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Generalized Extended C-Bochner Curvature Tensor on $(\grave{k}, \grave{\mu})$-Contact Metric Manifolds

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ABSTRACT

The object of the present paper is to study $(\grave{k}, \grave{\mu})$-contact metric manifolds with generalized extended C-Bochner curvature tensor.

Keywords: *$(\grave{k}; \grave{\mu})$-contact metric manifold, Sasakian manifold, C-Bochner curvature tensor, generalized extended C-Bochner curvature tensor.*
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1. Introduction

The Bochner curvature tensor was introduced by S. Bochner [4]. When he studied the Betti number of a Kähler manifold, he introduced a new tensor as an analogue of the Weyl conformal curvature tensor in a Riemannian manifold. Then D. Blair gave the geometric properties of the Bochner curvature tensor [1]. Then M. Matsumoto and G. Chūman introduced C-Bochner curvature tensor by using the Boothby-Wang's fibration and they studied its properties in a Sasakian manifold [9]. Some authors have studied vanishing C-Bochner curvature tensor in Sasakian manifolds ([7], [8]). On the other hand, the extended C-Bochner curvature tensor on a K-contact Riemannian manifold introduced by H. Endo and called it the *E-contact Bochner curvature tensor* [6] and he showed that a K-contact Riemannian manifold with vanishing the E- contact Bochner curvature tensor is a Sasakian manifold. Also the *generalized C-Bochner* (briefly GC-Bochner) curvature tensor defined by Shaikh and Baishya [11].

Motivated by these studies in this paper, firstly we introduce the generalized extended C-Bochner (briefly GEC-Bochner) curvature tensor by using H. Endo's method. Then we study vanishes the GEC-Bochner curvature tensor on a $(\grave{k}, \grave{\mu})$-contact metric manifold and show that the manifold is a Sasakian manifold. Also we consider a $(\grave{k}, \grave{\mu})$-contact metric manifold satisfying the conditions $R_{cur}(\partial_1, \partial_2) \cdot B^{GE} = 0$ and $B^{GE}(\partial_1, \partial_2) \cdot R_{cur} = 0$. We obtain that the manifold is a Sasakian manifold or locally isometric to $E^{n+1}(0) \times S^n(4)$ and the manifold is a Sasakian manifold or is an η-Einstein manifold, respectively.

2. Preliminaries

Let G^{2n+1} be a connected differentiable manifold which is said to admits an *almost contact structure* (ψ, ζ, η), where ψ is a tensor field of type $(1, 1)$, ζ is a vector field and η is a 1-form satisfying

$$\psi^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1, \quad \psi\zeta = 0, \ \eta \circ \psi = 0. \tag{2.1}$$

Let ρ be a compatible Riemannian metric with (ψ, ζ, η), that is,

$$\rho(\psi\partial_1, \psi\partial_2) = \rho(\partial_1, \partial_2) - \eta(\partial_1)\eta(\partial_2), \tag{2.2}$$

or,

$$\rho(\partial_1, \psi\partial_2) = -\rho(\psi\partial_1, \partial_2) \quad \text{and} \quad \eta(\partial_1) = \rho(\partial_1, \zeta),$$

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for all $\partial_1, \partial_2 \in \Gamma(TG)$. So, G is an almost contact metric manifold equipped with $(\psi, \zeta, \eta, \rho)$. If $\rho(\partial_1, \psi\partial_2) = d\eta(\partial_1, \partial_2)$ then an almost contact metric structure is a contact metric structure.

The 1-form η is then a contact form and ζ is its characteristic vector field. Also the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(\partial_1, \partial_2)\zeta = 0$ [2]. On the other hand, as we have noted [3], on a Sasakian manifold

$$R(\partial_1, \partial_2)\zeta = \eta(\partial_2)\partial_1 - \eta(\partial_1)\partial_2. \tag{2.3}$$

In a contact metric manifold, the $(1,1)$-tensor field $\tilde{\hbar}$ is symmetric and satisfies

$$\tilde{\hbar}\zeta = 0, \quad \tilde{\hbar}\psi + \psi\tilde{\hbar} = 0, \quad \nabla\zeta = -\psi - \psi\tilde{\hbar}, \quad tr\tilde{\hbar} = tr\psi\tilde{\hbar} = 0, \tag{2.4}$$

where ∇ is the Levi-Civita connection. A contact metric manifold is said to be η-Einstein if

$$R_{op} = aId + b\grave{w} \otimes \zeta,$$

where a, b are smooth functions and R_{op} is the Ricci operator on G.

