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An SIR Model of Influenza with the Effects of Treatment and Vaccination

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Abstract

We produced an SIR model of influenza which is a global infectious disease, by using Caputo fractional derivative. In this model, we separated S and I into different groups. Separation is made according to the group of people in S who get vaccinated and are protected from influenza, also people in S who get vaccinated but are not protected besides people in S who do not get vaccinated. Furthermore, infected people are separated as treated and untreated people in I. We did stability analysis of the model and produced the basic reproduction number. We emphasized the importance of influenza vaccine and treatment for infected people by varying the values of the parameters and was shown with graphics.

1. Introduction

Influenza is the virus that represents a persistent and significant threat to global public health, responsible for deaths of nearly half a million people annually worldwide [1]. Annually, approximately 5% to 10% of adults and 20% to 30% of children are infected [2]. Individuals with chronic diseases of these patients are at an increased risk of developing greater morbidity or mortality when infected by the influenza virus due to poor health conditions and impaired immune systems [3]. Furthermore, influenza-related hospitalization and mortality rates were significantly higher in individuals with chronic diseases [4]. The most effective method of preventing influenza is vaccination, which is recommended for all individuals aged six months or older, including pregnant and postpartum women, unless there are contraindications [1]. Thomas Francis and Jonas Salk at the University of Michigan developed the first inactivated influenza vaccine and this vaccine is licensed in 1945 for wider use [5]. The influenza virus undergoes annual mutations, which necessitate the annual revision of the vaccine content. Vaccination should occur at the beginning of the influenza season, which typically begins in October [1]. The classic symptoms of influenza are fever, fatigue, cough and body aches [1]. In the outpatient setting, diagnosis can be made based on clinical presentation, with optional confirmatory diagnostic testing. This disease, which many patients can recover without any treatment but can be fatal for people in risk groups such as chronically ill patients, babies and the elderly people. Some people need to take antiviral treatment within 24 to 48 hours after the onset of the disease rather than drug-free treatment such as drinking plenty of water [6]. It is recommended that drugs be used in accordance with the instructions provided by the prescribing physician.
The model presented in this study incorporates parameters that elucidate the impact of vaccination and treatment on the epidemiological dynamics of the disease. Once the model has been generated, the stability conditions for the disease-free and endemic equilibrium points are determined separately. Moreover, the basic reproduction number is found specifically for this model. Different values were assigned to the model parameters in order to assess the impact of vaccination and treatment on the course of influenza.

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2. Model

We constructed a model as follows:

$$\begin{aligned} {}^{C}D_t^{\alpha} S &= \mu N - \frac{(1-ab)\beta}{N} SI - (\mu + ab)S \\ {}^{C}D_t^{\alpha} I &= \frac{(1-ab)\beta}{N} SI - \big(\mu + c\gamma + (1-c)\kappa\big)I \\ {}^{C}D_t^{\alpha} R &= abS + \big(c\gamma + (1-c)\kappa\big)I - \mu R \end{aligned}$$

with assumptions:

- The rates of birth and death are equal.
- $S(t) \geq 0$, $I(t) \geq 0$, $R(t) \geq 0$ at any time t .
- All parameters take values between 0 and 1. So all parameters are non-negative.
- $\gamma > \kappa$.

In Table 2.1 , we gave all parameters.

Parameter	Description
μ	Birth rate and death rate
β	Contact rate between susceptible people and infected people
a	Rate of vaccination in S
b	Rate of vaccine efficiency
c	Rate of treatment in I
γ	Recovery rate for treated people in I
κ	Recovery rate for untreated people in I

