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Heaviside Fonksiyonu İçeren McKean Denklemindeki Süreksizliğin Haar Dalgacıkları ile Ele Alınması

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ÖZ

Bu çalışma süreksiz Heaviside adım fonksiyonunu sahip bir reaksiyon-difüzyon denklemi olan McKean denklemini çözmek için Haar dalgacıklarının uygulamasını araştırmaktadır. Kompakt destekleri ve ortogonallikleri ile Haar dalgacıkları denklemin lineer olmayan dinamiklerini ele almak için kullanılabilecek basit ama etkili bir araçtır. Burada, sistemin davranışını belirleyen eşik değeri kavramını anlamak için kritik olan McKean denkleminin zamandan bağımsız çözümüne odaklanıyoruz. McKean denkleminin zamandan bağımsız çözümünün analitik çözümü olmasına rağmen, daha karmaşık yapılar için bu tür çözümlerin kapalı formda elde edilmesi yaygın değildir ki bu durum Haar dalgacık yaklaşımının faydalılığını vurgular. Önerilen yöntem, en yüksek türevli ifadenin Haar serisi açılımını entegre ederek sistematik çözüm üretilmesini sağlar. Analitik çözümle ayrıntılı bir karşılaştırma yaparak Haar dalgacık yaklaşımının daha karmaşık reaksiyon-difüzyon sistemlerini çözmek için güvenilir ve hesaplama açısından uygun bir araç olduğunu gösteriyoruz. Ayrıca, sonuçlar yöntemin doğruluğunu ve verimliliğini göstermekte olup, özellikle süreksiz ve keskin geçişlere sahip olanlar olmak üzere, daha karmaşık reaksiyon-difüzyon sistemlerine daha geniş uygulanabilirliği hakkında bilgiler sunmaktadır.

The Efficacy of Haar Wavelets in Addressing Discontinuities of McKean Equations with Heaviside Functions

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ABSTRACT

This study explores the application of Haar wavelets to solve the McKean equation, a reaction-diffusion equation with discontinuous Heaviside step function. Haar wavelets, with their compact support and orthogonality, offer straightforward but yet powerful tools for addressing the equation's nonlinear dynamics. We focus on the time-independent solution of the McKean equation, which is crucial for understanding the threshold phenomenon that determines the system's behavior. Despite the existence of analytical time-independent solution to the McKean equation, achieving such solutions in closed form is uncommon for more complicated systems, highlighting the utility of the Haar wavelet approach. The proposed method integrates the Haar series expansion of the highest order derivative, enabling systematic solution derivation. Through a detailed comparison with analytical solution, we validate the

Haar wavelet approach as a robust and computationally feasible tool for solving complex reaction-diffusion systems. The results also demonstrate the method's accuracy and efficiency, offering insights into its broader applicability to more complex reaction-diffusion systems, especially those with discontinuity and sharp transitions.

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1. Introduction

Designing accurate and numerically stable approximations to the reaction-diffusion equation, there have been numerous methods introduced in recent years, each of which has its own advantages and limitations. From the 1980s, wavelets based on the Galerkin techniques or the collocation method have been considered to be an alternative direction in solving ordinary or partial differential equations. In this paper, we first review one of the popular families of the wavelet, namely Haar wavelet, and then we attempt the Haar wavelet solution for one of the famous one-component reaction-diffusion equations, namely McKean equation.

Nonlinear dynamics and their threshold behaviors are pivotal in understanding complex systems across various fields of science and engineering (Strogatz, 2018). Particularly, the study of how systems transition from one state to another upon reaching specific thresholds offers insights into phenomena ranging from phase transitions in physics to action potential generation in neuroscience. A mathematical framework that has been instrumental in exploring such phenomena is the McKean equation. Traditionally known for its stochastic interpretations, the McKean equation also presents a deterministic form that is crucial for modeling threshold-dependent dynamics in deterministic systems.