The $(\grave{k}, \grave{\mu})$-nullity distribution $N(\grave{k}, \grave{\mu})$ of a contact metric manifold G is defined by

$$\begin{aligned} N(\grave{k}, \grave{\mu}) \quad : \quad t \longrightarrow N_t(\grave{k}, \grave{\mu}) = \{\partial_3 \in T_tG \mid R(\partial_1, \partial_2)\partial_3 = \grave{k}[\rho(\partial_2, \partial_3)\partial_1 - \rho(\partial_1, \partial_3)\partial_2] \\ +\grave{\mu}[\rho(\partial_2, \partial_3)\tilde{\hbar}\partial_1 - \rho(\partial_1, \partial_3)\tilde{\hbar}\partial_2]\}, \end{aligned}$$

for all $\partial_1, \partial_2 \in \Gamma(TG)$, where $\grave{k}, \grave{\mu}$ real constants ([2], [10]). A contact metric manifold G with $\zeta \in N(\grave{k}, \grave{\mu})$ is called a $(\grave{k}, \grave{\mu})$-contact metric manifold, then we have

$$R(\partial_1, \partial_2)\zeta = \grave{k}\{\eta(\partial_2)\partial_1 - \eta(\partial_1)\partial_2\} + \grave{\mu}\{\eta(\partial_2)\tilde{\hbar}\partial_1 - \eta(\partial_1)\tilde{\hbar}\partial_2\}. \tag{2.5}$$

Also in a $(\grave{k}, \grave{\mu})$-contact metric manifold, we have

$$R_{cur}(\partial_1, \zeta) = 2nk\grave{w}(\partial_1), \tag{2.6}$$

$$R_{op}\zeta = 2nk\zeta, \tag{2.7}$$

$$\tilde{\hbar}^2 = (\grave{k} - 1)\psi^2, \ \grave{k} \leq 1, \tag{2.8}$$

$$R_{op}\psi - \psi R_{op} = 2[2(n-1) + \grave{\mu}]\tilde{\hbar}\psi, \tag{2.9}$$

$$R(\zeta, \partial_1)\partial_2 = \grave{k}\{\rho(\partial_1, \partial_2)\zeta - \eta(\partial_2)\partial_1\} + \grave{\mu}\{\rho(\tilde{\hbar}\partial_1, \partial_2)\zeta - \eta(\partial_2)\tilde{\hbar}\partial_1\}. \tag{2.10}$$

From (2.6)-(2.7), we have

$$tr\tilde{\hbar}^2 = 2n(1 - \grave{k}),$$

$$R_{cur}(\partial_1, \psi\partial_2) + R_{cur}(\psi\partial_1, \partial_2) = 2(2(n-1) + \grave{\mu})\rho(\tilde{\hbar}\psi\partial_1, \partial_2),$$

$$R_{cur}(\psi\partial_1, \psi\partial_2) = R_{cur}(\partial_1, \partial_2) - 2nk\eta(\partial_1)\eta(\partial_2) - 2(2(n-1) + \grave{\mu})\rho(\tilde{\hbar}\partial_1, \partial_2),$$

$$R_{op}\psi + \psi R_{op} = 2\psi R_{op} + 2(2(n-1) + \grave{\mu})\tilde{\hbar}\psi,$$

$$\psi R_{op}\psi = 2(2(n-1) + \grave{\mu})\tilde{\hbar} - R_{op} + 2nk\grave{w} \otimes \zeta,$$

$$R_{cur}(\psi\partial_1, \zeta) = 0,$$

$$tr(R_{op}\psi) = tr(\psi R_{op}) = 0.$$

Now we give the following:

Lemma 2.1. *[2]: In a non-Sasakian $(\grave{k}, \grave{\mu})$-manifold G, the Ricci operator R_{op} is given by*

$$R_{op} = (2(n-1) - n\mu)I + (2(n-1) + \grave{\mu})\tilde{\hbar} + (2(1-n) + n(2\grave{k} + \grave{\mu}))\eta \otimes \zeta. \tag{2.11}$$

Theorem 2.1. *[2], [3]: A contact metric manifold G satisfying $R(\partial_1, \partial_2)\zeta = 0$ is locally isometric to $E^{n+1} \times S^n(4)$ for $n > 1$ and flat for $n = 1$.*

3. GEC-Bochner curvature tensor

In [11], Shaikh and Baishya defined the GC-Bochner curvature tensor in a contact metric manifold as follows:

