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Semi MV-Algebras

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Özet. Bu makalede bulanık nokta MV-cebiri ve bulanık nokta MV-idealleri kavramlarını ortaya atıyor ve bunlarla MV-cebiri idealleri arasındaki bağlantıyı ele alıyoruz. Ayrıca iki bulanık nokta MV-cebirinin çarpımını inceliyoruz.[†]

Anahtar Kelimeler. MV-cebiri, bulanık nokta MV-cebiri, bulanık nokta idealleri, yarı cebir, bulanık idealler.

Abstract. In this paper we introduced the notions of fuzzy point MV-algebra and fuzzy point MV-ideals and discuss the relationship between them and the ideals of MV-algebra. Also we study the product of two fuzzy point MV-algebras.

Keywords. MV-algebra, fuzzy point MV-algebra, fuzzy point ideals, semi algebra, fuzzy ideals.

1. Preliminaries

MV-algebras are introduced by C. C. Chang in 1958 to provide an algebraic proof of completeness theorem of infinite valued Lukasewicz propositional calculus [2].

The concept of fuzzy sets was first initiated by Zadeh. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. The standard membership degrees of fuzzy sets, i.e. the real number from the unit interval, can in a natural way be understood as the truth degrees of an infinite valued logic, with [0,1] as its truth degree set. Pu and Liu [6] proposed a definition of a fuzzy point belonging to fuzzy subset under a natural equivalence on fuzzy subset.

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In this paper by using the notion of "fuzzy point" on an MV-algebra, the concept of "semi MV-algebra" is introduced and some properties of it are discussed.

Definition 1.1. [3] An algebra $M = (M, \oplus, ', 0)$ of type $< 2, 1, 0 >$ is called an MV-algebra, if for all $x, y, z \in M$, the following axioms hold:

1. $x \oplus (y \oplus z) = (x \oplus y) \oplus z,$ $(MV1)$
2. $x \oplus y = y \oplus x,$ $(MV2)$
3. $x \oplus 0 = x = 0 \oplus x,$ $(MV3)$
4. $(x')' = x,$ $(MV4)$
5. $x \oplus 0' = 0',$ $(MV5)$
6. $(x' \oplus y)' \oplus y = (x \oplus y')' \oplus x.$ $(MV6)$

Note. From now on, in this paper M denotes an MV-algebra and $0' = 1$.

Definition 1.2. [3] A nonempty subset of M is called an ideal if:

1. $0 \in M,$
2. if $x, y \in M$, then $x \oplus y \in M,$
3. if $y \in M$ and $x \ll y$, then $x \in M$, where $x \ll y \Leftrightarrow x' \oplus y = 1.$

Definition 1.3. [3] M is called "simple" if it has no ideal except $\{0\}$ and itself.

Definition 1.4. [3] A nonempty subset S of M is called a subalgebra of M if:

1. if $x \in S$, then $x' \in S,$
2. if $x, y \in S$, then $x \oplus y \in S.$

Definition 1.5. A fuzzy subset μ of M is called a "fuzzy subalgebra" of M if:

1. $\mu(x') \geq \mu(x),$
2. $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\}, \quad \forall x, y \in M.$

Definition 1.6. Let $(M, \oplus, ', 0)$ be a MV-algebra. Then the fuzzy set $\mu : M \to [0, 1]$ is called a "fuzzy MV-ideal" of M if:

1. $\mu(0) \geq \mu(x),$
2. $\mu(x) \geq \min\{\mu(x \oplus y), \mu(y)\}, \quad \forall x, y \in M.$

Definition 1.7. [6] A fuzzy set of a set X is called "fuzzy point" if it takes value 0 for all $y \in X$ except one, say $x \in X$. If it's value at x is t, $0 < t \leq 1$, then we denote the fuzzy point by x_t and the set of all fuzzy points of X by $FP(X)$.

2. Fuzzy Point MV-Algebras

Definition 2.1. Let $(M, \oplus, ', 0)$ be an MV-algebra. We define the operations "$\odot$" and "$*$" by:

$$\odot : FP(M) \times FP(M) \longrightarrow FP(M)$$
$$(x_\alpha, y_\beta) \mapsto (x \oplus y)_{\min\{\alpha,\beta\}}$$

for all $x_\alpha, y_\beta \in FP(M)$, and

$$* : FP(M) \longrightarrow FP(M)$$
$$(x_\alpha)^* = (x')_\alpha.$$

Definition 2.2. Let X be any set. For any $\alpha \in (0, 1]$, we define the set $FP_\alpha(X)$ as follows:

$$FP_\alpha(X) = \{x_\alpha \ : \ x \in X\}.$$

Proposition 2.3. *$(FP_\alpha(M), \odot, *, 0_\alpha)$, $\alpha \in (0, 1]$ is a MV-algebra, called "fuzzy point MV-algebra" of M.*

