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Solvable BCK-Algebras

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Abstract: In this paper, the notions of derived sub-algebras and solvable BCK-algebras are introduced and some properties are given. We introduce the notion of commutators in BCK-algebras and also discuss their properties. It has been found that the sub-algebras, isomorphic image and inverse image of a solvable BCK-algebras are still solvable BCK-algebras.

Keywords: BCK-algebras, commutators, derived sub-algebra, solvable BCK-algebras.

1. Introduction

In 1966, Y. Imai and K. Iseki [3,4] defined an algebra of type $(2,0)$, also known as BCK-algebra, as a generalization the notion of algebra sets with the subtraction set with the only a fundamental, non-nullary operation and the notion of implication algebra [7,8] on the other hand. This notion is derived using two different methodologies, one of which is based on set theory and the other on classical and non-classical propositional calculi.

Definition 1. An algebra $(X,*,0)$ of type $(2,0)$ is called a BCK-algebra, If it satisfies the following axiom: for all $x,y,z\in X$,

1. $((x*y)*(x*z))*(z*y)=0$,
2. $(x*(x*y))*y=0$,
3. $x*x=0$,
4. $0*x=0$,
5. $x*y=y*x=0$ implies $x=y$.

This definition is a dual form of the ordinary definition [1]. On any BCK-algebra $(X,*,0)$, the natural order can be defined by putting $x\leqslant y$ if and only if $x*y=0$, for all $x,y\in X$. In a BCK-algebra X, for all $x,y,z\in X$, the following identities hold [4,6,9].

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$$(x*y)*z = (x*z)*y, \tag{1}$$

$$(x*y) \leq x, \tag{2}$$

$$x \leq y \text{ implies } x*z \leq y*z \text{ and } z*y \leq z*x \tag{3}$$

$$x*0 = x. \tag{4}$$

BCK-algebra X is said to be bounded if there exists an element $1 \in X$ such that $x \leq 1$, for all $x \in X$. For elements x and y of a *BCK*-algebra X, we denote

$$x \wedge y = y*(y*x) \text{ and } x \vee y = N(Nx \wedge Ny) \text{ where } Nx = 1*x. \tag{5}$$

A *BCK*-algebra X is said to be commutative if it satisfies $x \wedge y = y \wedge x$, for all $x, y \in X$. A nonempty subset S of a *BCK*-algebra X is called a *BCK*-sub-algebra of X, if $x*y \in S$ whenever $x, y \in S$. Moreover, a nonempty subset I of a *BCK*-algebra X is called a *BCK*-ideal if [5]

$$0 \in I, \tag{6}$$

$$x*y \in I \text{ and } y \in I \text{ imply } x \in I \text{ for all } x, y \in X. \tag{7}$$

A *BCK*-algebra X is called implicative if and only if $x*(y*x) = x$, also X is called a positive implicative *BCK*-algebra if it satisfies in property $(x*z)*(y*z) = (x*y)*z$, The number of elements of a *BCK*-algebra is its order. Let $(X,*,0)$ and $(Y,.,0)$ be two *BCK*-algebras. A mapping $f : X \longrightarrow Y$ is called a homomorphism from X to Y if for any $x, y \in X$, $f(x*y) = f(x).f(y)$ holds.

In any commutative *BCK*-algebra, the following statements hold:

$$x \wedge x = x \vee x = x, \tag{8}$$

$$x \vee 0 = 0 \vee x = x \wedge 1 = 1 \wedge x = x, \tag{9}$$

$$x \wedge y = y \wedge x, \tag{10}$$

$$x \vee y = y \vee x, \tag{11}$$

$$x \vee 1 = 1 \vee x = 1, \tag{12}$$

$$0 \wedge x = x \wedge 0 = 0, \tag{13}$$

$$NNx = x. \tag{14}$$

For any fixed elements $a \leq b$ of a *BCK*-algebra X, the set

$$[a,b] = \{x \in X : a \leq x \leq b\} = \{x \in X : ax = xb = 0\}, \tag{15}$$

is called the segment of X [2]. Note that the segment $[0,b] = \{x \in X : x \leq b\} = \{x \in X : x*b = 0\}$ is called initial.