$$\begin{aligned} B(\partial_1,\partial_2)\partial_3 &= R_{cur}(\partial_1,\partial_2)\partial_3 + \rho(\psi\partial_2,\tilde{\hbar}\partial_3)\tilde{\hbar}\psi\partial_1 - \rho(\psi\partial_1,\tilde{\hbar}\partial_3)\tilde{\hbar}\psi\partial_2 \\ &+\frac{1}{\grave{m}+4}\{\frac{\grave{m}}{2}+\grave{\alpha}+\frac{tr\tilde{\hbar}^2}{\grave{m}+2}\}[\rho(\partial_1,\partial_3)\eta(\partial_2)\zeta - \rho(\partial_2,\partial_3)\eta(\partial_1)\zeta \\ &-\eta(\partial_2)\eta(\partial_3)\partial_1 + \eta(\partial_1)\eta(\partial_3)\partial_2] \\ &-\frac{1}{\grave{m}+4}\{\grave{\alpha}+\grave{m}+\frac{tr\tilde{\hbar}^2}{\grave{m}+2}\}[\rho(\psi\partial_1,\partial_3)\psi\partial_2 \\ &-\rho(\psi\partial_2,\partial_3)\psi\partial_1 + 2\rho(\psi\partial_1,\partial_2)\psi\partial_3] \\ &-\frac{1}{\grave{m}+4}\{\grave{\alpha}-4+\frac{tr\tilde{\hbar}^2}{\grave{m}+2}\}[\rho(\partial_1,\partial_3)\partial_2 - \rho(\partial_2,\partial_3)\partial_1] \\ &+\frac{1}{2(\grave{m}+4)}[\rho(\partial_1,\partial_3)R_{op}\partial_2 - \rho(\partial_2,\partial_3)R_{op}\partial_1 - R_{cur}(\partial_2,\partial_3)\partial_1 \\ &+R_{cur}(\partial_1,\partial_3)\partial_2 - \rho(\partial_1,\partial_3)\psi R_{op}\psi\partial_2 + \rho(\partial_2,\partial_3)\psi R_{op}\psi\partial_1 \\ &-R_{cur}(\psi\partial_2,\psi\partial_3)\partial_1 + R_{cur}(\psi\partial_1,\psi\partial_3)\partial_2 - R_{cur}(\psi\partial_2,\partial_3)\psi\partial_1 \\ &+R_{cur}(\partial_2,\psi\partial_3)\psi\partial_1 + R_{cur}(\psi\partial_1,\partial_3)\psi\partial_2 - R_{cur}(\partial_1,\psi\partial_3)\psi\partial_2 \\ &+2R_{cur}(\psi\partial_1,\partial_2)\psi\partial_3 - 2R_{cur}(\partial_1,\psi\partial_2)\psi\partial_3 \\ &+\rho(\psi\partial_1,\partial_3)(\psi R_{op}+R_{op}\psi)\partial_2 - \rho(\psi\partial_2,\partial_3)(\psi R_{op}+R_{op}\psi)\partial_1 \\ &+2\rho(\psi\partial_1,\partial_2)(\psi R_{op}+R_{op}\psi)\partial_3 - \eta(\partial_1)\eta(\partial_3)R_{op}\partial_2 \\ &+\eta(\partial_2)\eta(\partial_3)R_{op}\partial_1 + \eta(\partial_1)\eta(\partial_3)\psi R_{op}\psi\partial_2 \\ &-\eta(\partial_2)\eta(\partial_3)\psi R_{op}\psi\partial_1 + R_{cur}(\partial_2,\partial_3)\eta(\partial_1)\zeta - R_{cur}(\partial_1,\partial_3)\eta(\partial_2)\zeta \\ &+R_{cur}(\psi\partial_2,\psi\partial_3)\eta(\partial_1)\zeta - R_{cur}(\psi\partial_1,\psi\partial_3)\eta(\partial_2)\zeta], \end{aligned} \tag{3.1}$$

where $\grave{\alpha} = \frac{\tau+\grave{m}}{\grave{m}+2}$, $\grave{m} = 2n$. If the manifold is a Sasakian manifold, then we have $\tilde{\hbar} = 0$, $R_{op}\psi = \psi R_{op}$, $tr\tilde{\hbar}^2 = 0$, $R_{cur}(\psi\partial_1,\psi\partial_2) = R_{cur}(\partial_1,\partial_2) - m\eta(\partial_1)\eta(\partial_2)$ and hence (3.1) definities the C-Bochner curvature tensor.

From (3.1), we have the followings:

$$B(\partial_1,\partial_2)\partial_3 = -B(\partial_2,\partial_3)\partial_1, \tag{3.2}$$

$$B(\partial_1,\partial_2)\partial_3 + B(\partial_2,\partial_3)\partial_1 + B(\partial_3,\partial_1)\partial_2 = 0, \tag{3.3}$$

$$\rho(B(\partial_1,\partial_2)\partial_3,\partial_4) = -\rho(B(\partial_1,\partial_2)\partial_4,\partial_3), \tag{3.4}$$

$$\rho(B(\partial_1,\partial_2)\partial_3,\partial_4) = \rho(B(\partial_3,\partial_4)\partial_1,\partial_2), \tag{3.5}$$

for any vector fields $\partial_1,\partial_2,\partial_3,\partial_4 \in \Gamma(G)$. Also

$$B(\partial_1,\partial_2)\zeta = \frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\eta(\partial_2)\partial_1 - \eta(\partial_1)\partial_2] + \grave{\mu}[\eta(\partial_2)\tilde{\hbar}\partial_1 - \eta(\partial_1)\tilde{\hbar}\partial_2], \tag{3.6}$$

and

$$B(\zeta,\partial_1)\partial_2 = \frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_1,\partial_2)\zeta - \eta(\partial_2)\partial_1] + \grave{\mu}[\rho(\tilde{\hbar}\partial_1,\partial_2)\zeta - \eta(\partial_2)\tilde{\hbar}\partial_1], \tag{3.7}$$

Consequently, we have

$$B(\zeta,\partial_1)\zeta = \frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\eta(\partial_1)\zeta - \partial_1] + \grave{\mu}\tilde{\hbar}\partial_1, \tag{3.8}$$

$$\eta(B(\partial_1,\partial_2)\zeta) = 0, \tag{3.9}$$

$$\eta(B(\zeta,\partial_1)\partial_2) = \frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_1,\partial_2) - \eta(\partial_1)\eta(\partial_2)] + \mu\rho(\tilde{\hbar}\partial_1,\partial_2), \tag{3.10}$$

where $\grave{m} = 2n$.