Table 2.1: Table of parameters

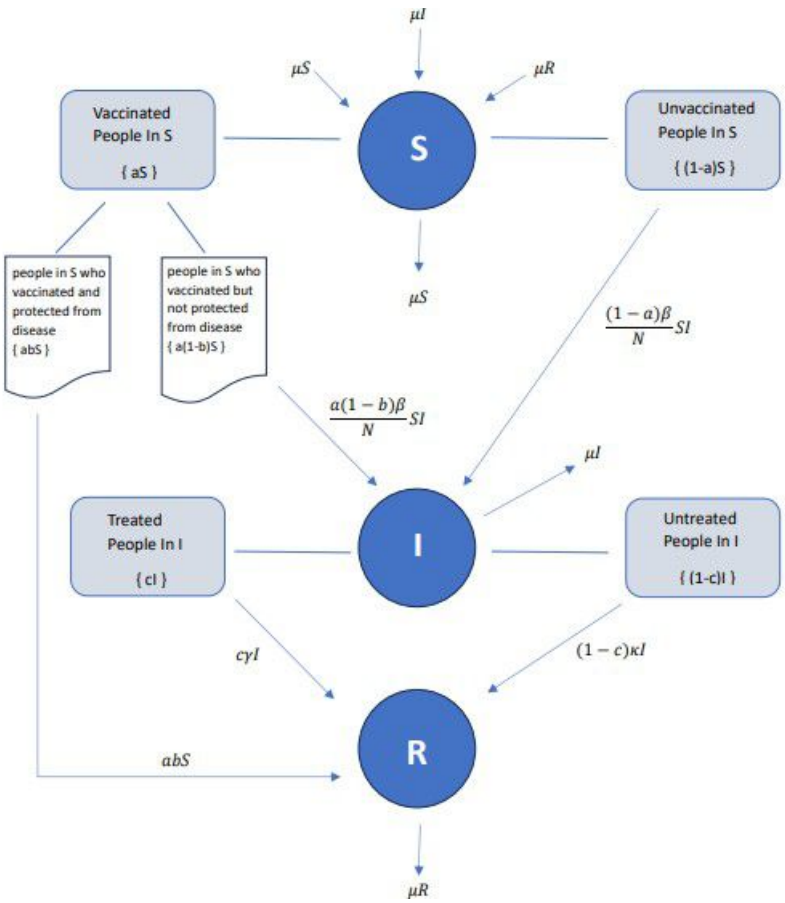

Figure 2.1: The model

3. Non-Negativity

Let us define the positive part of $\mathbb{R}^3$ as $\zeta \in \mathbb{R}^3 : \zeta \geq 0\}$ and $\zeta(t)$ as $(\zeta_1(t), \zeta_2(t), \zeta_3(t))^T$. We used Lemma 3.1 and Corollary 3.1 for proving the theorem that is related to non-negativity of the solutions [7].

Lemma 3.1 (Generalized Mean Value Theorem)**.** *Let $\psi(t)$ and $D^\alpha\psi(t)$ are continuous in the closed interval $[\rho,\sigma]$ for $0<\alpha\leq 1$. Then $\forall t\in[\rho,\sigma]$, $\psi(t)=\psi(\rho)+\frac{1}{\Gamma\alpha}D^\alpha\psi(\eta)(t-\rho)^\alpha$ holds with $0\leq\eta\leq t$ [8].*

Corollary 3.1. *Let $\psi(t)$ and $D^\alpha\psi(t)$ are continuous in the closed interval $[0,\sigma]$ for $0<\alpha\leq 1$. It can be seen from the lemma that $\psi(t)$ is non-decreasing $\forall t\in[0,\sigma]$ if $D^\alpha\psi(t)\geq 0$ for each $t\in(0,\sigma)$ and $\psi(t)$ is non-increasing $\forall t\in[0,\sigma]$ if $D^\alpha\psi(t)\leq 0$ for each $t\in(0,\sigma)$ [7].*

Theorem 3.1. *There exists a solution that unique and remains in the positive part of $\mathbb{R}^3$ for (1) such that $\zeta(t)=\big(S(t),I(t),R(t)\big)^T$.*

Proof. The existence and the uniqueness of $\zeta(t)$ in the interval $(0,\infty)$ can be obtained from [9] . It is necessary to explain that the domain $\mathbb{R}^3_+$ is positively invariant. Since
$D_t^\alpha S(t)=\mu N\geq 0$ where $S=0$,
$D_t^\alpha I(t)=0$ where $I=0$,
$D_t^\alpha R(t)=abS+\big(c\gamma+(1-c)\kappa\big)I\geq 0$ where $R=0$,
it can be obtained that by the corollary on each hyperplane bounding the non-negative octant, the vector field point into the positive part of $\mathbb{R}^3$. □

4. Stability Analysis

Two equilibrium points, the first one disease free equilibrium point and the second one endemic equilibrium point, are found as follows:

- $E_1=(\frac{\mu N}{\mu+ab},0,\frac{abN}{\mu+ab})$
- $E_2=\Big(\frac{(\mu+c\gamma+(1-c)\kappa)N}{(1-ab)\beta},N\big(\frac{\mu}{\mu+c\gamma+(1-c)\kappa}-\frac{(\mu+ab)}{(1-ab)\beta}\big),N\big(1+\frac{ab-c\gamma-(1-c)\kappa}{(1-ab)\beta}-\frac{\mu}{\mu+c\gamma+(1-c)\kappa}\big)\Big)$

by solving

$$\begin{aligned}\mu N-\frac{(1-ab)\beta}{N}\overline{SI}-(\mu+ab)\overline{S} &= 0\\ \frac{(1-ab)\beta}{N}\overline{SI}-\big(\mu+c\gamma+(1-c)\kappa\big)\overline{I} &= 0\\ ab\overline{S}+\big(c\gamma+(1-c)\kappa\big)\overline{I}-\mu\overline{R} &= 0.\end{aligned}$$

For this model the Jacobian matrix is obtained as follows:

$$\mathbf{J}=\begin{pmatrix}-\frac{(1-ab)\beta}{N}I-(\mu+ab) & -\frac{(1-ab)\beta}{N}S & 0\\ \frac{(1-ab)\beta}{N}I & \frac{(1-ab)\beta}{N}S-(\mu+c\gamma+(1-c)\kappa) & 0\\ ab & c\gamma+(1-c)\kappa & -\mu\end{pmatrix}.$$