Let I be an ideal of a BCK-algebra X. Define an equivalence relation $\sim$ on X by $x \sim y$ if and only if $x*y, y*x \in I$. Let C_x denote the class of $x \in X$. Then $C_0 = I$. Let X/I denote the set of all classes C_x, where $x \in X$. Then X/I is a BCK-algebra, and X/I is called the quotient BCK-algebra of X determined by I with $C_x * C_y = C_{x*y}$ and $C_x \leq C_y$ if and only if $x \leq y$, the notion of commutators in BCK -algebras is considered and some related results are obtained [10].

2. Pseudo Commutators in *BCK*-Algebras

From now on, for simply in this section X is a BCK–algebra, unless otherwise is stated.

Definition 2. [10] Let $x_1, x_2, x_3, ..., x_n$ be elements of X. Then the element $(x_1 \wedge x_2) * (x_2 \wedge x_1)$ of X is called pseudo commutator of x_1 and x_2 of weight 2 and denoted by $[x_1, x_2]$. i.e.,

$$[x_1, x_2] = (x_1 \wedge x_2) * (x_2 \wedge x_1) \tag{16}$$

In general, the element $[x_1, x_2, ..., x_n] = [[x_1, ..., x_{n-1}], x_n]$ is a commutator of weight $n \geq 2$, where by convention $[x_1] = x_1$. A useful shorthand notation is $[x, y]_n = [x, \underbrace{y, ..., y}_{\text{n times}}]$

Example 1. [10] Let $X = \{0, 1, 2, 3, 4\}$. Define $*$ by the following table

TABLE 1

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	3	4	1	0

Then $(X, *, 0)$ is a bounded positive implicative BCK-algebra with the largest element 4. We have $[2, 4] = 0 \neq 2 = [4, 2]$.

By Definition 2 and Example 1, in the general case there is $[x, y] \neq [y, x]$ in BCK-algebras, so this definition of commutators is not the commutators in the sense of usual commutators of group theory. Thus the commutators defined in this paper are essentially directed commutators.

Now, we list the properties of commutators in BCK algebras.

Lemma 1. [10] For any $x, y \in X$

1. if $x \leq y$, then $[x, y] = 0$,

2. $[x,0] = [0,x] = [x,x] = 0$,
3. $[x,y] * x = 0$,
4. $[x,y] * y = 0$.

3. Commutator of Sub-Algebras

It is useful to be able to compute the commutators of the subsets of BCK-algebras as well as their elements.

Definition 3. [10] Let $X_1, X_2, ..., X_n$ be nonempty subsets of X. Define the pseudo commutator of subset of X_1 and X_2 to be

$$[X_1, X_2] = \{[x_1, x_2] : x_1 \in X_1, x_2 \in X_2\}$$

More generally,

$$[X_1, ..., X_n] = [[X_1, ..., X_{n-1}], X_n]$$

where $n \geq 2$.

Furthermore, the subset $[X,X] = \{[a,b] | a,b \in X\}$ of X is called the derived subset of X.

In the general case, the pseudo commutators of the subsets of X are not a sub-algebra of X.

Example 2. Let $X = \{0,1,2,3,4\}$ and let the $*$ operation be defined by the following table

TABLE 2

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	1	1	0	1
4	4	4	4	4	0

Then $(X, *, 0)$ is a BCK-algebra. $A = \{4\}$ and $B = \{2\}$ are two subsets of X. However, $[A,B] = \{[4,2]\} = \{2\}$, therefore $2 \in [A,B]$ and $2 * 2 = 0$ is not member of $[A,B]$, i.e $[A,B]$ is not sub-algebra of X.

In the following, we give the definition of a derived sub-algebra.