H. Endo [6] defined the extended C-Bochner curvature tensor B^E on a contact metric manifold G by

$$B^E(\partial_1,\partial_2)\partial_3 = \tilde{B}(\partial_1,\partial_2)\partial_3-\eta(\partial_1)\tilde{B}(\zeta,\partial_2)\partial_3 -\eta(\partial_2)\tilde{B}(\partial_1,\zeta)\partial_3-\eta(\partial_3)\tilde{B}(\partial_1,\partial_2)\zeta, \tag{3.11}$$

where $\tilde{B}$ is the C-Bochner curvature tensor defined by Matsumoto and Chūman [9].

Now using this definition, we can define GEC-Bochner curvature tensor B^{GE} on a contact metric manifold G by

$$B^{GE}(\partial_1,\partial_2)\partial_3 = \hat{B}^E(\partial_1,\partial_2)\partial_3-\eta(\partial_1)\hat{B}^E(\zeta,\partial_2)\partial_3 -\eta(\partial_2)\hat{B}^E(\partial_1,\zeta)\partial_3-\eta(\partial_3)\hat{B}^E(\partial_1,\partial_2)\zeta, \tag{3.12}$$

where

$$\hat{B}^E(\partial_1,\partial_2)\partial_3 = B(\partial_1,\partial_2)\partial_3-\eta(\partial_1)B(\zeta,\partial_2)\partial_3 -\eta(\partial_2)B(\partial_1,\zeta)\partial_3-\eta(\partial_3)B(\partial_1,\partial_2)\zeta, \tag{3.13}$$

and B is the GC-Bochner curvature tensor defined by (3.1). Using (3.1), (3.6) and (3.7) in (3.13), from (3.12), we can write

$$\begin{aligned} B^{GE}(\partial_1,\partial_2)\partial_3 &= B(\partial_1,\partial_2)\partial_3-\eta(\partial_1)B(\zeta,\partial_2)\partial_3 \\ &\quad -\eta(\partial_2)B(\partial_1,\zeta)\partial_3-\eta(\partial_3)B(\partial_1,\partial_2)\zeta \\ &\quad +2[\eta(\partial_1)\eta(\partial_3)B(\zeta,\partial_2)\zeta-\eta(\partial_2)\eta(\partial_3)B(\zeta,\partial_1)\zeta]. \end{aligned} \tag{3.14}$$

Also using (3.1)-(3.8) in (3.14), we get

$$\begin{aligned} B^{GE}(\partial_1,\partial_2)\partial_3 &= B(\partial_1,\partial_2)\partial_3 \\ &\quad +\frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_2,\partial_3)\eta(\partial_1)\zeta-\rho(\partial_1,\partial_3)\eta(\partial_2)\zeta] \\ &\quad +\grave{\mu}[\rho(\tilde{\hbar}\partial_1,\partial_3)\eta(\partial_2)\zeta-\rho(\tilde{\hbar}\partial_2,\partial_3)\eta(\partial_1)\zeta]. \end{aligned} \tag{3.15}$$

4. Main results

Let G^{2n+1} $(n>1)$ be a $(\grave{k},\grave{\mu})$-contact metric manifold with vanishes the GEC-Bochner curvature tensor. Then we have $B^{GE}(\partial_1,\partial_2)\partial_3=0$ for all $\partial_1,\partial_2,\partial_3\in\Gamma(TG)$.

Now from (3.15), we can write

$$\begin{aligned} 0 &= \rho(B(\partial_1,\partial_2)\partial_3,\partial_4) \\ &\quad +\frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_2,\partial_3)\eta(\partial_1)\eta(\partial_4)-\rho(\partial_1,\partial_3)\eta(\partial_2)\eta(\partial_4)] \\ &\quad +\grave{\mu}[\rho(\tilde{\hbar}\partial_1,\partial_3)\eta(\partial_2)\eta(\partial_4)-\rho(\tilde{\hbar}\partial_2,\partial_3)\eta(\partial_1)\eta(\partial_4)]. \end{aligned} \tag{4.1}$$

Let $\{f_j : j=1,2,...,2n+1\}$ be an orthonormal basis of the tangent space T_tG at any point $t\in G$. Putting $\partial_1=\partial_4=f_j$ in (4.1) and taking summation over $1\le j\le \grave{m}+1$, $\grave{m}=2n$, we obtain

$$\begin{aligned} 0 &= \sum_{i=1}^{\grave{m}+1}\rho(B(f_j,\partial_2)\partial_3,f_j) \\ &\quad +\frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_2,\partial_3)-\eta(\partial_2)\eta(\partial_3)] \\ &\quad -\grave{\mu}\rho(\tilde{\hbar}\partial_2,\partial_3). \end{aligned} \tag{4.2}$$

From [11], it is known that

$$\begin{aligned} \sum_{i=1}^{\grave{m}+1}\rho(B(f_j,\partial_2)\partial_3,f_j) &= (\grave{m}-2+\grave{\mu})\rho(\tilde{\hbar}\partial_2,\partial_3)+\frac{(1-\grave{k})(3\grave{m}+8)}{2(\grave{m}+4)}\rho(\partial_2,\partial_3) \\ &\quad +\frac{(1-\grave{k})(1-\grave{m})(\grave{m}+8)}{2(\grave{m}+4)}\eta(\partial_2)\eta(\partial_3). \end{aligned} \tag{4.3}$$