Proof. We only prove $(MV6)$. The other parts are clear:

$$\begin{aligned}
(x^*_\alpha \odot y_\alpha)^* \odot y_\alpha &= ((x')_\alpha \odot y_\alpha)^* \odot y_\alpha \\
&= ((x' \oplus y)_\alpha)^* \odot y_\alpha \\
&= ((x' \oplus y)')_\alpha \odot y_\alpha \\
&= ((x' \oplus y)' \oplus y)_\alpha \\
&= ((x \oplus y')' \oplus x)_\alpha \\
&= ((x \oplus y')_\alpha)^* \odot x_\alpha \\
&= (x_\alpha \odot y^*_\alpha)^* \odot x_\alpha.
\end{aligned}$$

□

Proposition 2.4. *For any $x_\alpha, y_\beta, z_\gamma \in FP(M)$ we have the following:*

1. $x_\alpha \odot (y_\beta \odot z_\gamma) = (x_\alpha \odot y_\beta) \odot z_\gamma$,
2. $x_\alpha \odot y_\beta = y_\beta \odot x_\alpha$,
3. $(x^*_\alpha)^* = x_\alpha$,
4. $(x^*_\alpha \odot y_\beta)^* \odot y_\beta = (x_\alpha \odot y^*_\beta)^* \odot x_\alpha$.

Proof. The parts 1-3 are obvious. For part (4) we see that:

$$\begin{aligned}(x^*_\alpha \odot y_\beta)^* \odot y_\beta &= ((x')_\alpha \odot y_\beta)^* \odot y_\beta \\ &= [(x' \oplus y)_{\min\{\alpha,\beta\}}]^* \odot y_\beta \\ &= (x' \oplus y)'_{\min\{\alpha,\beta\}} \oplus y_\beta \\ &= [(x' \oplus y)' \oplus y]_{\min\{\alpha,\beta\}}. \qquad (1)\end{aligned}$$

On the other hand by similar way we can see that:

$$(x_\alpha \odot y^*_\beta)^* \odot x_\alpha = [(x \oplus y')' \oplus x]_{\min\{\alpha,\beta\}}.$$

By definition of a MV-algebra this is equivalent to (1). □

Note. We see that in $FP(M)$ if $\alpha < \beta$, then $x_\beta \odot 0_\alpha = x_\alpha \neq x_\beta$ and $x_\alpha \odot 1_\beta = 1_\alpha \neq 1_\beta$, so the conditions $(MV3)$ and $(MV5)$ do not always hold. Hence by Proposition 2.4, we say that $FP(M)$ is a "semi-MV-algebra".

Definition 2.5. A nonempty subset A of $FP_\alpha(M)$ is called a "fuzzy point subalgebra" of $FP_\alpha(M)$ if:

1. $x_\alpha \odot y_\alpha \in A$,
2. $(x_\alpha)^* \in A$,

for all $x_\alpha, y_\alpha \in A$.

Definition 2.6. A nonempty subset I_α of $FP_\alpha(M)$ is called a "fuzzy point ideal" of $FP_\alpha(M)$ if:

1. $0_\alpha \in I_\alpha$,
2. if $y_\alpha \in I_\alpha$, $x_\alpha \in FP_\alpha(M)$ and $x_\alpha \ll y_\alpha$, then $x_\alpha \in I_\alpha$,
3. if $x_\alpha, y_\alpha \in I_\alpha$, then $x_\alpha \odot y_\alpha \in I_\alpha$,

where $x_\alpha \ll y_\alpha \Leftrightarrow x^*_\alpha \odot y_\alpha = 1$.

Example 2.7. Let $M = [0, 1]$. Define:

$$x \oplus y = \min\{1, x + y\}, \quad x' = 1 - x \quad \text{for all } x, y \in M.$$

Then $(M, \oplus, ', 0)$ is a MV-algebra and the set $\{0_\alpha, 0.5_\alpha, 1_\alpha\}$ is a fuzzy point subalgebra of $FP_\alpha(M)$.

Example 2.8. Let $M = \{0, a, b, 1\}$. Consider the following operations:

$\oplus$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$'$	0
0	0
a	a
b	b
c	c

Then $(M, \oplus, ')$ is a MV-algebra and $\{0_\alpha, a_\alpha\}$ is a fuzzy point MV-ideal of $FP_\alpha(M)$ for all $\alpha \in (0, 1]$.

Lemma 2.9. *Let $x, y \in M$. Then $x \ll y \Leftrightarrow x_\alpha \ll y_\alpha$.*

Definition 2.10. Let $(FP_\alpha(M), \odot, *, 0)$ be a fuzzy point MV-algebra of MV-algebra $(M, \oplus, '0)$. Then for all $x_\alpha, y_\alpha \in FP_\alpha(M)$, we define $x_\alpha \otimes y_\alpha = (x^*_\alpha \odot y^*_\alpha)^*$.

Proposition 2.11. *Let $(FP_\alpha(M), \odot, *, 0_\alpha)$ be a fuzzy point MV-algebra of MV-algebra $(M, \oplus, ', 0)$. Then the following hold, for all $x_\alpha, y_\alpha \in FP_\alpha(M)$:*

1. $x^*_\alpha \odot x_\alpha = 1_\alpha$,
2. $x_\alpha \otimes x^*_\alpha = 0_\alpha$,
3. $x_\alpha \otimes 0_\alpha = 0_\alpha$,
4. $x_\alpha \otimes 1_\alpha = x_\alpha$,
5. $x_\alpha \odot y_\alpha = 0_\alpha \Leftrightarrow x_\alpha = 0_\alpha$ *and* $y_\alpha = 0_\alpha$,
6. $x_\alpha \otimes y_\alpha = 1 \Leftrightarrow x = 1$ *and* $y = 1$.