The study of traveling waves in excitable media has been a critical area of research, especially after the 1963 Nobel Prize-winning work by A. L. Hodgkin and A. F. Huxley, who mathematically modeled the electrical impulses of nerve cells (Hodgkin and Huxley, 1952). Given the complexity of their model, many researchers have focused on developing simpler, mathematically more tractable reduced-order systems. The McKean model, named after H. P. McKean (McKean, 1970), is one of notable contribution to this effort. This model simplifies the study of reaction-diffusion systems by suggesting a piecewise-linear representation for the reaction term, which makes it easier to analyze the dynamics of traveling waves and patterns in such systems (Tonnelier, 2003). Originally, the spatially expanded McKean model consists of two components defined as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u + \mathcal{H}(u-\alpha) + v,$$

$$\frac{\partial v}{\partial t} = \beta u - \gamma v\,, \quad t > 0,\ x \in (-\infty, \infty),$$

where $u(x,t)$ and $\mathrm{v}(x,t)$ are the state variables being excitation (fast) and recovery (slow) variables respectively, $\mathcal{H}$ is Heaviside step function, signifying an abrupt change in the system's behavior when the state variable exceeds the threshold parameter α and finally β and γ are constant parameters. Fast

subsystem of this equation is derived by neglecting the recovery component of the system and setting $v = 0$, $\beta = 0$, and $\gamma = 0$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u + \mathcal{H}(u - \alpha)\,. \tag{1}$$

This equation is known as McKean equation with one-component and this formulation illuminates the dynamics of systems under the influence of both diffusive processes and sharp, threshold-triggered transitions. Understanding these dynamics through the lens of the McKean equation provides a robust foundation for analyzing complex behaviors in deterministic systems, enriching our theoretical and practical grasp of threshold phenomena (Flores, 1991).

From a mathematical point of view, the McKean equation has a discontinuous right-hand side and this discontinuity introduces significant challenges in obtaining analytical and numerical solutions. Though this model has some analytical approaches, (see for example; McKean and Moll, 1986), it is usually not feasible to obtain analytical solutions, especially for multi-component cases due to nonlinearity and and the inherent complexity of the system. In such scenarios, numerical methods become essential for approximating solutions and understanding the behavior of the model. There are some numerical approaches employed to handle the discontinuities and complexities associated with such equations and some of them particularly need to be mentioned. For example, the study by Karniadakis and Sherwin provides a detailed analysis of the Discontinuous Galerkin (DG) method, demonstrating the effectiveness of the method in capturing sharp discontinuities while maintaining high accuracy (Karniadakis and Sherwin, 2005). However, despite their strengths, DG methods can face issues with computational cost and stability especially when dealing with highly complex systems. Adaptive numerical methods that enhance accuracy and convergence have also shown promise in dealing with such equations but they result in increased computational cost and require sophisticated implementation techniques (Deuflhard and Weiser, 2012). For new methodologies such as the weighted essentially non-oscillatory (WENO) method or Hybridizable Discontinuous Galerkin (HDG) method (Cockburn, Gopalakrishnan and Lazarov, 2009), please refer to the mentioned references. These advanced methods, while effective in handling discontinuities and enhancing accuracy, often come with significant drawbacks such as increased computational costs, stability issues, and the need for sophisticated implementation techniques. These drawbacks motivate us to introduce and employ the Haar wavelet method, which offers a balance between computational efficiency and accuracy, making it a promising alternative for solving equations with discontinuous right-hand sides.

2. Haar Wavelets

These wavelet transforms have been used effectively as a powerful mathematical tool in many scientific areas for instance signal analysis, quantum mechanism, and numerical analysis, just to mention a few. There are different types of wavelet families. However, among them, the Haar wavelet transform is the simplest and oldest introduced by the Hungarian mathematician Alfred Haar in 1910 (Haar, 1910). The

Haar wavelet transform is also considered to be the earliest orthonormal wavelet with compact support, i.e. it vanishes outside of a finite interval and attracts extensive attention due to its simplicity, high accuracy, and small computation cost (Lepik, 2007).

The Haar wavelet family for $x \in [0,1)$ is defined as a piecewise constant "square-shaped" function as follows:

$$h_i(x) = \begin{cases} 1 \text{ for } & x \in [\eta_1, \eta_2), \\ -1 \text{ for } & x \in [\eta_2, \eta_3), \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