Using (4.3) in (4.2), we have

$$\begin{aligned} 0 &= (\grave{m}-2)\rho(\tilde{\hbar}\partial_2, \partial_3) + 2(1-\grave{k})\rho(\partial_2, \partial_3) \\ & -\frac{\grave{m}(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}\eta(\partial_2)\eta(\partial_3). \end{aligned} \quad (4.4)$$

Now from (4.4), we get

$$(\grave{m}-2)\rho(\tilde{\hbar}\partial_2, \partial_3) + 2(1-\grave{k})\rho(\partial_2, \partial_3) - \frac{\grave{m}(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}\eta(\partial_2)\eta(\partial_3) = 0,$$

which implies that $\tilde{\hbar} = 0$ and hence $\grave{k} = 1$. Thus the manifold is a Sasakian manifold. Hence we can state the following:

Theorem 4.1. *Let $G^{2n+1}(n > 1)$ be a $(\grave{k}, \grave{\mu})$-contact metric manifold with vanishes GEC-Bochner curvature tensor. Then the manifold is a Sasakian manifold.*

Now we consider a $(\grave{k}, \grave{\mu})$-contact metric manifold G^{2n+1} satisfying the condition

$$R_{cur}(\partial_1, \partial_2) \cdot B^{GE} = 0, \quad (4.5)$$

where $R(\partial_1, \partial_2)$ is considered as a curvature tensor. Then from (4.5) it follows that

$$\begin{aligned} & R_{cur}(\zeta, \partial_2)B^{GE}(\partial_3, \partial_4)\partial_5 - B^{GE}(R_{cur}(\zeta, \partial_2)\partial_3, \partial_4)\partial_5 \\ & -B^{GE}(\partial_3, R_{cur}(\zeta, \partial_2)\partial_4)\partial_5 - B^{GE}(\partial_3, \partial_4)R_{cur}(\zeta, \partial_2)\partial_5 = 0. \end{aligned} \quad (4.6)$$

In view of (2.10), it follows from (4.6) that

$$\begin{aligned} & \grave{k}[\rho(B^{GE}(\partial_3, \partial_4)\partial_5, \partial_2)\zeta - \eta(B^{GE}(\partial_3, \partial_4)\partial_5)\partial_2 - \rho(\partial_2, \partial_3)B^{GE}(\zeta, \partial_4)\partial_5 \\ & +\eta(\partial_3)B^{GE}(\partial_2, \partial_4)\partial_5 - \rho(\partial_2, \partial_4)B^{GE}(\partial_3, \zeta)\partial_5 + \eta(\partial_4)B^{GE}(\partial_3, \partial_2)\partial_5 \\ & -\rho(\partial_2, \partial_5)B^{GE}(\partial_3, \partial_4)\zeta + \eta(\partial_5)B^{GE}(\partial_3, \partial_4)\partial_2] + \grave{\mu}[\rho(B^{GE}(\partial_3, \partial_4)\partial_5, \tilde{\hbar}\partial_2)\zeta \\ & -\eta(B^{GE}(\partial_3, \partial_4)\partial_5)\tilde{\hbar}\partial_2 - \rho(\tilde{\hbar}\partial_2, \partial_3)B^{GE}(\zeta, \partial_4)\partial_5 + \eta(\partial_3)B^{GE}(\tilde{\hbar}\partial_2, \partial_4)\partial_5 \\ & +\rho(\tilde{\hbar}\partial_2, \partial_4)B^{GE}(\partial_3, \zeta)\partial_5 + \eta(\partial_4)B^{GE}(\partial_3, \tilde{\hbar}\partial_2)\partial_5 - \rho(\tilde{\hbar}\partial_2, \partial_5)B^{GE}(\partial_3, \partial_4)\zeta \\ & +\eta(\partial_5)B^{GE}(\partial_3, \partial_4)\tilde{\hbar}\partial_2] = 0. \end{aligned} \quad (4.7)$$

Putting $\partial_3 = \zeta$ in (3.15), we obtain by virtue of (2.2), (2.4) and (3.1) that

$$B^{GE}(\partial_1, \partial_2)\zeta = \frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\eta(\partial_1)\partial_2 - \eta(\partial_2)\partial_1 - \grave{\mu}[\eta(\partial_1)\tilde{\hbar}\partial_2 - \eta(\partial_2)\tilde{\hbar}\partial_1]. \quad (4.8)$$

Using (3.5), we get from (4.8) that

$$B^{GE}(\zeta, \partial_1)\partial_2 = \frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\eta(\partial_2)\partial_1 - \eta(\partial_2)\eta(\partial_1)\zeta] - \mu\eta(\partial_2)\tilde{\hbar}\partial_1, \quad (4.9)$$

which implies

$$B^{GE}(\zeta, \partial_1)\zeta = \frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\partial_1 - \eta(\partial_1)\zeta] - \grave{\mu}\tilde{\hbar}\partial_1. \quad (4.10)$$