Proof. 1. In $(MV6)$ put $y_\alpha = 1_\alpha$, then $(x^*_\alpha \odot 1_\alpha)^* \odot 1_\alpha = (x_\alpha \odot 1^*_\alpha)^* \odot x_\alpha$, hence $(x'_\alpha \odot 1_\alpha)^* \odot 1_\alpha = (x_\alpha \odot 0_\alpha)^* \odot x_\alpha$, so $((x' \oplus 1)_\alpha))^* \odot 1_\alpha = (x \oplus 0)^*_\alpha \odot x_\alpha$, therefore $1_\alpha = x'_\alpha \odot x_\alpha$.

3. $(x_\alpha \otimes 0_\alpha) = (x^*_\alpha \odot 0^*_\alpha)^* = (x'_\alpha \odot 1_\alpha)^* = 1^*_\alpha = 0_\alpha$.

4. $(x_\alpha \otimes 1_\alpha) = (x^*_\alpha \odot 1^*_\alpha)^* = (x^*_\alpha \odot 0_\alpha)^* = (x^*_\alpha)^* = x_\alpha$.

5. Suppose that $x_\alpha \odot y_\alpha = 0_\alpha$, then $x^*_\alpha = x^*_\alpha \odot (x_\alpha \odot y_\alpha) = (x^*_\alpha \odot x_\alpha) \odot y_\alpha$, hence by part (1) $x^*_\alpha = 1_\alpha \odot y_\alpha = 1_\alpha$. So $x_\alpha = 0_\alpha$, similarly, $y_\alpha = 0_\alpha$.

6. If $x_\alpha \otimes y_\alpha = 1$, then by parts (2),(3),(4) we see that:

$$x^*_\alpha = (x^*_\alpha \otimes 1_\alpha) = (x^*_\alpha \otimes (x_\alpha \otimes y_\alpha)) = ((x^*_\alpha \otimes x_\alpha) \otimes y_\alpha) = 0_\alpha \otimes y_\alpha = 0_\alpha.$$

Hence $x_\alpha = 1_\alpha$. By similar way we can prove that $y_\alpha = 1_\alpha$. □

Proposition 2.12. *Let $(M, \oplus, ', 0)$ be a MV-algebra. Then $(FP(M), \odot)$ is an semi group.*

Proposition 2.13. *Let $(M, \oplus, ', 0)$ be a MV-algebra and $(FP(M), \odot, *, 0_\alpha)$ be its fuzzy point MV-algebra for all $\alpha \in (0, 1]$. Then:*

1. *T is subalgebra of $M \Leftrightarrow T_\alpha$ is a fuzzy point subalgebra of $FP_\alpha(M)$,*
2. *I is an ideal of $M \Leftrightarrow I_\alpha$ is a fuzzy point ideal of $FP_\alpha(M)$.*

Proof. 1. Let $x_\alpha, y_\alpha \in T_\alpha$ and T be a subalgebra of M. We know that $x_\alpha \odot y_\alpha = (x \oplus y)_\alpha$. But $x \oplus y \in T$, hence $(x \oplus y)_\alpha \in T_\alpha$ i.e. $x_\alpha \odot y_\alpha \in T$.

Now let $x_\alpha \in T_\alpha$. Then $x \in T$, hence $x' \in T$, so $(x')_\alpha = (x_\alpha)^* \in T_\alpha$.

Thus T_α is a subalgebra of $FP_\alpha(M)$.

Conversely, assume that T_α is a subalgebra of $FP_\alpha(M)$. Then $x \in T$ so $x_\alpha \in T_\alpha$, hence $x^*_\alpha \in T_\alpha$, and this means that $(x')_\alpha \in T_\alpha$. Therefore $x' \in T$.

Now, let $x, y \in T$. Then $x_\alpha, y_\alpha \in T_\alpha$ so $x_\alpha \odot y_\alpha \in T_\alpha$, hence $(x \oplus y)_\alpha \in T_\alpha$, thus $x \oplus y \in T$.

2. By Lemma 2.9 and similar to part 1. we can prove this part. □

Definition 2.14. Let μ be a fuzzy set in MV-algebra M and $\alpha \in (0, 1]$. Define:

$$FP_\alpha(\mu) := \{x_\alpha \ : \ \mu(x) \geq \alpha\} \ \text{ and } \ FP(\mu) = \bigcup_{\alpha \in (0,1]} FP_\alpha(\mu).$$

Note. $FP_\alpha(\mu)$ always is not a fuzzy point subalgebra of $FP_\alpha(M)$, for see this let M be a as Example 2.7 and μ be as follows:

$$\mu(x) = \begin{cases} 0.5 & \text{if } x = 0.6, \\ 0 & \text{otherwise }, \end{cases}$$

then $FP_{0.3}(\mu) = \{0.6_{0.3}\}$, but $0.6_{0.3} \odot 0.6_{0.3} = 1_{0.3} \notin FP_{0.3}(\mu)$.