Stability conditions for E_1: The Jacobian matrix is evaluated at disease free equilibrium point as follows :

$$\mathbf{J}|_{\mathbf{E_1}}=\begin{pmatrix}-(\mu+ab) & -\frac{\mu\beta(1-ab)}{\mu+ab} & 0\\ 0 & \frac{\mu\beta(1-ab)}{\mu+ab}-(\mu+c\gamma+(1-c)\kappa) & 0\\ ab & c\gamma+(1-c)\kappa & -\mu\end{pmatrix}.$$

Basic reproduction number :

Definition 4.1 (Next Generation Matrix Method, [10])**.** *Assume that $\mathscr{F}_i(x)$ is the new infections appearence rate in compartment i , $\mathscr{V}_i^+(x)$ is the transfer rate of individuals into compartment i by all other means and $\mathscr{V}_i^-(x)$ is the transfer rate of individuals out of compartment i. R_0 is defined as the spectral radius of FV^{-1} where*

$$F=\Big(\ \frac{\partial\mathscr{F}_i}{\partial x_j}(x_0)\ \Big)$$

and

$$V=\Big(\ \frac{\partial\mathscr{V}_i}{\partial x_j}(x_0)\ \Big).$$

The basic reproduction number (R_0) is found as $R_0=\frac{(1-ab)\mu\beta}{(\mu+ab)(\mu+c\gamma+(1-c)\kappa)}$ by using Next Generation Matrix method where

$$\begin{aligned}\mathscr{F}(S,I) &= \frac{(1-ab)\beta}{N}SI,\\ \mathscr{V}(I) &= -\big(\mu+c\gamma+(1-c)\kappa\big)I,\\ F &= \Big(\ \frac{(1-ab)\beta\mu}{\mu+ab}\ \Big),\\ V &= \big(\ -\big(\mu+c\gamma+(1-c)\kappa\big)\ \big).\end{aligned}$$

Stability analysis of the model can be done by using Matignon's conditions.

Theorem 4.2 (Matignon's Conditions)**.** *An equilibrium point is locally asymptotically stable if* $|arg(\lambda_i)| > \alpha\frac{\pi}{2}$ *for* $i = 1,2,3$ *where* λ_i*'s are the eigenvalues of Jacobian matrix that evaluated at the equilibrium point [11].*

Theorem 4.3. *If* $R_0 < 1$ *holds, then* E_1 *is locally asymptotically stable.*

Proof. Characteristic equation of $J|_{E_1}$ is obtained as follows:

$$\begin{aligned}
P(\lambda) &= \lambda^3 + \Big(\frac{3\mu^2+4\mu ab+a^2b^2-\mu\beta+\mu ab\beta+\mu c\gamma+(1-c)\kappa+abc\gamma+(1-c)\kappa}{\mu+ab}\Big)\lambda^2 \\
&+ \Big(\frac{-2\mu^2\beta+2\mu^2ab\beta+3\mu^3+5\mu^2ab+2\mu^2c\gamma+(1-c)\kappa+3\mu abc\gamma+(1-c)\kappa+2\mu a^2b^2-\mu ab\beta+\mu a^2b^2\beta+a^2b^2c\gamma+(1-c)\kappa}{\mu+ab}\Big)\lambda \\
&+ \Big(\frac{-\mu^3\beta+\mu^3ab\beta+\mu^4+\mu^3ab+\mu^3c\gamma+(1-c)\kappa+\mu^2abc\gamma+(1-c)\kappa-\mu^2ab\beta+\mu^2a^2b^2\beta+\mu^3ab}{\mu+ab} \\
&+ \frac{\mu^2a^2b^2+\mu^2abc\gamma+(1-c)\kappa+\mu a^2b^2c\gamma+(1-c)\kappa}{\mu+ab}\Big) = 0.
\end{aligned}$$

Eigenvalues of $J|_{E_1}$ are found by solving $P(\lambda) = 0$ as follows:

- $\lambda_1 = -\mu$
- $\lambda_2 = -(\mu+ab)$
- $\lambda_3 = \frac{-\mu ab+\mu\beta-\mu(c\gamma+(1-c)\kappa)-ab(c\gamma+(1-c)\kappa)-\mu^2-\mu ab\beta}{\mu+ab}$

Since all eigenvalues that obtained are real, according to the Matignon's Conditions the stability condition for E_1 is that all eigenvalues are negative. It is obvious that λ_1 and λ_2 are already negative.

$$\begin{aligned}
\frac{(1-ab)\mu\beta}{(\mu+ab)(\mu+c\gamma+(1-c)\kappa)} < 1 \quad &\Rightarrow \quad (1-ab)\mu\beta < (\mu+ab)(\mu+c\gamma+(1-c)\kappa) \\
&\Rightarrow \quad \mu\beta-\mu ab\beta < \mu^2+\mu c\gamma+(1-c)\kappa+\mu ab+abc\gamma+(1-c)\kappa \\
&\Rightarrow \quad -\mu ab+\mu\beta-\mu(c\gamma+(1-c)\kappa)-ab(c\gamma+(1-c)\kappa)-\mu^2-\mu ab\beta < 0
\end{aligned}$$