where $\eta_1 = \frac{k}{m}$, $\eta_2 = \frac{k+0,5}{m}$, $\eta_3 = \frac{k+1}{m}$. Here, the integer $m = 2^j$ $(j = 0,1, \dots J)$ indicates the level of the wavelet; the integer $k = 0,1, \dots, m-1$ is the translation parameter and the integer J is the maximal level of resolution. The index i is changed via the formula $i = m + k + 1$. For the minimal values of $m = 1$ and $k = 0$, we have $i = 2$ whereas the maximal value for i is $M = 2^m = 2^{J+1}$. As an exceptional consideration, h_1 is assumed to be equal to 1 in $[0,1)$ and 0 elsewhere. Note that h_1 corresponds to scaling function and the Haar wavelet functions are orthogonal to each other

$$\int_0^1 h_\alpha(x) h_\beta(x) \mathrm{d}x = \begin{cases} 2^{-j} & \text{for } \alpha = \beta = 2^j + k, \\ 0 & \text{for } \alpha \neq \beta. \end{cases} \tag{3}$$

Due to its piecewise constant function in nature, Haar is not continuous and the derivatives do not exist at the points of the discontinuities and therefore we cannot directly implement the Haar wavelet to the differential equations. One efficient way of overcoming this challenge is due to the work of Chen and Hsiao, who suggested focusing on the expansion of the highest derivative present in the differential equation using its Haar series (Chen and Hsiao, 1997). This method involves integrating the series expansion a number of times equal to the order of the highest derivative, thereby facilitating the derivation of the solution and all its derivatives through a systematic process of integration so that the solution itself is derived from this wavelet expansion of the highest derivative.

Consequently, any square-integrable function $y(x)$ can be expanded in terms of Haar wavelets with an infinite number of terms as follows:

$$y(x) = \sum_{i=1}^{\infty} c_i h_i(x) , \tag{4}$$

where the Haar coefficients can be obtained using orthogonality identity (3) as an infinite number of terms

$$c_i = 2^i \int_0^1 y(x) h_i \mathrm{d}x , \; c_0 = \int_0^1 y(x) \mathrm{d}x \tag{5}$$

are determined in such a way that the integral square error

$$\int_0^1 [\, y(x) - \sum_{i=1}^{M} c_i h_i(x) \,]^2 \, \mathrm{d}x. \tag{6}$$

Since $y(x)$ has infinite terms if piecewise constant or may be approximated as a piecewise constant during each subinterval, it can be terminated as finite terms

$$y(x) \cong \sum_{i=1}^{M} c_i h_i(x) = c_M^{\mathrm{T}} h_M(x), \tag{7}$$

where the superscript ⊤ represents transposition. The Haar coefficients vector $c_M^\top$ and Haar vector h_M are defined as

$$\begin{aligned} c_M^\top &= [c_1, c_2, \dots, c_M], \\ h_M(x) &= [h_1, h_2, \dots, h_M]^\top. \end{aligned} \tag{8}$$

The integrals of Haar function $h_i(x)$ can be evaluated as

$$\begin{aligned} p_{i,1}(x) &= \int_0^x h_i(x)\mathrm{d}x, \\ p_{i,\nu}(x) &= \int_0^x p_{i,\nu-1}(x)\mathrm{d}x, \quad \nu = 2,3,\dots \end{aligned} \tag{9}$$

Let us define the collocation points $x(l) = \frac{l-0,5}{M}$, $l = 1,2,\dots,M$ and discretize the Haar function $h_i(x)$ so that we get the coefficient matrix $H(i,l) = (h_i(x_l))$ which has dimension $M \times M$, the element of which can be evaluated according to (2). Specifically, the functions $p_{i,1}$ and $p_{i,2}$ are defined as follows:

$$p_{i,1} = \begin{cases} x - \eta_1 & \text{for } x \in [\eta_1, \eta_2), \\ \eta_3 - x & \text{for } x \in [\eta_2, \eta_3), \\ 0 & \text{otherwise,} \end{cases} \quad p_{i,2} = \begin{cases} 0 & \text{for } x \in [0, \eta_1), \\ 0{,}5(x - \eta_1)^2 & \text{for } x \in [\eta_1, \eta_2), \\ \frac{1}{4m^2} - \frac{1}{2}(\eta_3 - x)^2 & \text{for } x \in [\eta_2, \eta_3), \\ \frac{1}{4m^2} & \text{for } x \in [\eta_3, 1]. \end{cases} \tag{10}$$