Definition 4. $[X,X] = \{\prod[a_i,b_i] : a_i, b_i \in X\}$, where $\prod$ product represents a finite number of commutators. $[X,X]$ is called the derived sub-algebras of X and is denoted by X'. $X' = [X,X] = \{\prod[a_i,b_i] : a_i, b_i \in X\}$, $X'' = [X',X'] = \{\prod[a_i,b_i] : a_i, b_i \in X'\}$, $X^{(n)} = [X^{(n-1)}, X^{(n-1)}] = \{\prod[a_i,b_i] : a_i, b_i \in X^{(n-1)}\}$.

Generally, $[X_1,X_2] \neq [X_2,X_1]$, we write $[X,Y]_n$ for $[X,\underbrace{Y,...,Y}_{\text{n times}}]$.

For any $x \in X$, we have $[x,0] = [0,x] = [x,x] = 0$. For any two nonempty sub-algebras A,B of X, so $0 \in [A,B]$, that is $0 \in X^{'}$.

Example 3. Let $X = [0,1]$. Define $*$ on X by

$$x * y = \begin{cases} 0 & \text{if } x \leq y \\ x & \text{otherwise} \end{cases}$$

Then $(X,*,0)$ is a bounded *BCK*-algebra. Let $X_1 = [0,1], X_2 = [0,1/2], X_3 = [0,1/3], ..., X_n = [0,1/n]$. Then $[X_1,X_2] = [0,1/2], [X_2,X_1] = [0,1/2)$, $[X_1,X_2,...,X_n] = [0,1/n]$. Also for any $n \geq 1$, there is $X^{(n)} = 0$.

The notions of pseudo commutators in *BCK*-algebras and derived sub-algebra $X^{'} = [X,X] = \{[a_i,b_i] : a_i,b_i \in X\}$ are studied in [10]. In this paper, we generalized the notion of derived sub-algebra $X^{'}$ by $X^{'} = [X,X] = \{\prod[a_i,b_i] : a_i,b_i \in X\}$.

Example 4. Let the $X = \{0,a,b,c,d\}$ and $(*)$ operation be given by the following table.

TABLE 3

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	0	0
c	c	c	c	0	c
d	d	d	b	b	0

It is not difficult to verify that $(X,*,0)$ is a *BCK*-algebra. Consider $A = \{0,a\}$ and $B = \{0,c\}$, then A and B are sub-algebras of X. It is easy to check that $[A,B] = \{0\}$ and $[B,A] = \{0,a\}$, therefore $[A,B] \neq [B,A]$. Now, $X^{'} = \{0,a,b\}$ is a sub-algebra of X, but $X^{'}$ is not an ideal of X because $d * b = b \in X^{'}$ and $b \in X^{'}$ but $d \notin X^{'}$. Its initial segments of X are $[0,a] = \{0,a\}, [0,b] = \{0,b\}, [0,c] = \{0,a,b,c\}, [0,d] = \{0,b,d\}$. Hence $X^{'}$ is not an initial segment of X.

Remark 1. In Example 1, considering $A = \{0,2\}$ and $B = \{0,1,2,3,4\}$, we see A and B are two ideals of X and $[A,B] = \{0,1\}$, but $[A,B]$ is not a sub-algebra of B.

Theorem 1. 1. X is commutative if and only if $X^{'} = \{0\}$,

2. $X^{'}$ is sub-algebra of X.

Proof. 1. See Theorem 2.6. [10].

2. Let $x,y \in X^{'}$. Then $a_i,b_i,c_i,d_i, \in X$ exist for $i=1,2,...,n$ such that $x=\prod[a_i,b_i]$ and $y=\prod[c_i,d_i]$. Therefore, for $e_i,\, f_i \in X$ $(i=1,2,...,n)$, $x*y=(\prod[a_i,b_i])*(\prod[c_i,d_i])=\prod[e_i,\, f_i] \in X^{'}$, then $x*y \in X^{'}$ and $X^{'}$ is a sub-algebra of X.

Corollary 1. Let $(X,*,0)$ be a commutative BCK-algebra. If A and B are two subsets of X, then $[A,B]=[B,A]=\{0\}$.