When $R_0 < 1$, since $-\mu ab+\mu\beta-\mu(c\gamma+(1-c)\kappa)-ab(c\gamma+(1-c)\kappa)-\mu^2-\mu ab\beta < 0$ and $0 < \mu+ab$, λ_3 is negative. □

Stability conditions for E_2 **:** The Jacobian matrix is evaluated at endemic point as follows :

$$\mathbf{J}|_{\mathbf{E_2}} = \begin{pmatrix} -\frac{\mu\beta(1-ab)}{\mu+c\gamma+(1-c)\kappa} & -(\mu+c\gamma+(1-c)\kappa) & 0 \\ \frac{\mu\beta(1-ab)}{\mu+c\gamma+(1-c)\kappa}-(\mu+ab) & 0 & 0 \\ ab & c\gamma+(1-c)\kappa & -\mu \end{pmatrix}.$$

Theorem: Characteristic equation of $J|_{E_2}$ is obtained as follows :

$$\begin{aligned}
P(\lambda) &= \lambda^3 + \Big(\frac{\mu^2+\mu(c\gamma+(1-c)\kappa)+\mu\beta-\mu ab\beta}{\mu+c\gamma+(1-c)\kappa}\Big)\lambda^2 \\
&+ \Big(\frac{2\mu^2\beta-2\mu^2ab\beta-\mu^2ab-\mu(c\gamma+(1-c)\kappa)^2-2\mu^2(c\gamma+(1-c)\kappa)-ab(c\gamma+(1-c)\kappa)^2-\mu^3-2\mu ab(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa} \\
&+ \frac{\mu\beta(c\gamma+(1-c)\kappa)-\mu ab\beta(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa}\Big)\lambda \\
&+ \Big(\frac{-\mu^3ab+\mu^3\beta-\mu^2(c\gamma+(1-c)\kappa)^2-2\mu^3(c\gamma+(1-c)\kappa)-\mu ab(c\gamma+(1-c)\kappa)^2-\mu^4}{\mu+c\gamma+(1-c)\kappa} \\
&+ \frac{-2\mu^2ab(c\gamma+(1-c)\kappa)+\mu^2\beta(c\gamma+(1-c)\kappa)-\mu^3ab\beta-\mu^2ab\beta(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa}\Big) = 0.
\end{aligned}$$

Definition 4.4 ([7])**.** *The discriminant of the characteristic equation is defined as:*

$$D\big(P(\lambda)\big) = 18a_1a_2a_3+(a_1a_2)^2-4a_3a_1^3-4a_2^3-27a_3^2$$

where characteristic equation of the form

$$P(\lambda) = \lambda^3+a_1\lambda^2+a_2\lambda+a_3 = 0.$$

Discriminant of $P(\lambda)$ is evaluated as follows :

$$\begin{aligned} D\big(P(\lambda)\big) &= \frac{\big(\mu^2 ab+\mu(c\gamma+(1-c)\kappa)^2+\mu^2(c\gamma+(1-c)\kappa)+ab(c\gamma+(1-c)\kappa)^2+2\mu ab(c\gamma+(1-c)\kappa)}{(\mu+c\gamma+(1-c)\kappa)^4} \\ &- \frac{-\mu\beta(c\gamma+(1-c)\kappa)+\mu ab\beta(c\gamma+(1-c)\kappa)\big)^2\big(4\mu^4+4\mu^3 ab\beta+4\mu^3 ab-4\mu^3\beta+12\mu^3(c\gamma+(1-c)\kappa)}{(\mu+c\gamma+(1-c)\kappa)^4} \\ &- \frac{\mu^2a^2b^2\beta^2-2\mu^2ab\beta^2+8\mu^2ab\beta(c\gamma+(1-c)\kappa)+12\mu^2ab(c\gamma+(1-c)\kappa)+\mu^2\beta^2-8\mu^2\beta(c\gamma+(1-c)\kappa)+12\mu^2(c\gamma+(1-c)\kappa)^2}{(\mu+c\gamma+(1-c)\kappa)^4} \\ &- \frac{4\mu ab\beta(c\gamma+(1-c)\kappa)^2+12\mu ab(c\gamma+(1-c)\kappa)^2-4\mu\beta(c\gamma+(1-c)\kappa)^2+4\mu(c\gamma+(1-c)\kappa)^3+4ab(c\gamma+(1-c)\kappa)^3\big)}{(\mu+c\gamma+(1-c)\kappa)^4} \end{aligned}$$

