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N-Fractional Calculus Operator N^η Method Applied to a Gegenbauer Differential Equation

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Özet. Kesirli hesap tekniği yardımıyla, Gegenbauer denkleminin açık çözümleri elde edildi. Bu denklemlerin çözümlerini elde etmek için N-kesirli hesap operatörü olarak bilinen N^η metodu kullanıldı.

Anahtar Kelimeler. Kesirli hesap, Gegenbauer denklemi, adi diferensiyel denklem, genelleştirilmiş Leibniz kuralı.

Abstract. By means of fractional calculus techniques, we find explicit solutions of the Gegenbauer equation. We use the N-fractional calculus operator N^η method to derive the solutions of these equations.

Keywords. Fractional calculus, Gegenbauer equation, ordinary differential equation, generalized Leibniz rule.

1. Introduction, Definitions, and Preliminaries

The widely investigated subject of fractional calculus (that is, calculus of derivatives and integrals of any arbitrary real or complex order) has gained considerable importance and popularity during the past three decades or so, due chiefly to its demonstrated applications in numerous seemingly diverse fields of science and engineering (see, for details [1-6]). We can mention that fractional differential equations play an important role in fluid dynamics, traffic models with fractional derivatives, the measurement of viscoelastic material properties, modeling of viscoplasticity, control theory, economy, nuclear magnetic resonance, geometric mechanics, mechanics, optics, signal processing, and so on.

Fractional integration and fractional differentiation are generalizations of notions of integer order integration and differentiation and include kth derivatives and k-fold integrals (k is an integer) as particular cases. Some of the most obvious formulations

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based on the fundamental definitions of Riemann-Liouville fractional integration and fractional differentiation are, respectively [7],

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)} d\tau \quad (t > a,\, \alpha > 0), $$

and

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(k-\alpha)} \left(\frac{d}{dt}\right)^k \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-k+1}} d\tau \quad (k-1 \leq \alpha < k), $$

where $k \in \mathbb{N}$, $\mathbb{N}$ being the set of positive integers, and Γ stands for Euler's gamma function.

Recently, by applying the following definition of a fractional differintegral (that is, a fractional derivative and fractional integral) of the order $\upsilon \in \mathbb{R}$, many authors have explicitly obtained particular solutions of a number of families of homogeneous (as well as non-homogeneous) linear ordinary and partial differintegral equations (see, for details, [8]; see also [9-14]). An important example of Fuchsian differential equations is provided by the celebrated hypergeometric equation (or, more precisely, the Gauss hypergeometric equation)

$$ z(1-z)\frac{d^2u}{dz^2} + [\gamma - (\alpha+\beta+1)z]\frac{du}{dz} - \alpha\beta u = 0, $$

whose study can be traced back to L. Euler, C. F. Gauss and E. E. Kummer. On the other hand, a special limit (confluent) case of the Gauss hypergeometric equation, in the form [15]

$$ \frac{d^2u}{dz^2} + \left(-\frac{1}{4} + \frac{\varkappa}{z} - \frac{\ell(\ell+1)}{z^2}\right) u = 0, \quad \eta = \ell + \frac{1}{2} $$

is referred to as the Whittaker equation whose systematic study was initiated by E. T. Whittaker.

Other classes of non-Fuchsian differential equations which we shall consider in this investigation include the so-called Fukuhara equation [16]

$$ z^2\frac{d^2u}{dz^2} + z\frac{du}{dz} - \left(1 - z + z^2\right) u = 0, $$

the Tricomi equation [17]

$$ \frac{d^2u}{dz^2} + \left(\alpha + \frac{\beta}{z}\right)\frac{du}{dz} + \left(\gamma + \frac{\delta}{z} + \frac{\varepsilon}{z^2}\right) u = 0, $$

and the Bessel equation [18]

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}+\left(z^2-\nu^2\right)=0.$$

The Gegenbauer equation is in general written as

$$\left(1-z^2\right)\frac{d^2u}{dz^2}-(2\beta+1)\,z\frac{du}{dz}+n\,(n+2\beta)\,u=0. \tag{1}$$

For $\beta=1/2$, this equation reduces to the Legendre equation [19]. For the problem having analogous singularity, some questions of spectral analysis are given in [20].