Example 5. Let the $X=\{0,a,1\}$ and $(*)$ operation be given by the following table

TABLE 4

$*$	0	a	1
0	0	0	0
a	a	0	0
1	1	a	0

Then $(X,*,0)$ is a commutative bounded BCK-algebra with the largest element 1. Then $A=\{0,1\}$, $B=\{0,a\}$ are two sub-algebras of X and $[A,B]=[B,A]=\{0\}$.

Example 6. Let $X=\{0,1,2,...,n\}$ and $*$ is given

$$x*y=\begin{cases} 0 & \text{if } x \le y \\ x & \text{otherwise} \end{cases}$$

Then $(X,*,0)$ is a bounded positive implicative BCK-algebra of order n. $X^{'}=\{0,1,2,...,n-1\}$, therefore $X^{'}$ is of order $n-1$. Also, $X^{''}=\{0,1,2,...,n-3\}$ and $X^{(3)}=\{0,1,2,...,n-4\}$,..., $X^{(n-1)}=\{0\}$.

Theorem 2. If Y is a sub-algebra of X, then $Y^{'}$ is a sub-algebra of $X^{'}$.

Proof. Let $x \in Y^{'}$. Then there exist $a,b \in Y$ such that $x=[a,b]$, but $Y \subseteq X$, then $a,b \in X$ and $x=[a,b]$. Hence $x \in X^{'}$ and so $Y^{'} \subseteq X^{'}$. However, $Y^{'}$ and $X^{'}$ are two sub-algebras of X. Therefore $Y^{'}$ is a sub-algebra of $X^{'}$.

Theorem 3. Let I be an ideal of X. Then X/I are commutative BCK-algebras if and only if $X^{'} \subseteq I$.

Proof. X/I are commutative BCK-algebras if and only if for all $x,y \in X$, $C_x \wedge C_y = C_y \wedge C_x$ if and only if for all $x,y \in X$, $C_{x\wedge y}=C_{y\wedge x}$ if and only if for all $x,y \in X$, $C_{[x,y]}=C_{(x\wedge y)*(y\wedge x)}=C_{x\wedge y}*C_{y\wedge x}=C_0=I$ if and only if for all $x,y \in X$, $[x,y] \sim 0$ if and only if for all $x,y \in X$, $[x,y]*0 \in I$ and $0*[x,y] \in I$ if and only if for all $x,y \in X$, $[x,y] \in I$ if and only if $X^{'} \subseteq I$.

4. Main Result

Definition 5. X is called a solvable BCK-algebra, if there exists a non-negative real number n such that $X^{(n)} = \{0\}$.

Note that if X is a commutative BCK-algebra, then X is a solvable BCK-algebra. Since any implicative BCK-algebra X is a commutative BCK-algebra [9], then $X^{'} = \{0\}$ for any implicative BCK-algebra X, so any implicative BCK-algebra X is a solvable BCK-algebra.

Example 7. [9] Let $X = \{0,a,b,c,d,e,f,1\}$ and $(*)$ operation be given by the following table

TABLE 5

$*$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	a	0	0	0	a	0	0	0
b	b	a	0	0	b	a	0	0
c	c	a	a	0	c	a	a	0
d	d	d	d	d	a	0	0	0
e	e	d	d	d	a	0	0	0
f	f	e	d	d	b	a	0	0
1	1	e	e	d	c	a	a	0

It is not difficult to verify that $(X,*,0)$ is a bounded BCK-algebra of order 8. It is easy to check that $X^{'} = \{0,a\}$ and $X^{''} = \{0\}$. So X is a solvable BCK-algebra.

Example 8. Let $X = \{x \in R : 0 \leq x \leq 1\}$ and $(*)$ operation define on X by $x*y = max\{0,x-y\}$. Then $(X,*,0)$ is a BCK-algebra. We see that $X^{'} = \{0\}$. Therefore X is a solvable BCK-algebra. But if we define

$$x*y = \begin{cases} 0 & \text{if } x \leq y \\ x & \text{otherwise} \end{cases}$$

Then $(X,*,0)$ is a BCK-algebra and $X^{(n)} = [0,1)$ for any $n \geq 1$, so X is not a solvable BCK-algebra.