To facilitate the analysis, let us define the $M \times M$ matrices H, P, and Q. These matrices have the respective entries $H_{ij} = h_i(x_j), P_{ij} = p_{i,1}(x_j)\ and\ Q_{ij} = p_{i,2}(x_j)$, corresponding to the evaluations of the Haar functions and their integrals at specific points. In their study, Chen and Hsiao provided evidence supporting the matrix equation below for determining the matrix P of order M

$$P_{(M)} = \frac{1}{2M}\begin{pmatrix} 2MP_{(M/2)} & -H_{(M/2)} \\ H_{(M/2)}^{-1} & O \end{pmatrix}, \tag{11}$$

where O is a null matrix of order $\frac{M}{2} \times \frac{M}{2}$ (Chen and Hsiao, 1997). After computing the calculations for $P_{(M)}$ and $H_{(M)}$ once, they can subsequently be applied to any kind of differential equations. As H and H^{-1} contain many zeros, Haar transform is even faster than the Fourier transform and even the Walsh transform (Alkan et al, 2019). It should be also noted that calculations for $P_{(M)}$ and $H_{(M)}$ must be carried out once and therefore, it becomes universally applicable to any kind of differential equations.

3. Haar Wavelet on the McKean Equation

The aim of this article is to find the unique, time independent solution of the McKean equation by means of some numerical method in which we implement the Haar wavelet. This approach leverages the simplicity and computational efficiency of the Haar wavelets to effectively address the complexities inherent in the McKean equation. The time independent solution of the McKean equation is called 'critical nucleus' solution and it plays a key role in understanding the threshold phenomenon in one-component excitable media as its stable manifold divides the phase plane into two outcomes: propagating wave solution and resting state solution (Neu et al, 1997). To find the critical nucleus solution of the McKean equation, we consider the original equation (1) with the use of finite interval

$x \in [0, L]$ as an approximation the infinite interval $x \in [0, \infty)$. We then aim to investigate the behavior of the solutions of the equation subject to the following initial and boundary conditions

$$u(x,0) = u_0(x) = U_s\left(1 - \text{erf}\big(\gamma(x - x_s)\big)\right), \frac{\partial u}{\partial x}(0,t) = 0,\ u(L,t) = 0, \forall t > 0, \tag{12}$$

where erf is the error function defined as $\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2} dt$, ensuring a finite rectangular profile of a width x_s and U_s is the amplitude of this profile and γ is a constant parameter. We employ no-flux boundary condition at one end, and Dirichlet boundary condition at the other. To facilitate the use of the Haar wavelets, which are optimally defined in the domain $[0,1]$, we need to apply a domain transformation $y = x/L$. This transformation converts the McKean equation and its associated initial and boundary conditions into

$$\frac{\partial u}{\partial t} = \frac{1}{L^2}\frac{\partial^2 u}{\partial y^2} - u + \mathcal{H}(u - \alpha),\ y \in [0,1],\ t > 0,$$

$$u(y,0) = u_0(Ly),\ \ \frac{\partial u}{\partial y}(0,t) = 0,\ u(1,t) = 0, \forall t > 0. \tag{13}$$

In general, it is assumed that the highest derivatives appearing in any differential equations are expanded into the Haar series. In the transformed McKean equation (13), we use

$$\dot{u}''(y,t) = \sum_{i=1}^{M} c_i h_i(y) = c_M^{\mathsf{T}} h_M(y), \tag{14}$$

where $\cdot$ and $'$ refer differentiation with respect to t and y, respectively. The row vector c_M is constant in the subinterval $t \in [t_s, t_{s+1}]$. Integrating (14) with respect to t from t_s to t and twice with respect to y from 0 to 1 lets us obtain

$$u''(y,t) = (t - t_s)\sum_{i=1}^{M} c_i h_i(y) + u''(y,t_s), \tag{15}$$

$$u'(y,t) = (t - t_s)\sum_{i=1}^{M} c_i p_{i,1}(y) + u'(y,t_s), \text{with } u'(0,t) = u'(0,t_s) = 0 \tag{16}$$

$$u(y,t) = (t - t_s)\sum_{i=1}^{M} c_i p_{i,2}(y) + u(y,t_s) - u(0,t_s) + u(0,t), \tag{17}$$

$$\dot{u}(y,t) = \sum_{i=1}^{M} c_i p_{i,2}(y) + y\dot{u}'(0,t) + \dot{u}(0,t). \tag{18}$$