Definition. [21, 22] If the function $f(z)$ is analytic and has no branch point inside and on C, where

$$C:=\left\{C^-,C^+\right\},$$

C^- is a contour along the cut joining the points z and $-\infty+i\,\mathrm{Im}\,(z)$, which starts from the point at $-\infty$, encircles the point z once counter-clockwise, and returns to the point at $-\infty$, and C^+ is a contour along the cut joining the points z and $\infty+i\,\mathrm{Im}\,(z)$, which starts from the point at ∞, encircles the point z once counter-clockwise, and returns to the point at ∞,

$$f_\eta(z)=(f(z))_\eta:=\frac{\Gamma(\eta+1)}{2\pi i}\int_C\frac{f(t)\,dt}{(t-z)^{\eta+1}}\quad(\eta\neq-1,-2,\ldots) \tag{2}$$

and

$$f_{-n}(z):=\lim_{\eta\to-n}f_\eta(z)\quad\left(n\in\mathbb{Z}^+\right), \tag{3}$$

where $t\neq z$,

$$-\pi\leq\arg(t-z)\leq\pi\quad\text{for } C^-$$

and

$$0\leq\arg(t-z)\leq2\pi\quad\text{for } C^+,$$

then $f_\eta(z)$ $(\eta>0)$ is said to be the fractional derivative of $f(z)$ of order η and $f_\eta(z)$ $(\eta<0)$ is said to be the fractional integral of $f(z)$ of order $-\eta$, provided (in each case) that

$$|f_\eta(z)|<\infty\,(\eta\in\mathbb{R}). \tag{4}$$

We find it to be worthwhile to recall here the following useful lemmas and properties associated with the fractional differintegration which is defined above (cf. [21, 22]).

Lemma 1 (Linearity)**.** *Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If f_η and g_η exist, then*

$$(h_1 f(z))_\eta = h_1 f_\eta(z)$$

and

$$(h_1 f(z) + h_2 g(z))_\eta = h_1 f_\eta(z) + h_2 g_\eta(z) \tag{5}$$

where h_1 and h_2 are constants and $\eta \in \mathbb{R};\ z \in \mathbb{C}$.

Lemma 2 (Index Law)**.** *Let $f(z)$ be an analytic and single-valued function. If $(f_\rho)_\eta$ and $(f_\eta)_\rho$ exist, then*

$$(f_\rho(z))_\eta = f_{\rho+\eta}(z) = (f_\eta(z))_\rho, \tag{6}$$

where $\rho, \eta \in \mathbb{R};\ z \in \mathbb{C}$ and $\left|\dfrac{\Gamma(\rho+\eta+1)}{\Gamma(\rho+1)\Gamma(\eta+1)}\right| < \infty$.

Lemma 3 (Generalized Leibniz Rule)**.** *Let $f(z)$ and $g(z)$ be analytic and single-valued functions. If f_η and g_η exist, then*

$$(f(z)\ g(z))_\eta = \sum_{n=0}^{\infty} \frac{\Gamma(\eta+1)}{\Gamma(\eta-n+1)\Gamma(n+1)} f_{\eta-n}(z)\ g_n(z), \tag{7}$$

where $\eta \in \mathbb{R};\ z \in \mathbb{C}$ and $\left|\dfrac{\Gamma(\eta+1)}{\Gamma(\eta-n+1)\Gamma(n+1)}\right| < \infty$.

