1 - \cos(2\rho\tau)]\, q(\tau) d\tau \\ &\quad - \frac{\tan(\rho x)}{\rho^3} \int_0^x M(\tau,\tau) \cos^2(\rho\tau)\, d\tau + \frac{1}{2\rho^3} \int_0^x M(\tau,\tau) \sin(2\rho\tau)\, d\tau + o\left(\frac{1}{\rho^3}\right), \end{aligned}$$

and

$$\tan(\rho x)\left(1+o\left(\frac{1}{\rho}\right)\right)=\frac{1}{2\rho}\int_0^x[1-\cos(2\rho\tau)]\,q(\tau)d\tau-\frac{1}{2\rho^2}\int_0^x M(\tau,\tau)\sin(2\rho\tau)\,d\tau+o\left(\frac{1}{\rho^2}\right).$$

Now, we take $\rho=\rho_n$ and $x=x_i^{(n)}$ for large values of n. Hence by Taylor's theorem for the arctangent function for some integer i,

$$\rho_n x_i^{(n)}=i\pi+\frac{1}{2\rho_n}\int_0^{x_i^{(n)}}[1-\cos(2\rho_n\tau)]\,q(\tau)d\tau-\frac{1}{2\rho_n^2}\int_0^{x_i^{(n)}}M(\tau,\tau)\sin(2\rho_n\tau)\,d\tau+o\left(\frac{1}{\rho_n^2}\right).$$

Then, the Riemann Lebesque lemma implies that,

$$x_i^{(n)}=\frac{i\pi}{\rho_n}+\frac{1}{2\rho_n^2}\int_0^{x_i^{(n)}}[1-\cos(2\rho_n\tau)]\,q(\tau)d\tau-\frac{1}{2\rho_n^3}\int_0^{x_i^{(n)}}M(\tau,\tau)\sin(2\rho_n\tau)\,d\tau+o\left(\frac{1}{\rho_n^3}\right).$$

Therefore, the nodal length is

$$l_i^{(n)}=\frac{\pi}{\rho_n}+\frac{1}{2\rho_n^2}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}[1-\cos(2\rho_n\tau)]q(\tau)d\tau-\frac{1}{2\rho_n^3}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}M(\tau,\tau)\sin(2\rho_n\tau)\,d\tau+o(\frac{1}{\rho_n^3}),$$

or

$$l_i^{(n)}=\frac{\pi}{\rho_n}+\frac{1}{2\rho_n^2}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}[1-\cos(2\rho_n\tau)]q(\tau)d\tau+\frac{1}{2\rho_n^3}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}M(\tau,\tau)d\tau+o(\frac{1}{\rho_n^3}).$$

Lemma 2. Suppose that $G\in L^1(0,\pi)$. Then,

$$\lim_{n\to\infty}\rho_n\int_{x_i^{(n)}}^{x_{i+1}^{(n)}}G(t,t)dt=G(x,x),$$

for almost every $x\in(0,\pi)$, with $j=j_n(x)$.

Proof. It can be proved by using similar way with in [8].

Theorem 2. Suppose that $q\in L_1(0,\pi)$ and $M(x,t)\in W_1^2(D)$. Then, the function M satisfies

$$M(x,x)=\lim_{n\to\infty}\left[2\rho_n^3\left(\rho_n l_i^{(n)}-\pi\right)-\rho_n q(x)\right],$$

for almost every $x\in(0,\pi)$ with $j=j_n(x)$.

Proof: By Theorem 1,

$$l_i^{(n)} = \frac{\pi}{\rho_n} + \frac{1}{2\rho_n^2} \int\limits_{x_i^{(n)}}^{x_{i+1}^{(n)}} [1 - \cos(2\rho_n \tau)] q(\tau) d\tau + \frac{1}{2\rho_n^3} \int\limits_{x_i^{(n)}}^{x_{i+1}^{(n)}} M(\tau,\tau) d\tau + o(\frac{1}{\rho_n^3}).$$

By using some computations, we obtain

$$2\rho_n^2 \left(\rho_n l_i^{(n)} - \pi\right) = \rho_n \int\limits_{x_i^{(n)}}^{x_{i+1}^{(n)}} [1 - \cos(2\rho_n \tau)] q(\tau) d\tau + \int\limits_{x_i^{(n)}}^{x_{i+1}^{(n)}} M(\tau,\tau) d\tau + o(1).$$

For the large values of n, the terms $\rho_n \int\limits_{x_i^{(n)}}^{x_{i+1}^{(n)}} \cos(2\rho_n \tau) q(\tau) d\tau$ and $\rho_n \int\limits_{x_i^{(n)}}^{x_{i+1}^{(n)}} q(\tau) d\tau$ tend to zero and $q(x)$, respectively. It can be shown easily by considering [8] and Lemma 2. If we use this fact, we obtain

$$\rho_n \left[2\rho_n^2 \left(\rho_n l_i^{(n)} - \pi\right) - q(x)\right] = \rho_n \int\limits_{x_i^{(n)}}^{x_{i+1}^{(n)}} M(\tau,\tau) d\tau + o(1). \tag{8}$$

Taking the limit on both sides of (8) as $n \to \infty$, and using similar procedure, we obtain

$$M(x,x) = \lim_{n\to\infty} \rho_n \left[2\rho_n^2 \left(\rho_n l_i^{(n)} - \pi\right) - q(x)\right], \tag{9}$$

for almost every $x \in (0,\pi)$. This completes the proof.