Proposition 2.15. *Let μ be a fuzzy subalgebra of MV-algebra M and $0_\alpha \in FP_\alpha(\mu)$ for any $\alpha \in (0, 1]$. Then $FP_\alpha(\mu)$ is an subalgebra of $FP_\alpha(M)$.*

Proof. Let $x_\alpha \in FP_\alpha(\mu)$. Then $\mu(x') \geq \mu(x) \geq \alpha$. Hence $(x')_\alpha \in FP_\alpha(\mu)$, that is, $x^*_\alpha \in FP_\alpha(\mu)$.

Now let $x_\alpha, y_\alpha \in FP_\alpha(\mu)$. Then $\mu(x \oplus y) \geq \min\{\mu(x), \mu(y)\} \geq \alpha$, hence $(x \oplus y)_\alpha \in FP_\alpha(\mu)$, and this means that $x_\alpha \odot y_\alpha \in FP_\alpha(\mu)$. □

Lemma 2.16. *Let μ be a fuzzy subalgebra of M. Then $\mu(x') = \mu(x)$, for all $x \in M$.*

Proof. By definition $\mu(x') \geq \mu(x)$, then $\mu((x')') \geq \mu(x')$, so $\mu(x) \geq \mu(x')$, hence $\mu(x) = \mu(x')$. □

Definition 2.17. In M, define:

$S(FP(M)) = \{x_\alpha \in FP(M) : \ x_\alpha \odot x_\alpha = 0_\alpha\}$ and $S(M) = \{x \in M : \ x \oplus x = 0\}$.

Proposition 2.18. *Let $I \subseteq FP(M)$, for all $\alpha, \beta, \gamma \in (0, 1]$. Then*

$$x_\alpha \odot y_\beta \in I \ \Leftrightarrow \ x_\gamma \odot y_\gamma \in I.$$

Proof. Let $x_\gamma \odot y_\gamma \in I$, for all $\gamma \in (0, 1]$. Then:

$$x_\alpha \odot y_\beta = (x \oplus y)_{\min\{\alpha,\beta\}} = x_{\min\{\alpha,\beta\}} \odot y_{\min\{\alpha.\beta\}} \in I.$$

The converse is clear. □

Proposition 2.19. *Let μ be a fuzzy set in M. Then*

1. *if $x \in S(M)$, then $x_\alpha \in S(FP(M))$, for all $x \in M$, $\alpha \in (0, 1]$,*
2. *if $x_\alpha \in S(FP(M))$ for some $\alpha \in (0, 1]$, then $x \in S(M)$,*
3. *if $x_\alpha \in S(FP(M))$ for $\alpha \in (0, 1]$, then $x_\beta \in S(FP(M))$, for all $\beta \in (0, 1]$,*
4. *$S(FP(M)) \subseteq FP(\mu) \Leftrightarrow \mu(x) = 1$, for all $x \in S(M)$,*
5. *$FP(\mu) \subseteq S(FP(M)) \Leftrightarrow FP(M) = S(FP(M))$.*

Proof. 1. Let $x \in S(M)$ and $\alpha \in (0, 1]$. Then $x_\alpha \odot x_\alpha = (x \oplus x)_\alpha = 0_\alpha$, hence $x_\alpha \in S(FP(M))$.

2. If $\alpha \in (0, 1]$ and $x_\alpha \in S(FP(M))$, then $(x \oplus x)_\alpha = x_\alpha \odot x_\alpha = 0_\alpha$, so $x \oplus x = 0$, hence $x \in S(M)$.

3. Let $x_\alpha \in S(FP(M))$, for $\alpha \in (0, 1]$. Then by 2., $x \in S(M)$, by (1.), $x_\beta \in S(FP(M))$, for all $\beta \in (0, 1]$.

4. If $x \in S(M)$, thus $x_\alpha \in S(FP(M)) \subseteq FP(\mu)$ $\forall \alpha \in (0, 1]$, hence $\mu(x) = 1$.

Conversely, let $x_\alpha \in S(FP(M))$ and $\mu(x) = 1$, for some $x \in S(M)$. Then $\mu(x) = 1 \geq \alpha$, $\forall \alpha \in (0, 1]$, hence $x_\alpha \in FP_\alpha(\mu) \subseteq FP(\mu)$.

5. If $x_\alpha \in FP(M)$ and $\mu(x) = \beta$, $\beta \in (0, 1]$, then $x_\beta \in FP(\mu)$, by hypothesis $x_\beta \in S(FP(M)))$; and by part (3) this implies that $x_\alpha \in S(FP(M))$. Hence $FP(M) \subseteq S(FP(M))$, so $FP(M) = S(FP(M))$.

The converse is clear. □

Corollary 2.20. *M is simple if and only if $FP_\alpha(M)$ is simple.*

Definition 2.21. Let X, Y be any sets, $\alpha \in (0,1]$ and $f : X \to Y$ be a function. Then we define $f^\alpha(x)$ by $f^\alpha(x) = [f(x)]_\alpha$ and we call it "fuzzy point pre-image" of f at α.