Theorem 4.5. *Fractional Routh Hurwitz Conditions [12]:*

- *If* $D\big(P(\lambda)\big)>0$ *, then locally asymptotically stability conditions for the equilibrium point are* $a_1>0$ *,* $a_3>0$ *,* $a_1a_2>a_3$*.*
- *If* $D\big(P(\lambda)\big)<0$ *,* $a_1\geq 0$ *,* $a_2\geq 0$ *,* $a_3>0$ *then the equilibrium point is locally asymptotically stable for* $\alpha<\frac{2}{3}$*.*
- *If* $D\big(P(\lambda)\big)<0$ *,* $a_1<0$ *,* $a_2<0$ *and* $\alpha>\frac{2}{3}$ *then all roots of the characteristic equation of the endemic equilibrium point satisfy the condition* $|arg(\lambda_i)|<\alpha\frac{\pi}{2}$*.*

Taking into account the Fractional Routh Hurwitz Conditions, it can be seen that the conditions of stability are as follows:

- If $D\big(P(\lambda)\big)>0$, then locally asymptotically stability conditions for the equilibrium point are
  $\left(\frac{\mu^2+\mu(c\gamma+(1-c)\kappa)+\mu\beta-\mu ab\beta}{\mu+c\gamma+(1-c)\kappa}\right)>0,$
  $\left(\frac{-\mu^3ab+\mu^3\beta-\mu^2(c\gamma+(1-c)\kappa)^2-2\mu^3(c\gamma+(1-c)\kappa)-\mu ab(c\gamma+(1-c)\kappa)^2-\mu^4-2\mu^2ab(c\gamma+(1-c)\kappa)+\mu^2\beta(c\gamma+(1-c)\kappa)-\mu^3ab\beta-\mu^2ab\beta(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa}\right)>0,$
  $\left(\frac{\mu^2+\mu(c\gamma+(1-c)\kappa)+\mu\beta-\mu ab\beta}{\mu+c\gamma+(1-c)\kappa}\right)$
  $\left(\frac{2\mu^2\beta-2\mu^2ab\beta-\mu^2ab-\mu(c\gamma+(1-c)\kappa)^2-2\mu^2(c\gamma+(1-c)\kappa)-ab(c\gamma+(1-c)\kappa)^2-\mu^3-2\mu ab(c\gamma+(1-c)\kappa)+\mu\beta(c\gamma+(1-c)\kappa)-\mu ab\beta(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa}\right)>$
  $\left(\frac{-\mu^3ab+\mu^3\beta-\mu^2(c\gamma+(1-c)\kappa)^2-2\mu^3(c\gamma+(1-c)\kappa)-\mu ab(c\gamma+(1-c)\kappa)^2-\mu^4-2\mu^2ab(c\gamma+(1-c)\kappa)+\mu^2\beta(c\gamma+(1-c)\kappa)-\mu^3ab\beta-\mu^2ab\beta(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa}\right).$
- If $D\big(P(\lambda)\big)<0$, $\left(\frac{\mu^2+\mu(c\gamma+(1-c)\kappa)+\mu\beta-\mu ab\beta}{\mu+c\gamma+(1-c)\kappa}\right)\geq 0$,
  $\left(\frac{2\mu^2\beta-2\mu^2ab\beta-\mu^2ab-\mu(c\gamma+(1-c)\kappa)^2-2\mu^2(c\gamma+(1-c)\kappa)-ab(c\gamma+(1-c)\kappa)^2-\mu^3-2\mu ab(c\gamma+(1-c)\kappa)+\mu\beta(c\gamma+(1-c)\kappa)-\mu ab\beta(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa}\right)\geq 0,$
  $\left(\frac{-\mu^3ab+\mu^3\beta-\mu^2(c\gamma+(1-c)\kappa)^2-2\mu^3(c\gamma+(1-c)\kappa)-\mu ab(c\gamma+(1-c)\kappa)^2-\mu^4-2\mu^2ab(c\gamma+(1-c)\kappa)+\mu^2\beta(c\gamma+(1-c)\kappa)-\mu^3ab\beta-\mu^2ab\beta(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa}\right)>0$
  then the equilibrium point is locally asymptotically stable for $\alpha<\frac{2}{3}$.
- If $D\big(P(\lambda)\big)<0$, $\left(\frac{\mu^2+\mu(c\gamma+(1-c)\kappa)+\mu\beta-\mu ab\beta}{\mu+c\gamma+(1-c)\kappa}\right)<0$,
  $\left(\frac{2\mu^2\beta-2\mu^2ab\beta-\mu^2ab-\mu(c\gamma+(1-c)\kappa)^2-2\mu^2(c\gamma+(1-c)\kappa)-ab(c\gamma+(1-c)\kappa)^2-\mu^3-2\mu ab(c\gamma+(1-c)\kappa)+\mu\beta(c\gamma+(1-c)\kappa)-\mu ab\beta(c\gamma+(1-c)\kappa)}{\mu+c\gamma+(1-c)\kappa}\right)<0$
  and $\alpha>\frac{2}{3}$ then all roots of the characteristic equation of the endemic equilibrium point satisfy the condition $|arg(\lambda_i)|<\alpha\frac{\pi}{2}$.