Property 1. *For a constant λ,*

$$\left(e^{\lambda z}\right)_\eta = \lambda^\eta e^{\lambda z} \quad (\lambda \neq 0;\ \eta \in \mathbb{R};\ z \in \mathbb{C}). \tag{8}$$

Property 2. *For a constant λ,*

$$\left(e^{-\lambda z}\right)_\eta = e^{-i\pi\eta} \lambda^\eta e^{-\lambda z} \quad (\lambda \neq 0;\ \eta \in \mathbb{R},\ z \in \mathbb{C}). \tag{9}$$

Property 3. *For a constant λ,*

$$\left(z^\lambda\right)_\eta = e^{-i\pi\eta} \frac{\Gamma(\eta-\lambda)}{\Gamma(-\lambda)} z^{\lambda-\eta} \quad \left(\eta \in \mathbb{R};\ z \in \mathbb{C};\ \left|\frac{\Gamma(\eta-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \tag{10}$$

The purpose of our study is to give the fractional solutions of Gegenbauer equation by using Nishimoto method.

2. The N^η Method Applied to a Gegenbauer Equation

Theorem 1. *Let $\phi \in \{\phi : 0 \neq |\phi_\eta| < \infty; \eta \in \mathbb{R}\}$ and $f \in \{f : 0 \neq |f_\eta| < \infty; \eta \in \mathbb{R}\}$. Then the non-homogeneous Gegenbauer differential equation*

$$L[\phi, r; \beta, n] = \phi_2\left(1 - r^2\right) - \phi_1(2\beta+1) r + \phi n(n+2\beta) = f \quad (r \neq \{-1, 1\}) \tag{11}$$

has particular solutions of the forms:

$$\phi^{I}=\left\{\left[f_{\frac{n(n+2\beta)}{2\beta+1}}\left(1-r^{2}\right)^{\frac{1}{2}\left(\frac{\left(4\beta^{2}-1\right)+2n(n+2\beta)}{2\beta+1}\right)}\right]_{-1}\left(1-r^{2}\right)^{-\frac{1}{2}\left(\frac{(2\beta+1)^{2}+2n(n+2\beta)}{2\beta+1}\right)}\right\}_{-\left(\frac{(2\beta+1)+n(n+2\beta)}{2\beta+1}\right)} \tag{12}$$

$$\phi^{II}=(1-r)^{\frac{1-2\beta r}{1+r}}\left\{\left(\left[f\,(1-r)^{\frac{2\beta r-1}{1+r}}\right]_{\frac{n(n+2\beta)}{1-2\beta}}\left(\frac{1-r}{1+r}\right)\left(1-r^{2}\right)^{-\frac{1}{2}\left(\frac{\left(4\beta^{2}-1\right)+2n(n+2\beta)}{2\beta-1}\right)}\right)_{-1}\left(\frac{1+r}{1-r}\right)\left(1-r^{2}\right)^{\frac{1}{2}\left(\frac{(2\beta-1)^{2}+2n(n+2\beta)}{2\beta-1}\right)}\right\}_{-\left(\frac{(1-2\beta)+n(n+2\beta)}{1-2\beta}\right)} \tag{13}$$

$$\phi^{III}=(1+r)^{\frac{1+2\beta r}{1-r}}\left\{\left(\left[f\,(1+r)^{\frac{2\beta r+1}{r-1}}\right]_{\frac{n(n+2\beta)}{1-2\beta}}\left(\frac{1+r}{1-r}\right)\left(1-r^{2}\right)^{-\frac{1}{2}\left(\frac{\left(4\beta^{2}-1\right)+2n(n+2\beta)}{2\beta-1}\right)}\right)_{-1}\left(\frac{1-r}{1+r}\right)\left(1-r^{2}\right)^{\frac{1}{2}\left(\frac{(2\beta-1)^{2}+2n(n+2\beta)}{2\beta-1}\right)}\right\}_{-\left(\frac{(1-2\beta)+n(n+2\beta)}{1-2\beta}\right)} \tag{14}$$

where $\phi_n=\dfrac{d^n\phi}{dr^n}$ $(n=0,1,2)$, $\phi_0=\phi=\phi(r)$, $r\in\mathbb{C}$, β *and* n *are given constants,* $\beta\neq\left\{-\dfrac{1}{2},\dfrac{1}{2}\right\}$.