Lemma 2.22. *Let $f : M_1 \to M_2$ be a homomorphism of MV-algebras. Then $f(1) = 1$.*

Proof. $f(1) = f(0') = (f(0))' = 0' = 1$. □

Proposition 2.23. *Let $f : M_1 \to M_2$ be an onto homomorphism of MV-algebras. Then:*

1. *if J is an ideal of M_2, then $(f^{-1})^\alpha(J)$ is an ideal of $FP_\alpha(M_1)$ for all $\alpha \in (0,1]$,*
2. *if I is an ideal of M_1, then $f^\alpha(I)$ is an ideal of $FP_\alpha(M_2)$ for all $\alpha \in (0,1]$.*

Proof. 1. First we see that $0_\alpha \in (f^{-1})^\alpha(0) \subseteq (f^{-1})^\alpha(J)$. Now if $y_\alpha \in (f^{-1})^\alpha(J)$ and $x_\alpha \in FP(M_1)$ such that $x_\alpha \ll y_\alpha$, so $x \ll y$ and $y \in f^{-1}(J)$, hence $f(y) \in J$.
But $x \ll y$ means that $x' \oplus y = 1$, hence $f(x)' \oplus f(y) = f(x' \oplus y) = 1$ so $f(x) \ll f(y)$. Therefore $f(x) \in J$, and this implies that $x \in f^{-1}(J)$, hence $x_\alpha \in (f^{-1})^\alpha(J)$.

2. Since $0 \in I$, we have $0_\alpha \in f^\alpha(I)$. Now, let $x_\alpha \in FP_\alpha(M_2)$, $y_\alpha \in f^\alpha(I)$ such that $x_\alpha \ll y_\alpha$. Then $y_\alpha \in f^\alpha(I)$ implies that $y \in f(I)$. So there exists a $z \in M_1$ such that $f(z) = y$. Since $x_\alpha \in FP_\alpha(M_2)$ and f is onto, we have $x = f(s)$, for some $s \in M_1$. Now,

$$f(s' \oplus z) = f(s)' \oplus f(z) = x' \oplus y = 1 = f(1).$$

So $s' \oplus z = 1$, hence $s \ll z$. Thus $s \in I$, so $x_\alpha = f^\alpha(s) \in f^\alpha(I)$. □

Proposition 2.24. *Let μ be a fuzzy set of M. Then $FP(\mu)$ is a semi-subalgebra of $FP(M)$ if and only if $FP_\alpha(\mu)$ is a fuzzy point MV-subalgebra of $FP_\alpha(M)$, for all $\alpha \in (0,1]$.*

Proof. If $FP(\mu)$ is a semi-subalgebra of $FP(M)$ and $x_\alpha, y_\alpha \in FP_\alpha(\mu)$, hence $x_\alpha, y_\alpha \in FP(\mu)$ therefore $x_\alpha \odot y_\alpha \in FP(\mu)$ i.e. $(x \oplus y)_\alpha \in FP(\mu) = \bigcup_{\alpha \in (0,1]} FP_\alpha(\mu)$, thus $x_\alpha \oplus y_\alpha \in FP_\alpha(\mu)$.

On the other hand, let $x_\alpha \in FP_\alpha(\mu) \subseteq FP(\mu)$. Hence:

$$x^*_\alpha \in FP(\mu) = \bigcup_{\alpha \in (0,1]} FP_\alpha(\mu) \Rightarrow x^*_\alpha \in FP_\alpha(\mu).$$

Conversely, assume that $FP_\alpha(\mu)$ is a fuzzy point subalgebra of semi-MV-algebra $FP_\alpha(M)$ and $x_\alpha, y_\beta \in FP(\mu)$ hence

$$\begin{cases} \mu(x) \geq \alpha \geq \min\{\alpha, \beta\} = \gamma, \\ \mu(y) \geq \beta \geq \min\{\alpha, \beta\} = \gamma. \end{cases}$$

Thus by hypothesis $(x \oplus y)_\gamma = x_\gamma \odot y_\gamma \in FP(\mu)$. Therefore:

$$x_\alpha \odot y_\beta = (x \oplus y)_{\gamma=\min\{\alpha,\beta\}} \in F_\gamma(\mu) \subseteq FP(\mu).$$

Now, let $x_\alpha \in FP(\mu)$. Then $x_\alpha \in FP_\alpha(\mu)$, hence $x^*_\alpha \in FP_\alpha(\mu)$ thus $x^*_\alpha \in FP(\mu)$. □

Lemma 2.25. *Let μ be a fuzzy ideal of M. Then $\mu(x) \geq \mu(1) \quad \forall x \in M$.*

Proposition 2.26. *μ is a fuzzy ideal of M if and only if*

1. *$\mu(0) \geq \mu(x)$, for all $x \in M$,*
2. *$0_\alpha \in FP_\alpha(\mu)$, for all $\alpha \in Im(\mu)$,*
3. *$x_\alpha \odot y_\beta \in FP(\mu)$ and $y_\beta \in FP(\mu)$ implies that $x_{\min\{\alpha,\beta\}} \in FP(\mu)$, for all $\alpha, \beta \in (0, 1]$.*

Proof. 1. If μ is a fuzzy ideal of M, then by definition is clear.