5. Numerical Approximation

We used Fractional Backward Euler Method to give numerical results with graphics.
Fractional backward Euler method. [13] The solution of

$$\begin{cases} D^\alpha f(t)=\Big(g\big(t,f(t)\big)\Big) \\ f(0)=f_0 \end{cases}$$

is as follows by using fractional backward Euler formula :

$$f_{k+1}=f_0+h^\alpha\sum_{j=0}^{k}a_{j,k+1}g(t_{j+1},f_{j+1}),\quad k=0,1,...,K-1.$$

Here,

$$a_{j,k+1}=\frac{(k-j+1)^\alpha-(k-j)^\alpha}{\Gamma(1+\alpha)},\quad k=0,1,...,K-1,\quad j=0,1,...,k.$$

The fractional backward Euler formula for the model is obtained as

$$\begin{aligned} S_{k+1} &= S_0+h^\alpha\sum_{j=0}^{k}a_{j,k+1}\big(\mu N-\frac{(1-ab)\beta}{N}S_{j+1}I_{j+1}-(\mu+ab)S_{j+1}\big) \\ I_{k+1} &= I_0+h^\alpha\sum_{j=0}^{k}a_{j,k+1}\big(\frac{(1-ab)\beta}{N}S_{j+1}I_{j+1}-\big(\mu+c\gamma+(1-c)\kappa\big)I_{j+1}\big) \\ R_{k+1} &= R_0+h^\alpha\sum_{j=0}^{k}a_{j,k+1}\big(abS_{j+1}+\big(c\gamma+(1-c)\kappa\big)I_{j+1}-\mu R_{j+1}\big) \end{aligned}$$

where

$$a_{j,k+1}=\frac{(k-j+1)^\alpha-(k-j)^\alpha}{\Gamma(1+\alpha)},\quad k=0,1,...,K-1,\quad j=0,1,...,k.$$

6. Case Studies and Numerical Results

This study demonstrates the impact of vaccines and treatments on influenza through graphs generated by adjusting model parameters.