3. Conclusions

In this study, we have attempted to reconstruct the given operator L with a dense subset of the nodal points provided that the function q is known. We have expressed a reconstruction formula for the function M.

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References

[1] J. R. McLaughlin, Inverse Spectral Theory Using Nodal Points as Data-a Uniqueness Result, *Journal of Differential Equations*, **73**, (1988), 354-362.

[2] C. L. Shen, On the Nodal Sets of the Eigenfunctions of the String Equations, *SIAM Journal on Mathematical Analysis*, **19**, (1988), 1419-1424.

[3] V. A. Yurko, An Inverse Problem for Integro-differential Operators, *Matematicheskie Zametki*, **50**(5), (1991), 134–146 (Russian); *English Translation in Mathematical Notes*, **50**(5–6), (1991), 1188–1197.

[4] Y. V. Kuryshova and C. T. Shieh, An Inverse Nodal Problem for Integro-Differential Operators, *Journal of Inverse and III-posed Problems*, **18**, (2010), 357–369.

[5] O. Hald and J. R. McLaughlin, Solutions of Inverse Nodal Problems, *Inverse Problems*, **5**, (1989), 307-347.

[6] P. J. Browne and B. D. Sleeman, Inverse Nodal Problem for Sturm-Liouville Equation with Eigenparameter Depend Boundary Conditions, *Inverse Problems*, **12**, (1996), 377-381.

[7] Y. T. Chen, Y. H. Cheng, C. K. Law and J. Tsay, L_1 Convergence of the Reconstruction Formula for the Potential Function, *Proceedings of the American Mathematical Society*, **130**, (2002), 2319-2324.

[8] C. K. Law, C. L. Shen and C. F. Yang, The Inverse Nodal Problem on the Smoothness of the Potential Function, *Inverse Problems*, **15**(1), (1999), 253-263 (Erratum, *Inverse Problems*, **17,** (2001), 361-363).

[9] E. Yilmaz and H. Koyunbakan, Reconstruction of Potential Function and its Derivatives for Sturm-Liouville Problem with Eigenvalues in Boundary Conditions, *Inverse Problems in Science and Engineering*, **18**(7), (2010), 935-944.

[10] H. Koyunbakan and E. Yilmaz, Reconstruction of the Potential Function and its Derivatives for the Diffusion Operator, *Verlag der Zeitschrift für Naturforch*, **63**(a), (2008), 127-130.

[11] C. T. Shieh and V. A. Yurko, Inverse Nodal and Inverse Spectral Problems for Discontinuous Boundary Value Problems, *Journal of Mathematical Analysis and Applications*, **347**(1),(2008), 266-272.

[12] V. S. Gerdjikov, On the Spectral Theory of the Integro-Differential Operator a Generating Nonlinear Evolution Equations, *Letters in Mathematical Physics*, **6**(5), (1982), 315-324.

[13] Y. V. Kuryshova, Inverse Spectral Problem for Integro-Differential Operators, *Mathematical Notes*, **81**(6), (2007), 767-777.

[14] B. Wu and J. Yu, Uniqueness of an Inverse Problem for an Integro-Differential Equation Related to the Basset Problem, *Boundary Value Problems*, **229**, (2014).

[15] C. F. Yang and X. P. Yang, Inverse Nodal Problems for Differential Pencils on a Star-Shaped Graph, *Inverse Problems*, **26**, (2010).

[16] A. S. Ozkan and B. Keskin, Inverse Nodal Problems for Sturm–Liouville Equation with Eigenparameter-Dependent Boundary and Jump Conditions, *Inverse Problems in Science and Engineering*, (2014).

[17] S. A. Buterin, On an Inverse Spectral Problem for a Convolution Integro-Differential Operator, *Results in Mathematics*, **50**, (2007), 173-181.

[18] S. A. Buterin, The Inverse Problem of Recovering the Volterra Convolution Operator from the Incomplete Spectrum of its Rank-One Perturbation, *Inverse Problems*, **22**, (2006), 2223–2236.

[19] G. Freiling and V. A. Yurko, Inverse Sturm-Liouville Problems and their Applications, *NOVA Science Publishers*, New York, (2001).