2. By definition we have $\mu(0) \geq \mu(x)$, for all $x \in M$, if $\alpha \in Im(\mu)$ and $\mu(x) = \alpha$, then $\mu(0) \geq \mu(x) = \alpha$, hence $0_\alpha \in FP_\alpha(\mu)$.

3. If $x_\alpha \odot y_\beta \in FP(\mu)$ and $y_\beta \in FP(\mu)$, then $\mu(y) \geq \beta$.

Since $(x \oplus y)_{\min\{\alpha,\beta\}} = x_\alpha \odot y_\beta \in FP(\mu)$, we have $\mu(x \oplus y) \geq \min\{\alpha, \beta\}$. Hence $\mu(x) \geq \min\{\mu(x \oplus y), \mu(y)\} = \min\{\alpha, \beta\}$, so $x_{\min\{\alpha,\beta\}} \in FP(\mu)$.

Conversely, let $\min\{\mu(x \oplus y), \mu(y)\} = \alpha$. Then $\mu(y) \geq \alpha$ and $\mu(x \oplus y) \geq \alpha$, hence:

$$y_\alpha \in FP(\mu) \quad \text{and} \quad x_\alpha \odot y_\alpha \in FP(\mu).$$

By part (2.) this implies that $x_\alpha \in FP(\mu)$. And this implies that $\mu(x) \geq \alpha = \min\{\mu(x \oplus y), \mu(y)\}$. □

Definition 2.27. Let X be a any nonempty set. We define:

$$FP^t(X) = \{x_\alpha : \ \alpha \geq t, \ x \in X, \ \alpha, t \in (0, 1]\}$$

and

$$FP^t_\alpha(M) = \{x_\alpha : \ t \ll x, \ x, t \in M\}$$

for any MV-algebra M. We call $FP^t_\alpha(M)$ as t-cut of $FP_\alpha(M)$.

Lemma 2.28. $FP^0(M) = \bigcup_{\alpha\in(0,1]} FP^0_\alpha(M)$.

Proof. By definition we see that $FP^0(M) = FP(\mu)$ and $FP^0_\alpha(M) = FP_\alpha(M)$, hence the proof is clear. □

Proposition 2.29. *$FP^t(M)$ is a semi-subalgebra of semi-MV-algebra $FP(M)$ for any $t \in (0,1]$.*

Proof. 1. If $x_\alpha \in FP^t(M)$, then it is clear that $x^*_\alpha \in FP^t(M)$.

2. If $x_\alpha, y_\beta \in FP^t(M)$. Then $x_\alpha \odot y_\beta = (x \oplus y)_{\min\{\alpha,\beta\}} \in FP^t(M)$. □

Proposition 2.30. *I is an ideal of M if and only if $FP^t(I)$ is an ideal of $FP^t(M)$, for all $t \in (0,1]$.*

Proof. Let I be an ideal of M, then we have $0_t \in FP^t(I)$. Now, let $y_t \in FP^t(I)$, $x_t \in FP_t(M)$ and $x_t \ll y_t$. Then by Lemma 2.9 $x \ll y$. We have $y \in I$. So $x \in I$, hence $x_t \in FP^t(I)$.

Now, let $x_t, y_t \in FP^t(I)$, then $x, y \in I$, so $x \oplus y \in I$. Hence $x_t \odot y_t \in FP^t(I)$.

The converse can prove by similar way. □

Proposition 2.31. *The nonempty subset S of M is a subalgebra of M if and only if $FP^t(S)$ is a semi-subalgebra of $FP(M)$, for all $t \in (0,1]$.*

Proof. Let $x_\alpha \in FP^t(S)$, so $x \in S$, then $x' \in S$, hence $x^*_\alpha \in FP^t(S)$.

Now, let $x_\alpha, y_\beta \in FP^t(S)$. Then $x, y \in S$, so $x \oplus y \in S$, thus

$$x_\alpha \odot y_\beta = (x \oplus y)_{\min\{\alpha,\beta\}} \in FP^t(S).$$

By similar way we can prove the converse. □

3. Product of Fuzzy Point MV-Algebras

Definition 3.1. Let $(M_1, \ominus_1, *_1, 0_1)$ and $(M_2, \ominus_2, *_2, 0_2)$ be two MV-algebras, $M = M_1 \times M_2$ and $(FP_\alpha(M_1), \oplus_1, *)$, $(FP_\beta(M_2), \oplus_2, \diamond)$ be two fuzzy point MV-algebras for $\alpha, \beta \in (0,1]$. Then $FP_{(\alpha,\beta)}(M) = (FP_\alpha(M_1) \times FP_\beta(M_2), \oplus, \prime)$ is a fuzzy point MV-algebra, which called product of fuzzy point MV-algebras $FP_\alpha(M_1)$ and $FP_\beta(M_2)$, with the following operations:

1. $(x_\alpha, y_\beta) \oplus (r_\alpha, s_\beta) = ((x_\alpha \oplus_1 r_\alpha), (y_\beta \oplus_2 s_\beta))$,

2. $(x_\alpha, y_\beta)' = (x^*_\alpha, y^\diamond_\beta)$,
3. $0 = ((0_1)_\alpha, (0_2)_\beta)$,
4. $(x_\alpha, y_\beta) \ll (r_\alpha, s_\beta) \Leftrightarrow x_\alpha \ll r_\alpha\ ,\ y_\beta \ll s_\beta$.