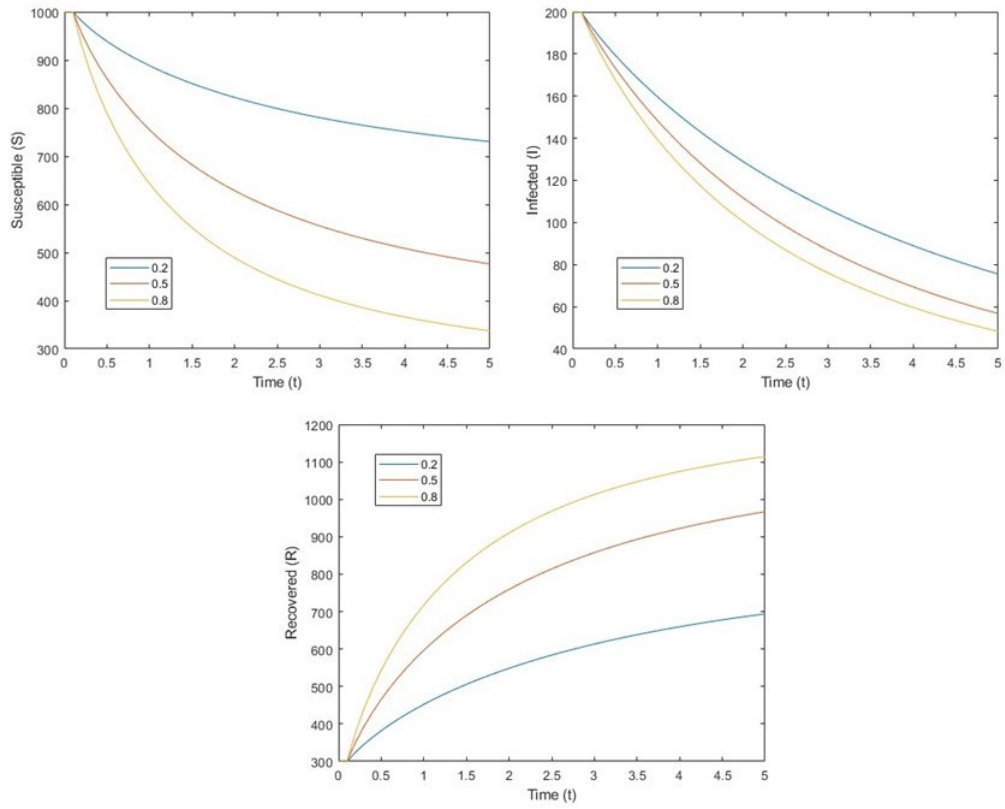

Figure 6.1: Graphs obtained by varying parameter a

In the graphs in Figure 6.1, the curves labelled "0.2" are for $a = 0.2, b = 0.7, c = 0.4, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.3445512821 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.0021 and $a_1 = 0.1827$, $a_3 = -0.0082$, $a_1 a_2 = -0.0134$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The curves labelled "0.5" are for $a = 0.5, b = 0.7, c = 0.4, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.1388888889 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.0317 and $a_1 = 0.1625$, $a_3 = -0.0201$, $a_1 a_2 = -0.0317$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The curves labelled "0.8" are for $a = 0.8, b = 0.7, c = 0.4, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.0641025641 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.1280 and $a_1 = 0.1423$, $a_3 = -0.0321$, $a_1 a_2 = -0.0451$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The graphs represent the case where parameter a values change and other parameters remain constant. Figure 6.1 demonstrate that as the value of the parameter a increases, the number of susceptible individuals is getting lower, the number of infected individuals is getting lower, and the number of recovered individuals is getting higher. This means that as individuals get vaccinated, transmission is reduced and therefore the number of infected is getting lower , according to our model . People in S who vaccinated and protected from the disease and do not get sick so they go to group R and the number of recovered individuals is getting higher in this case ; since people are significantly go to group R and protected from the disease, their possibilities of getting sick decrease, that is, the number of people in S is getting lower. In this comparison,it is shown that vaccination of susceptible people significantly reduces transmission.

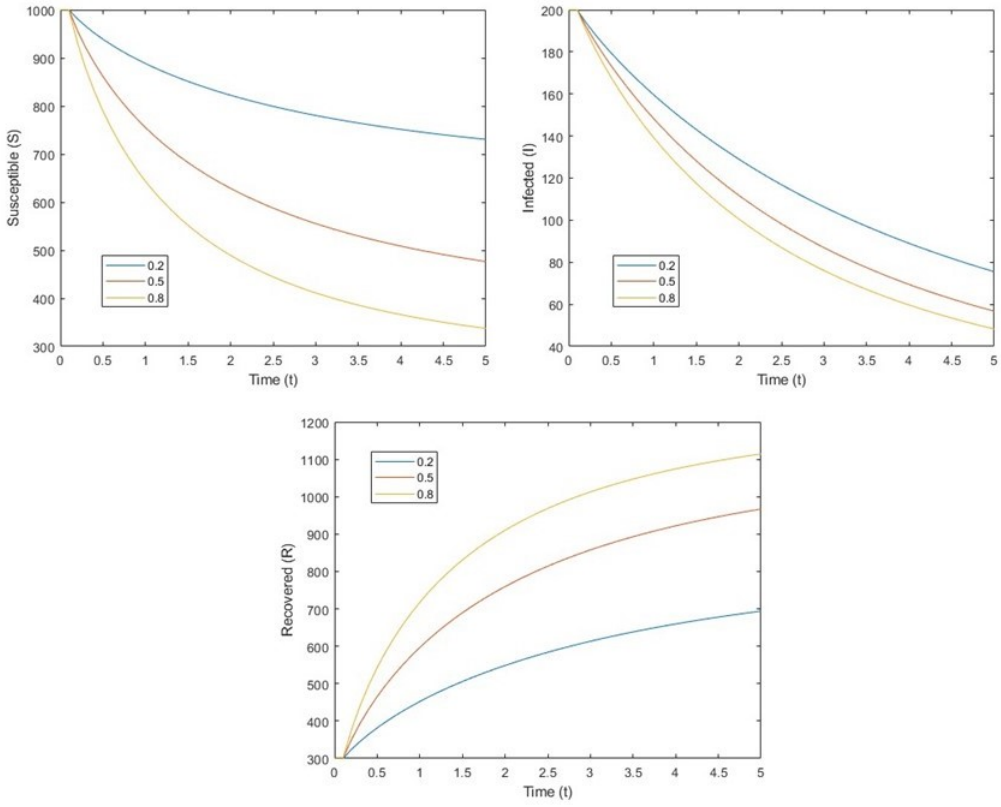

Figure 6.2: Graphs obtained by varying parameter b

In the graphs in Figure 6.2, the curves labelled "0.2" are for and $a = 0.7, b = 0.2, c = 0.4, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.3445512821 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.0021 and $a_1 = 0.1827$, $a_3 = -0.0082$, $a_1 a_2 = -0.0134$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The curves labelled $a = 0.7, b = 0.5, c = 0.4, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.1388888889 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.0317 and $a_1 = 0.1625$, $a_3 = -0.0201$, $a_1 a_2 = -0.0317$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The curves labelled "0.8" are for $a = 0.7, b = 0.8, c = 0.4, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.0641025641 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.1280 and $a_1 = 0.1423$, $a_3 = -0.0321$, $a_1 a_2 = -0.0451$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The graphs represent the case where parameter b values change and other parameters remain constant. Figure 6.2 demonstrate that as the value of the parameter b increases, the number of susceptible individuals is getting lower, the number of infected individuals is getting lower, and the number of recovered individuals is getting higher. This means that as individuals get more effective vaccines, transmission is reduced and therefore the number of infected is getting lower , according to our model. People in S who vaccinated and protected from the disease and do not get sick so they go to group R and the number of recovered individuals is getting higher in this case ; since people are significantly go to group R and protected from the disease, their possibilities of getting sick decrease, that is, the number of people in S is getting lower.