Let f be a homomorphism from BCK-algebras $(X,*,0)$ to BCK-algebra $(Y,*^{'},0^{'})$ and $x,y \in X$. Then $f([x,y]) = f((x\wedge y)*(y\wedge x)) = f(x\wedge y)*^{'} f(y\wedge x) = f(y*(y*x))*^{'} f(x*(x*y)) = (f(y)*^{'} f(y*x))*^{'} (f(x)*^{'} f(x*y)) = (f(y)*^{'} (f(y)*^{'} f(x)))*^{'} (f(x)*^{'} (f(x)*^{'} f(y)) = (f(y)\wedge f(x))*^{'} (f(x)\wedge f(y)) = [f(x),f(y)]$. i.e, if $[x,y] \in X^{'}$, then $f([x,y]) = [f(x),f(y)] \in Y^{'}$.

Theorem 4. Let f be an isomorphism from X to BCK-algebra Y. X is a solvable BCK-algebra if and only if Y is a solvable BCK-algebra.

Proof. Since $f(X) = Y$, therefore $f(X^{'}) = Y^{'}$, because if $y \in f(X^{'})$, then there exists a $x \in X^{'}$ such that $f(x) = y$. But $x \in X^{'}$, then there exist $a_i, b_i \in X$ such that $x = \prod[a_i, b_i]$. Consequently $f(x) = f(\prod[a_i, b_i]) = \prod f[a_i, b_i] = \prod[f(a_i), f(b_i)] = y \in Y^{'}$. So, $f(X^{'}) \subseteq Y^{'}$.

Conversely, let $h \in Y^{'}$. Then, there exists $h_i, k_i \in Y$ such that $h = \prod[h_i, k_i]$. However, f is an isomorphism, so there exist $a_i, b_i \in X$ such that $h_i = f(a_i), k_i = f(b_i)$. Thus, $h = \prod[h_i, k_i] = \prod[f(a_i), f(b_i)] = \prod f[a_i, b_i] = f(\prod[a_i, b_i]) \in f(X^{'})$. By induction, we can show that $f(X^{(n)}) = Y^{(n)}$. Since X is a solvable *BCK*-algebra, there exists $n \in N$ such that $X^{(n)} = \{0\}$. Therefore, $\{0\} = f(\{0\}) = f(X^{(n)}) = Y^{(n)}$; that is, Y is a solvable *BCK*-algebra.

Conversely, let Y be a solvable *BCK*-algebra. Since $f(X) = Y$, then $f(X^{(n)}) = Y^{(n)} = \{0\}$, Therefore $f(X^{(n)}) = \{0\} = f(\{0\})$, so $X^{(n)} = \{0\}$; that is X is a solvable *BCK*-algebra.

Theorem 5. Sub-algebras and homomorphic images of solvable *BCK*-algebras also are solvable.

Proof Let Y be a sub-algebra of X. Then for any $n \in N, Y^{(n)}$ is sub-algebra of $X^{(n)}$. Since X is a solvable *BCK*-algebra, there exists n such that $X^{(n)} = \{0\}$ and so $Y^{(n)} = \{0\}$; that is, Y is a solvable *BCK*-algebras.

Now, let $(X, *, 0)$ and $(Y, *^{'}, 0^{'})$ be two *BCK*-algebras and X be a solvable *BCK*-algebra, $f : X \longrightarrow Y$ be a epimorphism (homomorphism) from X to Y. Then for some non-negative real number n, we have $f(X^{(n)}) = Y^{(n)}$. Hence $\{0\} = f(\{0\}) = f(X^{(n)}) = Y^{(n)}$. So, $f(X^{(n)}) = Y^{(n)} = \{0\}$; that is, Y is a solvable *BCK*-algebra.