Note. From now on M_1, M_2 are two MV-algebras and $M = M_1 \times M_2$.

Proposition 3.2. *$(FP_{(\alpha,\beta)}(M), \oplus, \prime)$ which is defined in Definition 3.1, is an MV-algebra.*

Proof. We prove $(MV1)$ and $(MV6)$, the other conditions are clear.

$$\begin{aligned}(MV1):\quad (x_\alpha, y_\beta) \oplus ((r_\alpha, s_\beta) \oplus (p_\alpha, q_\beta)) &= (x_\alpha, y_\beta) \oplus ((r_\alpha \oplus_1 p_\alpha), (s_\beta \oplus_2 q_\beta)) \\ &= (x_\alpha \oplus_1 (r_\alpha \oplus_1 p_\alpha), y_\beta \oplus_2 (s_\beta \oplus_2 q_\beta)) \\ &= ((x_\alpha \oplus_1 r_\alpha) \oplus_1 p_\alpha, (y_\beta \oplus_2 s_\beta) \oplus_2 q_\beta) \\ &= ((x_\alpha \oplus_1 r_\alpha), (y_\beta \oplus_2 s_\beta)) \oplus (r_\alpha, q_\beta) \\ &= ((x_\alpha, y_\beta) \oplus (r_\alpha, s_\beta)) \oplus (p_\alpha, q_\beta).\end{aligned}$$

$$\begin{aligned}(MV6):\quad ((x_\alpha, y_\beta)' \oplus (r_\alpha, s_\beta))' \oplus (r_\alpha, s_\beta) &= ((x^*_\alpha, y^\diamond_\beta) \oplus (r_\alpha, s_\beta))' \oplus (r_\alpha, s_\beta) \\ &= ((x^*_\alpha \oplus_1 r_\alpha), (y^\diamond_\beta \oplus_2 s_\beta))' \oplus (r_\alpha, s_\beta) \\ &= ((x^*_\alpha \oplus_1 r_\alpha)^*, (y_\beta \oplus_2 s_\beta)^\diamond) \oplus (r_\alpha, s_\beta) \\ &= ((x^*_\alpha \oplus_1 r_\alpha)^* \oplus_1 r_\alpha, (y^\diamond_\beta \oplus_2 s_\beta)^\diamond \oplus_2 s_\beta) \\ &= ((x_\alpha \oplus_1 r^*_\alpha)^* \oplus_1 x_\alpha, (y_\beta \oplus_2 s^\diamond_\beta)^\diamond \oplus_2 y_\beta) \\ &= ((x_\alpha \oplus_1 r^*_\alpha)^*, (y_\beta \oplus_2 s^\diamond_\beta)^\diamond) \oplus (x_\alpha, y_\beta) \\ &= ((x_\alpha \oplus_1 r^*_\alpha), (y_\beta \oplus_2 s^\diamond_\beta))^\diamond \oplus (x_\alpha, y_\beta) \\ &= ((x_\alpha, y_\beta) \oplus (r^*_\alpha, s^\diamond_\beta))' \oplus (x_\alpha, y_\beta) \\ &= ((x_\alpha, y_\beta) \oplus (r_\alpha, s_\beta)')' \oplus (x_\alpha, y_\beta).\end{aligned}$$

□

Proposition 3.3. *Let $M = M_1 \times M_2$, $(FP_\alpha(M_1), \oplus_1, *)$ and $(FP_\alpha(M_2), \oplus_2, \diamond)$ be two fuzzy point MV-algebras. Then $K = I_\alpha \times J_\beta$ is an ideal of $(FP_{(\alpha,\beta)}, \oplus, \prime)$ if and only if I_α, J_β are MV-ideals of FP_α, FP_β, respectively, for $\alpha, \beta \in (0, 1]$.*

Proof. Let K be an ideal of $FP_{\alpha,\beta}(M)$. Then since $((0_1)_\alpha, (0_2)_\beta) = 0 \in K$, we have $(0_1)_\alpha \in I_\alpha$.

Suppose that $x_\alpha, y_\alpha \in I_\alpha$, then $(x_\alpha, (0_2)_\beta), (y_\alpha, (0_2)_\beta) \in K$, hence $(x_\alpha \oplus_1 y_\alpha, (0_2)_\beta) \in K$. Therefore $x_\alpha \oplus_1 y_\alpha \in I_\alpha$.

Now, let $y_\alpha \in I_\alpha$ and $x_\alpha \ll y_\alpha$. Then $(x_\alpha, (0_2)_\beta) \ll (y_\alpha, (0_2)_\beta)$ and $(y_\alpha, (0_2)_\beta) \in K$. Therefore $(x_\alpha, (0_2)_\beta) \in K$, hence $x_\alpha \in I_\alpha$, thus I_α is an MV-ideal of $FP_\alpha(M_1)$.

By similar way we can prove that J_β is an MV-ideal of $FP_\beta(M_2)$.