From (4.8), it follows that

$$\eta\left(B^{GE}(\partial_1, \partial_2)\zeta\right) = 0.$$

Again (4.8) yields

$$B^{GE}(\tilde{\hbar}\partial_1, \partial_2)\zeta = -\frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}\eta(\partial_2)\tilde{\hbar}\partial_1 + \grave{\mu}(1-\grave{k})[\eta(\partial_2)\partial_1 - \eta(\partial_2)\eta(\partial_1)\zeta], \quad (4.11)$$

which implies by (3.2) that

$$B^{GE}(\partial_1, \tilde{\hbar}\partial_2)\zeta = \frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}\eta(\partial_1)\tilde{\hbar}\partial_2 + \grave{\mu}(1-\grave{k})[\eta(\partial_1)\eta(\partial_2)\zeta - \eta(\partial_1)\partial_2]. \quad (4.12)$$

Putting $\partial_5 = \zeta$ in (4.7) and then using (4.8)-(4.12), we get

$$\begin{aligned}&\grave{k}\{\frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_2,\partial_4)\partial_3-\rho(\partial_2,\partial_3)\partial_4]+\grave{\mu}[\rho(\partial_2,\tilde{\hbar}\partial_3)\eta(\partial_4)\zeta\\&-\rho(\partial_2,\tilde{\hbar}\partial_4)\eta(\partial_3)\zeta+\rho(\partial_2,\partial_3)\tilde{\hbar}\partial_4-\rho(\partial_2,\partial_4)\tilde{\hbar}\partial_3]+B^{GE}(\partial_3,\partial_4)\partial_2\}\\&+\grave{\mu}\{\frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\tilde{\hbar}\partial_2,\partial_4)\partial_3-\rho(\tilde{\hbar}\partial_2,\partial_3)\partial_4]+\grave{\mu}[\rho(\tilde{\hbar}\partial_2,\tilde{\hbar}\partial_3)\eta(\partial_4)\zeta\\&-\rho(\tilde{\hbar}\partial_2,\tilde{\hbar}\partial_4)\eta(\partial_3)\zeta+\rho(\tilde{\hbar}\partial_2,\partial_3)\tilde{\hbar}\partial_4-\rho(\tilde{\hbar}\partial_2,\partial_4)\tilde{\hbar}\partial_3]+B^{GE}(\partial_3,\partial_4)\tilde{\hbar}\partial_2\}=0,\end{aligned} \tag{4.13}$$

which implies that

$$\begin{aligned}&\grave{k}\{\frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\partial_2,\partial_4)\rho(\partial_3,\partial_1)-\rho(\partial_2,\partial_3)\rho(\partial_4,\partial_1)]\\&+\grave{\mu}[\rho(\partial_2,\tilde{\hbar}\partial_3)\eta(\partial_4)\rho(\zeta,\partial_1)-\rho(\partial_2,\tilde{\hbar}\partial_4)\eta(\partial_3)\rho(\xi,\partial_1)\\&+\rho(\partial_2,\partial_3)\rho(\tilde{\hbar}\partial_4,\partial_1)-\rho(\partial_2,\partial_4)\rho(\tilde{\hbar}\partial_3,\partial_1)]+\rho(B^{GE}(\partial_3,\partial_4)\partial_2,\partial_1)\}\\&+\grave{\mu}\{\frac{(\grave{k}-1)(\grave{m}+8)}{2(\grave{m}+4)}[\rho(\tilde{\hbar}\partial_2,\partial_4)\rho(\partial_3,\partial_1)-\rho(\tilde{\hbar}\partial_2,\partial_3)\rho(\partial_4,\partial_1)]\\&+\grave{\mu}[\rho(\tilde{\hbar}\partial_2,\tilde{\hbar}\partial_3)\eta(\partial_4)\rho(\zeta,\partial_1)-\rho(\tilde{\hbar}\partial_2,\tilde{\hbar}\partial_4)\eta(\partial_3)\rho(\zeta,\partial_1)\\&+\rho(\tilde{\hbar}\partial_2,\partial_3)\rho(\tilde{\hbar}\partial_4,\partial_1)-\rho(\tilde{\hbar}\partial_2,\partial_4)\rho(\tilde{\hbar}\partial_3,\partial_1)]+\rho(B^{GE}(\partial_3,\partial_4)\tilde{\hbar}\partial_2,\partial_1)\}=0.\end{aligned} \tag{4.14}$$

Let $\{f_j : \ j = 1, 2, ..., 2n + 1\}$ be an orthonormal basis of the tangent space T_tG at any point $t \in G$. Putting $\partial_1 = \partial_3 = f_j$ in (4.14) and taking summation over $1 \leq j \leq \grave{m} + 1$, $\grave{m} = 2n$, we obtain

$$\begin{aligned}&\frac{(1-\grave{k})}{2(\grave{m}+4)}\{\grave{k}(\grave{m}+1)(\grave{m}+8)+\grave{k}(3\grave{m}+8)+2\mu(\grave{m}+4)(\grave{m}-2)\}\rho(\partial_2,\partial_4)\\&+\frac{1}{2(\grave{m}+4)}\{2k(\grave{m}+4)(\grave{m}-2)+\grave{\mu}(1-\grave{k})[(\grave{m}+8)(1+\grave{m})+(3\grave{m}+8)]\}\rho(\tilde{\hbar}\partial_4,\partial_2)\\&+\frac{(1-\grave{k})}{2(\grave{m}+4)}\{-k\grave{m}(\grave{m}+8)-2\mu(\grave{m}+4)(\grave{m}-2)\}\eta(\partial_4)\eta(\partial_2)=0.\end{aligned} \tag{4.15}$$