Theorem 6. Let I be an ideal of X. If I and X/I are solvable *BCK*-algebras, then X is a solvable *BCK*-algebra.

Proof. Let f be the natural homomorphism from X onto X/I. Since X/I is solvable, so for some n, $f(X^{(n)}) = (X/I)^{(n)} = \{I\}$. Hence $X^{(n)}$ is a sub-algebra of $ker(f) = I$. By Theorem 5, $X^{(n)}$ is solvable. Therefore there exists a positive integer number k such that $X^{(n+k)} = (X^{(n)})^{(k)} = \{0\}$. That is X is solvable.

Lemma 2. Let $A, B \subseteq X$. Then

1. $A^{'} \cup B^{'} \subseteq (A \cup B)^{'}$,
2. $(A \cap B)^{'} \subseteq A^{'} \cap B^{'}$.

Proof. 1. Since A and B are two subsets of $A \cup B$, $A^{'} \subseteq (A \cup B)^{'}$ and $B^{'} \subseteq (A \cup B)^{'}$. Therefore, $A^{'} \cup B^{'} \subseteq (A \cup B)^{'}$.

2. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$. Then $(A \cap B)^{'} \subseteq A^{'}$ and $(A \cap B)^{'} \subseteq B^{'}$. So $(A \cap B)^{'} \subseteq A^{'} \cap B^{'}$.

In general, $(A \cup B)^{'} \nsubseteq A^{'} \cup B^{'}$ and $A^{'} \cap B^{'} \nsubseteq (A \cap B)^{'}$. For example, in the *BCK*-algebra in Example 4, consider $A = \{0, a\}$ and $B = \{0, c\}$. Then $A^{'} = \{0\}$, $B^{'} = \{0\}$ and $(A \cup B)' = \{0, a\} \nsubseteq A^{'} \cup B^{'} = \{0\}$.

Moreover, in Example 5, consider $A = \{0,1\}$ and $B = \{a\}$, then $A^{'} = B^{'} = \{0\}$ and $A^{'} \cap B^{'} = \{0\} \nsubseteq (A \cap B)^{'} = \phi$.

Theorem 7. The intersection of any two solvable sub-algebras of X, is solvable.

Proof. Let X_1 and X_2 be the two solvable sub-algebras of X. Since $X_1 \cap X_2 \subseteq X_1$ and X_1 is solvable, there exists $n \in N$ such that $X_1^{(n)} = \{0\}$. Therefore, $(X_1 \cap X_2)^{(n)} \subseteq X_1^{(n)} = \{0\}$. Then $(X_1 \cap X_2)^{(n)} = \{0\}$, that is $(X_1 \cap X_2)$ is a solvable *BCK*-algebra.

The above result can be generalized such that the intersection of any arbitrary family of sub-algebras of a solvable *BCK*-algebra, is again a solvable *BCK*-algebra. However, in general, the union of two solvable *BCK*-sub-algebras of a *BCK*-algebra may not be a solvable *BCK*-algebra. For example, consider the *BCK*-algebra $(N,*,0)$ together with $x * y = max\{0, x-y\}$, for all $x, y \in N$. Let X_1 be the sub-algebra of multiples of two with the operation $*$ and X_2 be the sub-algebra of multiples of three with operation $*$. Observe that $X_1 \cup X_2$ is not a sub-algebra, because it is not closed under $*$.

We have only one *BCK*-algebra of order one, that is, $X = \{0\}$, for this algebra $X^{'} = \{0\}$. Also, there is a unique *BCK*-algebra of order two, that is, $X = \{0,1\}$ with the following operation [9] for this algebra $X^{'} = \{0\}$. Up to isomorphism, there exist only three *BCK*-algebras of order 3

TABLE 6

$*$	0	1
0	0	0
1	1	0

and fourteen *BCK*-algebras of order 4 and eighty-eight *BCK*-algebras of order 5 all being solvable *BCK*-algebras, so all *BCK*-algebras with order less than 6 are solvable.

Open Problem: Is any finite *BCK*-algebra a solvable *BCK*-algebra?

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