Proof. (i) Operate $N-$fractional calculus operator N^{η} directly to the both sides of (11), we then obtain

$$\left[\phi_2\left(1-r^2\right)\right]_\eta-\left[\phi_1\left(2\beta+1\right)r\right]_\eta+\left[\phi n\left(n+2\beta\right)\right]_\eta=(f)_\eta. \tag{15}$$

Using $(5),(6),(7)$ we have

$$\left[\phi_2\left(1-r^2\right)\right]_\eta=\phi_{2+\eta}\left(1-r^2\right)-2\eta\phi_{1+\eta}r \tag{16}$$

and

$$\left[\phi_1\left(2\beta+1\right)r\right]_\eta=\left(2\beta+1\right)\phi_{1+\eta}r+\left(2\beta+1\right)\eta\phi_\eta. \tag{17}$$

Making use of the relations (16) and (17), we may write (15) in the following form:

$$\phi_{2+\eta}\left(1-r^2\right)-\phi_{1+\eta}\left[\left(2\eta+2\beta+1\right)r\right]+\phi_\eta\left[n^2+2\beta n-\left(2\beta+1\right)\eta\right]=f_\eta. \tag{18}$$

Chose η such that

$$\eta = \frac{n\left(n+2\beta\right)}{2\beta+1}, \tag{19}$$

we have then

$$\phi_{2+\frac{n(n+2\beta)}{2\beta+1}} - \phi_{1+\frac{n(n+2\beta)}{2\beta+1}} \left[\left(\frac{\left(2\beta+1\right)^2 + 2n\left(n+2\beta\right)}{2\beta+1}\right)\frac{r}{1-r^2}\right] = f_{\frac{n(n+2\beta)}{2\beta+1}}\left(1-r^2\right)^{-1} \tag{20}$$

from (18).

Therefore, setting

$$\phi_{1+\frac{n(n+2\beta)}{2\beta+1}} = y = y\left(r\right) \quad \left(\phi = \left(y\right)_{-1-\frac{n(n+2\beta)}{2\beta+1}}\right), \tag{21}$$

we have

$$y_1 - y\left[\left(\frac{\left(2\beta+1\right)^2 + 2n\left(n+2\beta\right)}{2\beta+1}\right)\frac{r}{1-r^2}\right] = f_{\frac{n(n+2\beta)}{2\beta+1}}\left(1-r^2\right)^{-1} \tag{22}$$

from (20). This is an ordinary differential equation of the first order which has a particular solution:

$$y = \left[f_{\frac{n(n+2\beta)}{2\beta+1}}\left(1-r^2\right)^{\frac{1}{2}\left(\frac{\left(4\beta^2-1\right)+2n(n+2\beta)}{2\beta+1}\right)}\right]_{-1}\left(1-r^2\right)^{-\frac{1}{2}\left(\frac{(2\beta+1)^2+2n(n+2\beta)}{2\beta+1}\right)}. \tag{23}$$

Thus we obtain the solution (12) from (21) and (23).

Inversely, the function given by (23) satisfies (22) clearly. Hence (12) satisfies equation (20). Therefore, the function (12) satisfies equation (11).

(ii) Set

$$\phi = \left(1-r\right)^{\tau}\psi, \quad \psi = \psi\left(r\right), \tag{24}$$

hence

$$\phi_1 = -\tau\left(1-r\right)^{\tau-1}\psi + \left(1-r\right)^{\tau}\psi_1 \tag{25}$$

and

$$\phi_2 = \tau\left(\tau-1\right)\left(1-r\right)^{\tau-2}\psi - 2\tau\left(1-r\right)^{\tau-1}\psi_1 + \left(1-r\right)^{\tau}\psi_2. \tag{26}$$

Substitute (24), (25) and (26) into (11), we have

$$\begin{aligned}&\left(1-r^2\right)\psi_2 - \left[\left(2\tau+2\beta+1\right)r + 2\tau\right]\psi_1 \\ &+ \left\{\left[\tau\left(\tau-1\right)\left(1+r\right) + \tau\left(2\beta+1\right)r\right]\left(1-r\right)^{-1} + n\left(n+2\beta\right)\right\}\psi = f\left(1-r\right)^{-\tau}.\end{aligned} \tag{27}$$

Here, we choose τ such that

$$\tau(\tau-1)(1+r)+\tau(2\beta+1)r=0,$$

that is,

$$\tau_1=0, \quad \tau_2=\frac{1-2\beta r}{1+r}.$$

In the case $\tau=0$, we have the same results as (i).