Applying the boundary condition $u(1,t) = 0$ to the equations at $y = 1$, we have

$$(t - t_s)\sum_{i=1}^{M} c_i p_{i,2}(1) + u(1,t_s) - u(0,t_s) + u(0,t) = 0, \tag{19}$$

$$\dot{u}(1,t) = \sum_{i=1}^{M} c_i p_{i,2}(1) + \dot{u}(0,t). \tag{20}$$

Given the boundary condition $u(1,t) = 0$ and its implication that $\dot{u}(1,t) = 0$, the equations at $y = 1$ simplifies as follows:

$$(t - t_s)\sum_{i=1}^{M} c_i p_{i,2}(1) - u(0,t_s) + u(0,t) = 0, \tag{21}$$

$$\sum_{i=1}^{M} c_i p_{i,2}(1) + \dot{u}(0,t) = 0. \tag{22}$$

These corrections account for the Dirichlet boundary condition at $y = 1$ (or at x= L in the original domain), ensuring that both the function u and its time derivative $\dot{u}$ are zero at this boundary, reflecting a state of equilibrium or a condition where the quantity of interest is neutralized.

Rearranging the equations to express $u(0,t_s) - u(0,t)$ and $\dot{u}(0,t)$ in terms of other variables, we find

$$u(0,t_s) - u(0,t) = (t - t_s)\sum_{i=1}^{M} c_i p_{i,2}(1), \tag{23}$$

$$\dot{u}(0,t) = -\sum_{i=1}^{M} c_i p_{i,2}(1). \tag{24}$$

It follows from (10) that

$$\tilde{p}_{i,2} = p_{i,2}(1) = \begin{cases} 0{,}5 & \text{if } i = 1, \\ \frac{1}{4m^2} & \text{if } i > 1. \end{cases} \tag{25}$$

Substituting these into formulae (15), (17) and (18) and discretizing the results $y \to y_l$ and $t \to t_{s+1}$, we obtain

$$u''(y_l,t_{s+1}) = (t_{s+1} - t_s)\sum_{i=1}^{M} c_i h_i(y_l) + u''(y_l,t_s), \tag{26}$$

$$u(y_l,t_{s+1}) = (t_{s+1} - t_s)\sum_{i=1}^{M} c_i[p_{i,2}(y) - \tilde{p}_{i,2}] + u(y_l,t_s), \tag{27}$$

$$\dot{u}(y_l,t_{s+1}) = \sum_{i=1}^{M} c_i[p_{i,2}(y) - \tilde{p}_{i,2}]. \tag{28}$$

Applying these into (13) yields

$$\sum_{i=1}^{M} c_i[p_{i,2}(y) - \tilde{p}_{i,2}] = \frac{1}{(2L)^2}u''(y_l,t_s) - u(y_l,t_s) + \mathcal{H}(u(y_l,t_s) - \alpha). \tag{29}$$

From this formula, we can successively calculate the wavelet coefficients.

4. Numerical Results

In this section, we will demonstrate the method's effectiveness and how the solutions it produces are not only highly accurate but also it requires minimal computational resources in terms of both time and space. McKean equation with the threshold parameters in the range $0 < \alpha < 1/2$ has the following unique critical nucleus solution found in (McKean and Moll, 1986) as

$$\tilde{u}(x) = \begin{cases} \alpha e^{x+m}, & \text{if } x \le -m, \\ 1 - e^{-m}\cosh(x), & \text{if } x < m, \\ \alpha e^{-x+m}, & \text{if } x \ge m, \end{cases} \tag{30}$$

where $m = -\ln(1-2\alpha)/2$. In their article, they also showed that there exists a threshold surface in the space of initial data that separates initial conditions tending to the equilibrium value from those leading to the excited value. For further details, please see (Bezekci, 2017). This threshold surface has been studied by many researchers and in one of the major studies, Moll and Rosencrans pursued a numerical method to determine this threshold surface with the use of a rectangular initial pulse condition. They then observed if the solution decays to rest or develops into a propagation wave. Adjusting the pulse's amplitude based on the outcome, they iteratively identified the threshold surface as a function of the initial pulse's amplitude and width (Moll and Rosencrans, 1990). The critical nucleus solution of the McKean equation acts as a pivotal threshold, distinguishing between initial conditions that evolve into excited or resting states, fundamentally determining the system's dynamics.