Conversely, let I_α and J_β be two MV-ideals of $FP_\alpha(M_1), FP_\beta(M_2)$. Then $0 = ((0_1)_\alpha, (0_2)_\beta) \in I_\alpha \times J_\beta = K$. Let $(x_\alpha, y_\beta), (r_\alpha, s_\beta) \in K$, then $x_\alpha, r_\alpha \in I_\alpha$ and $y_\beta, s_\beta \in J_\beta$, hence $x_\alpha \oplus_1 r_\alpha \in I_\alpha$ and $y_\beta \oplus_2 s_\beta \in J_\beta$. So:

$$(x_\alpha, y_\beta) \oplus (r_\alpha, s_\beta) = (x_\alpha \oplus_1 r_\alpha, y_\beta \oplus_2 s_\beta) \in I_\alpha \oplus J_\beta.$$

Now, let $(r_\alpha, s_\beta) \in K$ and $(x_\alpha, y_\beta) \in M$ such that $(x_\alpha, y_\beta) \ll (r_\alpha, s_\beta)$. Then $x_\alpha \ll r_\alpha \in I_\alpha$ and $y_\beta \ll s_\beta \in J_\beta$. So $x_\alpha \in I_\alpha$ and $y_\beta \in J_\beta$. Thus $(x_\alpha, y_\beta) \in I_\alpha \times J_\beta = K$, so K is an MV-ideal of M. □

Definition 3.4. Let μ, υ be two fuzzy subset of M_1, M_2 respectively. Then the product of μ, υ is defined by:

$$\mu \times \upsilon(x, y) = \min\{\mu(x), \upsilon(y)\}, \quad \forall (x, y) \in M_1 \times M_2.$$

Proposition 3.5. *Let μ, υ be two fuzzy ideals of M_1, M_2 respectively. Then $\mu \times \upsilon$ is a fuzzy ideal of $M_1 \times M_2$.*

Proof. Let $(x, y), (r, s) \in M_1 \times M_2$. Then

$$\begin{aligned}
\mu \times \upsilon(x, y) &= \min\{\mu(x), \upsilon(y)\} \\
&\geq \min\{\min\{\mu(r), \mu(x \oplus_1 r)\}, \min\{\upsilon(s), \upsilon(y \oplus_2 s)\}\} \\
&= \min\{\min\{\mu(r), \upsilon(s)\}, \min\{\mu(x \oplus_1 r), \upsilon(y \oplus_2 s)\}\} \\
&= \min\{\mu \times \upsilon(r, s), \mu \times \upsilon((x \oplus_1 r), (y \oplus_2 s))\} \\
&= \min\{\mu \times \upsilon(r, s), \mu \times \upsilon((x, y) \oplus (r, s))\}.
\end{aligned}$$

□

Proposition 3.6. *There is an isomorphism*

$$f : FP_\alpha(M_1) \times FP_\alpha(M_2) \to FP_\alpha(M)$$

$$(x_\alpha, y_\alpha) \mapsto (x, y)_\alpha$$

where $\alpha \in (0, 1]$.

Proposition 3.7. *The following are equivalence:*

1. I, J *are* MV*-ideals of* M_1, M_2 *respectively;*
2. $I \times J$ *is an* MV*-ideal of* $M_1 \times M_2$*;*
3. $(I \times J)_\alpha$ *is an* MV*-ideal of* $FP_\alpha(M)$*;*
4. I_α, J_α *are* MV*-ideals of* $FP_\alpha(M_1), FP_\alpha(M_2)$ *respectively,*

for all $\alpha \in (0, 1]$.

Proof. $(2 \to 3)$: By Proposition 3.6, is clear.

$(3 \to 4)$: is obvious.

$(1 \to 2)$: Let I and J are MV-ideals of M_1 and M_2 respectively. Then $((0_1)_\alpha, (0_2)_\beta) \in I \times J$.
Suppose that $(x, y), (r, a) \in I \times J$, then $x, r \in I$ and $y, s \in J$. Hence $x \oplus_1 r \in I$ and $y \oplus_2 s \in J$, so $(x \oplus_1 r, y \oplus_2 s) \in I \times J$.

Now, let $(x, y) \ll (r, s)$ where $(r, s) \in I \times J$. Then $x \ll r, y \ll s$ and $r \in I, s \in J$, hence $x \in I$ and $y \in J$, thus $(x, y) \in I \times J$.

$(4 \to 1)$: Let I_α be an ideal of $FP_\alpha(M_1)$. Then $(0_1)_\alpha \in I_\alpha$, hence $0_1 \in I$. Suppose that $x, y \in I$, then $x_\alpha, y_\alpha \in I_\alpha$. So $(x \ominus y)_\alpha = x_\alpha \oplus_1 y_\alpha \in I_\alpha$, therefore $x \ominus y \in I$.

Now, let $x \ll y$ and $y \in I$. Then $x_\alpha \ll y_\alpha$ and $y_\alpha \in I_\alpha$. So $x_\alpha \in I_\alpha$, hence $x \in I$. □

4. Conclusion

In this work we generalized the notion of MV-algebra by using fuzzy points. Also, we obtained some results in fuzzy point MV-algebras.

In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as lattices and Lie algebras. It is our hope that this work would form foundations for further study of the theory of fuzzy MV-algebra.

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