Let $\tau=(1-2\beta r)/(1+r)$. From (24) and (27) we have

$$\phi=(1-r)^{\frac{1-2\beta r}{1+r}}\psi \tag{28}$$

and

$$\left(1-r^2\right)\psi_2+\left[(2\beta-1)r-2\right]\psi_1+\left[n(n+2\beta)\right]\psi=f(1-r)^{\frac{2\beta r-1}{1+r}}, \tag{29}$$

respectively.

Applying the operator N^η to both members of (29), we have

$$\begin{aligned}\psi_{2+\eta}\left(1-r^2\right)&+\left[(2\beta-2\eta-1)r-2\right]\psi_{1+\eta}\\&+\left[(2\beta-1)\eta+n(n+2\beta)\right]\psi_\eta=\left[f(1-r)^{\frac{2\beta r-1}{1+r}}\right]_\eta.\end{aligned} \tag{30}$$

Here we choose η such that

$$(2\beta-1)\eta+n(n+2\beta)=0$$

that is

$$\eta=\frac{n(n+2\beta)}{1-2\beta}. \tag{31}$$

Substituting (31) into (30), we have

$$\begin{aligned}\psi_{2+\frac{n(n+2\beta)}{1-2\beta}}&+\left[\left(\frac{(2\beta-1)^2+2n(n+2\beta)}{2\beta-1}\right)\frac{r}{1-r^2}-\frac{2}{1-r^2}\right]\psi_{1+\frac{n(n+2\beta)}{1-2\beta}}\\&=\left[f(1-r)^{\frac{2\beta r-1}{1+r}}\right]_{\frac{n(n+2\beta)}{1-2\beta}}\left(1-r^2\right)^{-1}.\end{aligned} \tag{32}$$

Set

$$\psi_{1+\frac{n(n+2\beta)}{1-2\beta}}=g=g(r) \quad \left(\psi=(g)_{-1-\frac{n(n+2\beta)}{1-2\beta}}\right), \tag{33}$$

we have then

$$g_1 + g\left[\left(\frac{(2\beta-1)^2 + 2n(n+2\beta)}{2\beta-1}\right)\frac{r}{1-r^2} - \frac{2}{1-r^2}\right] = \left[f(1-r)^{\frac{2\beta r-1}{1+r}}\right]_{\frac{n(n+2\beta)}{1-2\beta}}\left(1-r^2\right)^{-1} \quad (34)$$

from (32). A particular solution of ordinary differential equation (34) is given by

$$g = \left\{\left[f(1-r)^{\frac{2\beta r-1}{1+r}}\right]_{\frac{n(n+2\beta)}{1-2\beta}}\left(\frac{1-r}{1+r}\right)\left(1-r^2\right)^{-\frac{1}{2}\left(\frac{\left(4\beta^2-1\right)+2n(n+2\beta)}{2\beta-1}\right)}\right\}_{-1} \times \left(\frac{1+r}{1-r}\right)\left(1-r^2\right)^{\frac{1}{2}\left(\frac{(2\beta-1)^2+2n(n+2\beta)}{2\beta-1}\right)}. \quad (35)$$

Therefore, we obtain the solution (13) from (28), (33) and (35).

Inversely, (35) satisfies (34), then

$$\psi = (g)_{-1-\frac{n(n+2\beta)}{1-2\beta}},$$

satisfies (32). Therefore (13) satisfies (11), since we have (24).