In order to find this critical nucleus numerically, we solve the transformed McKean equation (13) by means of the wavelet transform, which yields equations (26)-(29). This is achieved by fixing x_s and methodically varying U_s via the implementation of a bisection loop, starting with known superthreshold

and subthreshold values. By averaging these, we establish a new amplitude, adjusting based on excitation outcomes to precisely identify critical nucleus parameters. Delving deeper into bisection algorithm and threshold surface is beyond the scope of this study but interested readers may find more information in (Bezekci et al., 2015).

For the numerical simulations, the following parameters are used: $L = 10, \alpha = 0{,}25, \gamma = 20, x_s = 0{,}1$, $J = 8{,}9{,}10$. The parameter U_s needs to be found through the bisection loop using the described numerical method. The initial condition we used and its second derivative are (first derivative is not needed for the computation)

$$u(y,0) = u_0 = U_s\left(1 - \mathrm{erf}\big(\gamma(y - x_s)\big)\right),\ \mathrm{u}''(y,0) = u_0'' = \frac{4\gamma^3 U_s}{\sqrt{\pi}}(y - x_s)\mathrm{e}^{-\gamma^2(y-x_s)^2}.$$

The solution and the second derivative of the solution need to be calculated in every time step. Setting $\Delta_\mathrm{t} = t_{s+1} - t_s$, $R_{ij} = [Qij - \tilde{p}_{i,2}]$, $S_{ij} = (R^{-1})_{ij}$ as these calculations need to be performed just a single time, prior to the time loop's execution, which significantly reduces the overall computation time. Following Algorithm 1 below provides the required steps we employ to solve the McKean equation using Haar wavelet methods. It is important to note that the calculation for the matrices used during computation is outside of the main time loop, which is why this method is rather efficient to use in terms of the simplicity and the computation time required.

Require: Set L, α, γ, x_s, U_s, J, Δ_t as initial inputs
1: Define the matrices H, R, S
2: $u_i \leftarrow u_0$, $u_i'' \leftarrow u_0''$ for $i = 1, \ldots, N$ ▷ set initial condition and its second derivative
3: **for** $t = 1, 2, \ldots$ **do**
4: $b_i \leftarrow \frac{1}{L^2} u_i'' - u_i + \mathcal{H}(u_i - \alpha)$ ▷ Right hand side of equation (29)
5: $c_i \leftarrow b_i \times S$ ▷ wavelet coefficients
6: $u_i'' \leftarrow \Delta_t(c_i \times H) + u_i''$ ▷ update the second derivative
7: $u_i \leftarrow \Delta_t(c_i \times R) + u_i$ ▷ update the solution
8: **end for**

Algorithm 1. Haar wavelet numerical solution of the critical nucleus solution of the McKean equation.

In the analysis of numerical approximation to find the critical nucleus solution of the McKean equation using the Haar wavelet method, the accuracy and convergence of the solutions are of foremost importance. Figure 1 shows the comparison between the numerical solution approximating the critical nucleus of the McKean equation using the Haar wavelet method obtained using varying maximal level of resolutions, J, and the analytical solution described by equation (30). The numerical solutions represented by u_8, u_9 and u_{10} respectively correspond to the cases where $J = 8$, $J = 9$ and $J = 10$, thus indicating finer resolutions. The right panel focuses on the differences between these numerical solutions and the analytical critical nucleus $\tilde{u}(x)$, emphasizing the importance of selecting the right

maximal level of resolution if more accurate approximation is desired. While optimizing the performance of the numerical methods, it is known that fine-tuning numerical methods leads to closer approximation but the computational cost increases.