The relation (4.15) gives us either $\grave{k} = 1$, or

$$\grave{k}(\grave{m}+1)(\grave{m}+8)+\grave{k}(3\grave{m}+8)+2\mu(\grave{m}+4)(\grave{m}-2)=0, \tag{4.16}$$

either $\tilde{\hbar} = 0$, or

$$2k(\grave{m}+4)(\grave{m}-2)+\grave{\mu}(1-\grave{k})[(\grave{m}+8)(1+\grave{m})+(3\grave{m}+8)]=0, \tag{4.17}$$

either $\grave{k} = 1$, or

$$-\grave{k}\grave{m}(\grave{m}+8)-2\mu(\grave{m}+4)(\grave{m}-2)=0. \tag{4.18}$$

If $\grave{k} = 1$ then $\tilde{\hbar} = 0$ and therefore the manifold is a Sasakian manifold. Thus we have either the manifold is a Sasakian manifold (for $\grave{k} = 1$) or (4.16)-(4.18) holds (for non-Sasakian case). From (4.16) and (4.18), it follows that $\grave{k} = 0$. If $\grave{k} = 0$, then (4.16) or (4.18) implies that $\grave{\mu} = 0$. Hence in the non-Sasakian case, we have $\grave{k} = 0 = \grave{\mu}$. Then by Theorem 2.1 the manifold is locally isometric to $E^{n+1}(0) \times S^n(4)$.

Also from (4.16) and (4.17), we get

$$\grave{k}^2(\grave{m}+1)(\grave{m}+8)+\grave{k}^2(3\grave{m}+8)+\grave{\mu}^2(\grave{k}-1)(\grave{m}^2+12\grave{m}+16)=0. \tag{4.19}$$

Hence from (4.17) and (4.18), it follows that

$$\grave{k}^2\grave{m}(\grave{m}+8)+\grave{\mu}^2(\grave{k}-1)(\grave{m}^2+12\grave{m}+16)=0. \tag{4.20}$$

Thus from (4.19) and (4.20), we get $\grave{k} = 0$. If $\grave{k} = 0$, then (4.16) or (4.18) implies that $\grave{\mu} = 0$.

The above discussion leads us to state the following:

Theorem 4.2. *Let $G^{2n+1}(n>1)$ be a $(\grave{k},\grave{\mu})$-contact metric manifold satisfying the condition $R(\partial_1,\partial_2)\cdot B^{GE}=0$. Then the manifold is a Sasakian manifold or locally isometric to $E^{n+1}(0)\times S^n(4)$.*

Let us consider a $(\grave{k},\grave{\mu})$-contact metric manifold G^{2n+1} satisfying the condition

$$B^{GE}(\partial_1,\partial_2)\cdot R_{cur}=0. \tag{4.21}$$

Then from (4.5) it follows that

$$\begin{aligned}&B^{GE}(\zeta,\partial_2)R_{cur}(\partial_3,\partial_4)\partial_5-R_{cur}(B^{GE}(\zeta,\partial_2)\partial_3,\partial_4)\partial_5\\&-R_{cur}(\partial_3,B^{GE}(\zeta,\partial_2)\partial_4)\partial_5-R_{cur}(\partial_3,\partial_4)B^{GE}(\zeta,\partial_2)\partial_5=0.\end{aligned} \tag{4.22}$$

Using (4.9) in (4.22), we have

$$\begin{aligned}&\grave{k}[\rho(B^{GE}(\zeta,\partial_2)\partial_3,\partial_5)\partial_4-\rho(\partial_3,\partial_4)B^{GE}(\zeta,\partial_2)\partial_5\\&-\rho(B^{GE}(\zeta,\partial_2)\partial_4,\partial_5)\partial_3+\rho(\partial_3,\partial_5)B^{GE}(\zeta,\partial_2)\partial_4\\&-\rho(\partial_4,B^{GE}(\zeta,\partial_2)\partial_5)\partial_3+\rho(\partial_3,B^{GE}(\zeta,\partial_2)\partial_5)\partial_4]\\&+\grave{\mu}[\rho(B^{GE}(\zeta,\partial_2)\partial_3,\partial_5)\tilde{\hbar}\partial_4-\rho(\partial_3,\partial_4)B^{GE}(\zeta,\partial_2)\tilde{\hbar}\partial_5\\&-\rho(B^{GE}(\zeta,\partial_2)\tilde{\hbar}\partial_4,\partial_5)\partial_3+\rho(\partial_3,\partial_5)B^{GE}(\zeta,\partial_2)\tilde{\hbar}\partial_4\\&-\rho(\tilde{\hbar}\partial_4,B^{GE}(\zeta,\partial_2)\partial_5)\partial_3+\rho(\partial_3,B^{GE}(\zeta,\partial_2)\partial_5)\tilde{\hbar}\partial_4]=0.\end{aligned} \tag{4.23}$$