(iii) Set

$$\phi = (1+r)^{\tau}\psi, \quad \psi = \psi(r), \quad (36)$$

hence

$$\phi_1 = \tau(1+r)^{\tau-1}\psi + (1+r)^{\tau}\psi_1 \quad (37)$$

and

$$\phi_2 = \tau(\tau-1)(1+r)^{\tau-2}\psi + 2\tau(1+r)^{\tau-1}\psi_1 + (1+r)^{\tau}\psi_2. \quad (38)$$

Substitute (36), (37) and (38) into (11), we have

$$\left(1-r^2\right)\psi_2 + \left[-(2\tau+2\beta+1)r + 2\tau\right]\psi_1 + \left\{\left[\tau(\tau-1)(1-r) - \tau(2\beta+1)r\right](1+r)^{-1} + n(n+2\beta)\right\}\psi = f(1+r)^{-\tau}. \quad (39)$$

Here, we choose τ such that

$$\tau(\tau-1)(1-r) - \tau(2\beta+1)r = 0,$$

that is,

$$\tau_1 = 0, \quad \tau_2 = \frac{1+2\beta r}{1-r}.$$

Let $\tau = (1+2\beta r)/(1-r)$. From (36) and (39) we have

$$\phi = (1+r)^{\frac{1+2\beta r}{1-r}}\,\psi \tag{40}$$

and

$$\left(1-r^2\right)\psi_2 + \left[(2\beta-1)\,r+2\right]\psi_1 + \left[n\,(n+2\beta)\right]\psi = f\,(1+r)^{\frac{2\beta r+1}{r-1}}, \tag{41}$$

respectively.

Applying the operator N^η to both members of (41), we have

$$\psi_{2+\eta}\left(1-r^2\right) + \left[(2\beta-2\eta-1)\,r+2\right]\psi_{1+\eta} + \left[(2\beta-1)\,\eta + n\,(n+2\beta)\right]\psi_\eta = \left[f\,(1+r)^{\frac{2\beta r+1}{r-1}}\right]_\eta. \tag{42}$$

Here we choose η such that

$$(2\beta-1)\,\eta + n\,(n+2\beta) = 0$$

that is

$$\eta = \frac{n\,(n+2\beta)}{1-2\beta}. \tag{43}$$

Substituting (43) into (42), we have

$$\psi_{2+\frac{n(n+2\beta)}{1-2\beta}} + \left[\left(\frac{(2\beta-1)^2+2n\,(n+2\beta)}{2\beta-1}\right)\frac{r}{1-r^2}+\frac{2}{1-r^2}\right]\psi_{1+\frac{n(n+2\beta)}{1-2\beta}} = \left[f\,(1+r)^{\frac{2\beta r+1}{r-1}}\right]_{\frac{n(n+2\beta)}{1-2\beta}}\left(1-r^2\right)^{-1}. \tag{44}$$

Set

$$\psi_{1+\frac{n(n+2\beta)}{1-2\beta}} = h = h\,(r) \quad \left(\psi = (h)_{-1-\frac{n(n+2\beta)}{1-2\beta}}\right), \tag{45}$$

we have then

$$h_1 + h\left[\left(\frac{(2\beta-1)^2+2n\,(n+2\beta)}{2\beta-1}\right)\frac{r}{1-r^2}+\frac{2}{1-r^2}\right] = \left[f\,(1+r)^{\frac{2\beta r+1}{r-1}}\right]_{\frac{n(n+2\beta)}{1-2\beta}}\left(1-r^2\right)^{-1} \tag{46}$$

from (44). A particular solution of ordinary differential equation (46) is given by

$$h=\left\{\left[f\left(1+r\right)^{\frac{2\beta r+1}{r-1}}\right]_{\frac{n(n+2\beta)}{1-2\beta}}\left(\frac{1+r}{1-r}\right)\left(1-r^{2}\right)^{-\frac{1}{2}\left(\frac{\left(4\beta^{2}-1\right)+2n(n+2\beta)}{2\beta-1}\right)}\right\}_{-1}$$
$$\times\left(\frac{1-r}{1+r}\right)\left(1-r^{2}\right)^{\frac{1}{2}\left(\frac{(2\beta-1)^{2}+2n(n+2\beta)}{2\beta-1}\right)}. \quad (47)$$

Therefore, we obtain the solution (14) from (40), (45) and (47).