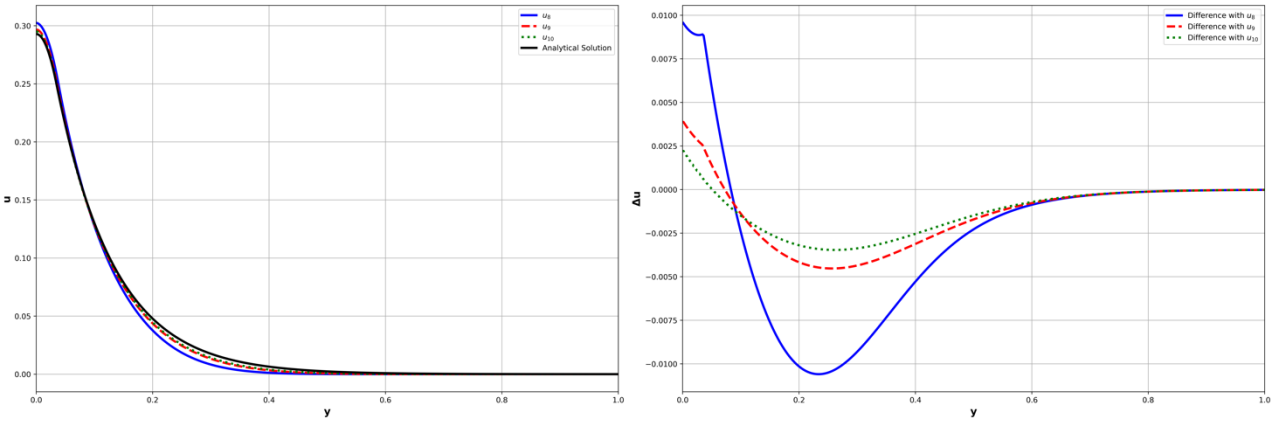

Figure 1. Left panel: Comparison between the numerical solution approximating the critical nucleus of the McKean equation using the Haar wavelet method for different J values and the analytical solution described by equation (30). Right panel: The difference between each numerical solution and the analytical solution.

Further analysis can be performed in terms of the accuracy and two of the most widely-used error metrics are $\mathcal{L}_2$ and $\mathcal{L}_\infty$ norms. Figure 2 graphically illustrates these error metrics for three numerical solutions, allowing us to assess how well the numerical solutions approximate the analytical one on average as well as the significance of the maximum deviations. As emphasized before, increasing the maximum level of resolution tends to improve the accuracy but also causes higher computational costs.

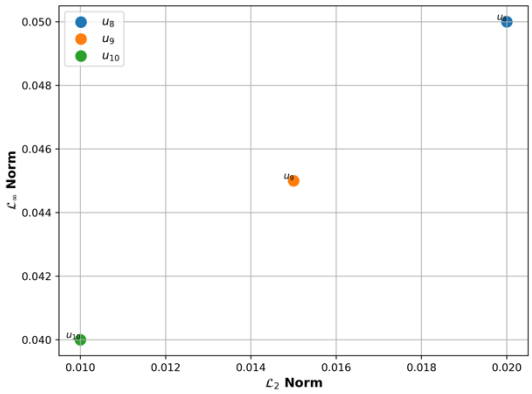

Figure 2. The sketch of least square deviations $\mathcal{L}_2$ and maximum deviations $\mathcal{L}_\infty$ norms for the numerical solutions.

5. Conclusion and Further Works

The main aim of this article is to propose the Haar wavelet-based numerical method for obtaining time-independent, also known as critical nucleus, solution of the McKean equation with discontinuous heaviside step function. The Haar wavelets are characterized by their compact support and orthogonality properties, making them especially suitable for numerical solutions of differential equations with

discontinuous or sharp transitions. This motivated us to employ them in tackling the McKean equation with the aim of handling the complexity introduced by its nonlinearity. Our approach successively finds numerical critical nucleus solutions that closely align with their analytical counterparts.

Despite the McKean equation having its own analytical solution, obtaining such solution in a closed form is more likely uncommon. For more complex scenarios, analytical approaches are difficult to establish and the use of numerical methods become essential. This challenge motivates us to employ the Haar wavelet method, which has shown the capability to approximate differential equations effectively. In adopting this methodology, we validate the effectiveness of the Haar wavelet-based methods in accurately solving the reaction-diffusion equations, demonstrating its utility in addressing more complex mathematical challenges.

This article investigated the Haar wavelet approach in one-dimensional, one-component spatially extended excitable media, specifically the McKean equation. It could be of interest for further researches to investigate the effectiveness of the method in a wider range of problem settings, including more complex reaction-diffusion equations and higher-dimensional problems, and to perform a comparative analysis with other established numerical methods mentioned above. This would help to evaluate the Haar wavelet method's relative strengths and weaknesses, providing a clearer understanding of its performance in various discontinuous and nonlinear scenarios.

Statement of Conflict of Interest

The author has declared no conflict of interest.

Author's Contributions

The author contributed to this manuscript 100%.

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