Putting $\partial_3=\zeta$ in (4.23) and taking the inner product with ∂_6, we get

$$\begin{aligned}&\frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[\eta(R_{cur}(\zeta,\partial_4)\partial_5)g(\partial_2,\partial_6)-\eta(R_{cur}(\zeta,\partial_4)\partial_5)\eta(\partial_2)\eta(\partial_6)\\&-\rho(R_{cur}(\partial_2,\partial_4)\partial_5,\partial_6)+\eta(\partial_2)\rho(R_{cur}(\zeta,\partial_4)\partial_5,\partial_6)\\&-\eta(\partial_4)\rho(R_{cur}(\zeta,\partial_2)\partial_5,\partial_6)-\eta(\partial_5)\rho(R_{cur}(\zeta,\partial_4)\partial_2,\partial_6)\\&+\eta(\partial_5)\eta(\partial_2)\rho(R_{cur}(\zeta,\partial_4)\zeta,\partial_6)]\\&-\grave{\mu}[\eta(R_{cur}(\zeta,\partial_4)\partial_5)\rho(\tilde{\hbar}\partial_2,\partial_6)-\rho(R_{cur}(\tilde{\hbar}\partial_2,\partial_4)\partial_5,\partial_6)\\&-\eta(\partial_4)\rho(R_{cur}(\zeta,\tilde{\hbar}\partial_2)\partial_5,\partial_6)-\eta(\partial_5)\rho(R_{cur}(\zeta,\partial_4)\tilde{\hbar}\partial_2,\partial_6)]=0.\end{aligned} \tag{4.24}$$

Let $\{f_j:\ j=1,2,...,2n+1\}$ be an orthonormal basis of the tangent space T_tG at any point $t\in G$. Putting $\partial_2=\partial_6=f_j$ in (4.24) and taking summation over $1\le j\le \grave{m}+1$, $\grave{m}=2n$, we obtain

$$\begin{aligned}&\frac{(1-\grave{k})(\grave{m}+8)}{2(\grave{m}+4)}[(\grave{m}+1)\grave{k}\rho(\partial_4,\partial_5)-R_{cur}(\partial_4,\partial_5)-\grave{k}\eta(\partial_4)\eta(\partial_5)\\&+(\grave{m}+1)\mu\rho(\tilde{\hbar}\partial_4,\partial_5)]+\grave{\mu}[-\grave{k}\rho(\tilde{\hbar}\partial_5,\partial_4)\\&-\grave{\mu}(\grave{k}-1)(\grave{m}-1)\rho(\partial_4,\partial_5)+\grave{\mu}(\grave{k}-1)(\grave{m}-1)\eta(\partial_4)\eta(\partial_5)]=0.\end{aligned} \tag{4.25}$$

Now we can say if $\grave{k}=1$, then $\tilde{\hbar}=0$. Thus the manifold is a Sasakian manifold. If $\grave{k}\neq 1$, then we have

$$\begin{aligned}R_{cur}(\partial_4,\partial_5)\;=\;&\left[(\grave{m}+1)\grave{k}+\frac{2\mu^2(\grave{m}-1)(\grave{m}+4)}{\grave{m}+8}\right]\rho(\partial_4,\partial_5)\\&+\left[-\frac{2\mu^2(\grave{m}-1)(\grave{m}+4)}{\grave{m}+8}-\grave{k}\right]\eta(\partial_4)\eta(\partial_5)\\&+\left[(\grave{m}+1)\grave{\mu}-\frac{2\mu k(\grave{m}+4)}{(\grave{m}+8)(1-\grave{k})}\right]\rho(\tilde{\hbar}\partial_4,\partial_5).\end{aligned} \tag{4.26}$$

Using (2.11) in (4.26), we get

$$R_{cur}(\partial_4,\partial_5)=\left[\frac{A_1-A_3A_5}{1-A_3A_4}\right]\rho(\partial_4,\partial_5)+\left[\frac{A_2-A_3A_6}{A_3A_4-1}\right]\eta(\partial_4)\eta(\partial_5),$$

where

$$A_1 = (\grave{m}+1)\grave{k} + \frac{2\mu^2(\grave{m}-1)(\grave{m}+4)}{\grave{m}+8}, \qquad A_2 = \grave{k} + \frac{2\mu^2(\grave{m}-1)(\grave{m}+4)}{\grave{m}+8},$$
$$A_3 = (\grave{m}+1)\grave{\mu} - \frac{2\mu\grave{k}(\grave{m}+4)}{(1-\grave{k})(\grave{m}+8)}, \qquad A_4 = \frac{1}{\grave{m}-2+\grave{\mu}},$$
$$A_5 = \frac{\grave{m}-2-\frac{\grave{m}}{2}\grave{\mu}}{\grave{m}-2+\grave{\mu}}, \qquad A_6 = \frac{2-\grave{m}+\grave{m}\grave{k}+\frac{\grave{m}}{2}\grave{\mu}}{\grave{m}-2+\grave{\mu}}.$$

Then the manifold is an η-Einstein manifold. Hence we can say the following:

Theorem 4.3. *Let G^{2n+1} be an $(\grave{k}, \grave{\mu})$-contact metric manifold satisfying $B^{GE}(\partial_1, \partial_2) \cdot R = 0$. Then G^{2n+1} is a Sasakian manifold or is an η-Einstein manifold.*

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