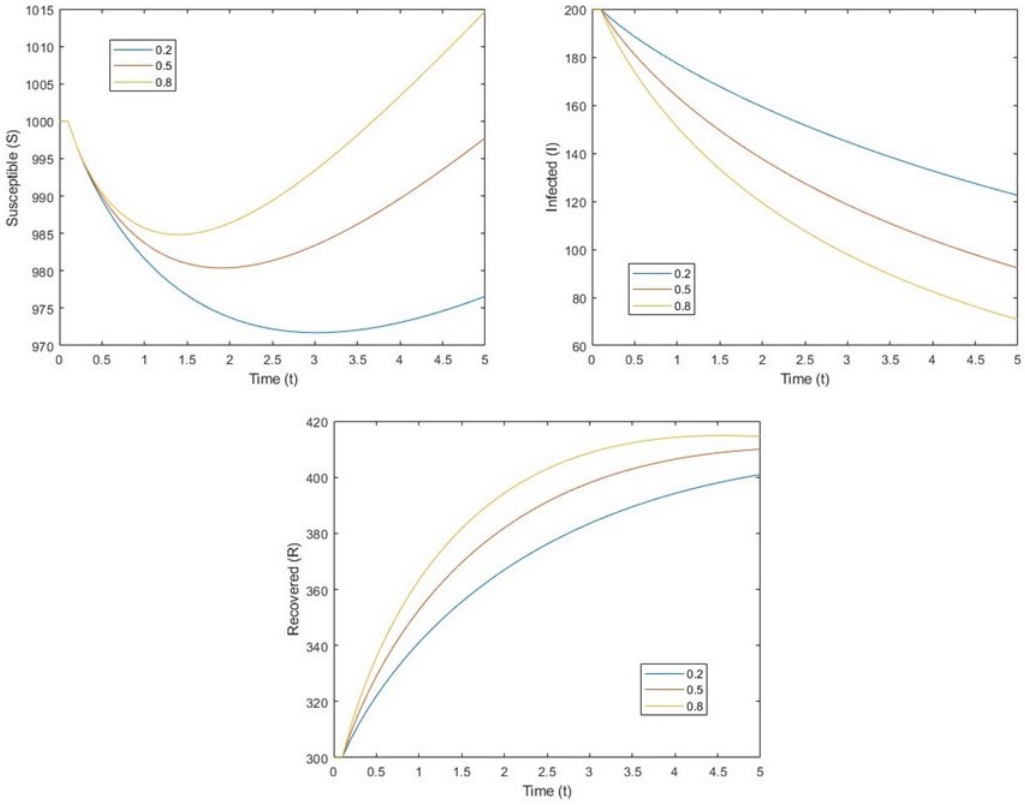

Figure 6.3: Graphs obtained by varying parameter c

In the graphs in Figure 6.3, the curves labelled "0.2" are for $a = 0.1, b = 0.1, c = 0.2, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.9782608696 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.00000006147321429 and $a_1 = 0.2076$, $a_3 = -0.00011$, $a_1 a_2 = 0.0020$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The curves labelled "0.5" are for $a = 0.1, b = 0.1, c = 0.5, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.8181818182 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.00000521 and $a_1 = 0.1900$, $a_3 = -0.0011$, $a_1 a_2 = -0.00038$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The curves labelled "0.8" are for $a = 0.1, b = 0.1, c = 0.8, \mu = 0.1, \gamma = 0.6, \kappa = 0.3, \beta = 0.5$. $R_0 = 0.703125 < 1$ for this case. So disease free equilibrium point is locally asymptotically stable. But discriminant of $P(\lambda)$ for endemic equilibrium point is obtained as 0.000031106 and $a_1 = 0.1773$, $a_3 = -0.0021$, $a_1 a_2 = -0.0023$. Since $a_3 < 0$, endemic equilibrium point is unstable according to Fractional Routh Hurwitz Conditions.
The graphs obtained represent the case where parameter c values change and other parameters remain constant. Figure 6.3 demonstrate that as the value of the parameters c increases, the number of susceptible individuals is getting higher, the number of infected individuals is getting lower, and the number of recovered individuals is getting higher. This means that as individuals in the infected group are treated they recovered and therefore the number of recovered people is getting higher ; since more infected individuals go to group R the number of infected people is getting lower.

7. Conclusion

In Figure 6.1, it is shown that getting vaccination of susceptible people significantly reduces transmission. In Figure 6.2, it is shown that getting efficient vaccination of susceptible people significantly reduces transmission.
In Figure 6.1 and Figure 6.2, parameter values except a and b are taken equal to emphasize that getting vaccine and getting effective vaccines are equally important. The correctness of this can be seen from the fact that exactly the same data are obtained in the graphs of Figure 6.1 and Figure 6.2.
In Figure 6.3, it is shown that treatment of infected people significantly increases recovery.

Article Information

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