Inversely, (47) satisfies (46), then

$$\psi=(h)_{-1-\frac{n(n+2\beta)}{1-2\beta}},$$

satisfies (44). Therefore (14) satisfies (11), since we have (36). □

Theorem 2. *Let* $\phi\in\{\phi:0\neq|\phi_{\eta}|<\infty;\eta\in\mathbb{R}\}$. *Then the homogeneous Gegenbauer differential equation*

$$L[\phi,r;\beta,n]=\phi_{2}\left(1-r^{2}\right)-\phi_{1}\left(2\beta+1\right)r+\phi n\left(n+2\beta\right)=0\quad(r\neq\{-1,1\}) \quad (48)$$

has solutions of the forms:

$$\phi^{(I)}=k\left\{\left(1-r^{2}\right)^{-\frac{1}{2}\left(\frac{(2\beta+1)^{2}+2n(n+2\beta)}{2\beta+1}\right)}\right\}_{-\left(\frac{(2\beta+1)+n(n+2\beta)}{2\beta+1}\right)} \quad (49)$$

$$\phi^{(II)}=k\left(1-r\right)^{\frac{1-2\beta r}{1+r}}\left[\left(\frac{1+r}{1-r}\right)\left(1-r^{2}\right)^{\frac{1}{2}\left(\frac{(2\beta-1)^{2}+2n(n+2\beta)}{2\beta-1}\right)}\right]_{-\left(\frac{(1-2\beta)+n(n+2\beta)}{1-2\beta}\right)} \quad (50)$$

$$\phi^{(III)}=k\left(1+r\right)^{\frac{1+2\beta r}{1-r}}\left[\left(\frac{1-r}{1+r}\right)\left(1-r^{2}\right)^{\frac{1}{2}\left(\frac{(2\beta-1)^{2}+2n(n+2\beta)}{2\beta-1}\right)}\right]_{-\left(\frac{(1-2\beta)+n(n+2\beta)}{1-2\beta}\right)} \quad (51)$$

where k *is an arbitrary constant.*

Proof. When $f=0$ in Theorem 1, we have

$$y_{1}-y\left[\left(\frac{\left(2\beta+1\right)^{2}+2n\left(n+2\beta\right)}{2\beta+1}\right)\frac{r}{1-r^{2}}\right]=0, \quad (52)$$

$$g_{1}+g\left[\left(\frac{\left(2\beta-1\right)^{2}+2n\left(n+2\beta\right)}{2\beta-1}\right)\frac{r}{1-r^{2}}-\frac{2}{1-r^{2}}\right]=0, \quad (53)$$

and

$$h_1 + h\left[\left(\frac{(2\beta-1)^2 + 2n(n+2\beta)}{2\beta-1}\right)\frac{r}{1-r^2} + \frac{2}{1-r^2}\right] = 0 \tag{54}$$

instead of (22), (34) and (46), respectively.

Therefore, we obtain (49) for (52), (50) for (53) and (51) for (54). $\square$

Theorem 3. *Let* $\phi \in \{\phi : 0 \neq |\phi_\eta| < \infty; \eta \in \mathbb{R}\}$ *and* $f \in \{f : 0 \neq |f_\eta| < \infty; \eta \in \mathbb{R}\}$. *Then the non-homogeneous Gegenbauer differential equation* (11) *is satisfied by the fractional differintegrated functions (for example)*

$$\phi = \phi^I + \phi^{(I)}. \tag{55}$$

Proof. It is clear by Theorems 1 and 2. $\square$

3. Conclusion

In this paper, we apply the Nishimoto method for a Gegenbauer equation. The most important advantage of the method is that it can be applied for singular equations.

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