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W-Coatomic Modules

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Özet. Bu makalede w-koatomik modül tanımı verildi ve özellikleri incelendi. W-koatomik modüllerin sonlu toplamının da w-koatomik olduğu ispatlandı. W-koatomik modüllerin genişleme altında kapalı olduğu gösterildi. Ayrıca indirgenmiş bir M modülünde her yarıbasit alt modülü bir tümleyene sahipse, M modülünün w-koatomik olduğu ispatlandı.

Anahtar Kelimeler. Koatomik modül, w-koatomik modül, yarıbasit modül.

Abstract. In this paper we introduce w-coatomic modules and some of their properties. We prove that the finite sum of w-coatomic modules is w-coatomic. It is shown that w-coatomic modules are closed under extensions. It is proved that if an R-module M is reduced and every proper semisimple submodule N of M has a supplement in M, then M is w-coatomic.

Keywords. Coatomic modules, w-coatomic modules, semisimple modules.

1. Introduction

Throughout this note, we assume that R is an associative ring with unity and all modules are unital left R-modules, unless otherwise stated. Let R be a ring and M be an R-module. $\mathrm{Rad}(M)$ and $\mathrm{Soc}(M)$ will denote the Jacobson radical and socle of M, respectively. A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M, equivalently, for a submodule N of M, whenever $\mathrm{Rad}\,(M/N) = M/N$, then $M = N$; see [1, 2, 3]. A module M is said to be *semisimple* if every submodule of M is a direct summand in M. Semisimple modules, finitely generated modules, hollow modules and local modules are coatomic modules. A module M is called *finitely coatomic*, or simply *f-coatomic*, if every finitely generated proper submodule of M is contained in a maximal submodule of

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M; see [4]. We say that a module M is *w-coatomic*, if every nonzero proper semisimple submodule of M is contained in a maximal submodule of M or equivalently, for a semisimple submodule N of M, if $\mathrm{Rad}\,(M/N) = M/N$, then $M = N$.

In this paper, we will assume that M has always a nonzero socle, that is, M contains a nonzero simple submodule. Otherwise, if a module M has no nonzero simple submodule, then it has only 0 as a semisimple submodule. Thus, whenever $\mathrm{Rad}\,(M/N) = M/N$ where $N = 0$, then $M = N = 0$. So, M has no maximal submodules. Thus, M cannot be w-coatomic.

Clearly, any coatomic module which has a nonzero socle is w-coatomic but the converse is not true.

Example 1.1. Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rational integers, respectively. Let us consider the $\mathbb{Z}$-module $M = (\mathbb{Z}/8\mathbb{Z}) \oplus \mathbb{Q}$. Then $\mathrm{Soc}(M)$ is nonzero, because $\mathrm{Soc}(\mathbb{Z}/8\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Soc}(\mathbb{Q}) = 0$. So, every proper semisimple submodule of M is of the form $K \oplus 0$ where K is a proper semisimple submodule of $\mathbb{Z}/8\mathbb{Z}$. Note that $K = \mathbb{Z}/2\mathbb{Z}$ and K is contained in the maximal submodule $\langle 2 + 8\mathbb{Z}\rangle$ of $\mathbb{Z}/8\mathbb{Z}$. So, $K \oplus 0$ is contained in the maximal submodule $\langle 2 + 8\mathbb{Z}\rangle \oplus \mathbb{Q}$. We claim that M is not coatomic, because otherwise by [1, Lemma 4], $\mathbb{Q}$ must be coatomic. This is impossible since $\mathbb{Q}$ has no maximal submodule, that is, $\mathbb{Q}$ is not coatomic, which is a contradiction. Thus M is not coatomic.

2. Properties of W-Coatomic Modules

In [1, Corollary 5], it is proved that any finite direct sum of coatomic modules is coatomic. We show that this property holds also for w-coatomic modules.

Proposition 2.1. *Let $M = M_1 + M_2$. If M_1 and M_2 are w-coatomic, then so is M.*

Proof. Let U be a proper semisimple submodule of M. Let us consider $U \cap M_1$. If $U \cap M_1 = M_1$, then $M_1 \subseteq U$ and so M_1 is semisimple. If $U \cap M_2 = M_2$, then M_2 is semisimple. Thus $M = M_1 + M_2$ is semisimple and so M is w-coatomic. In case $M_1 \subseteq U$ and $U \cap M_2 \neq M_2$, since M_2 is w-coatomic, there exists a maximal submodule K_2 of M_2 such that $U \cap M_2 \subseteq K_2$. Clearly, $M_1 + K_2$ is a maximal submodule of $M_1 + M_2$ and $U \subseteq M_1 + K_2$ because every member of U is either in M_1 or in $U \cap M_2$. If $U \cap M_1 \neq M_1$, then M_1 has a maximal submodule K_1 such that $U \cap M_1 \subseteq K_1$. Thus $K_1 + M_2$ is maximal in $M_1 + M_2$, so $U \subseteq K_1 + M_2$, because parts of U that are included in M_1 are in K_1 and parts of U staying in M_2 are in M_2. □

Corollary 2.2. *Any finite direct sum of w-coatomic modules is w-coatomic.*

Lemma 2.3. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a short exact sequence. If both L and N are w-coatomic, then so is M.

Proof. Let U be a proper semisimple submodule of M. Let's consider N as M/L. Then $(U+L)/L$ is a semisimple submodule of M/L. Suppose that $(U+L)/L = M/L$, then $M = U + L$. Since every semisimple module and L are w-coatomic, M is w-coatomic, by Proposition 2.1. Let $(U+L)/L$ be a proper submodule of M/L. Since M/L is w-coatomic, there exists a maximal submodule K/L in M/L such that $(U+L)/L \subseteq K/L$ for some submodule K of M containing L. Thus, K is a maximal submodule in M containing U. □

Unfortunately, no factor module of a w-coatomic module is w-coatomic.

Example 2.4. Let $\mathbb{Z}$ be the set of integers. Let $M = \mathbb{Z} \oplus \mathbb{Z}_{(p^\infty)}$, where $\mathbb{Z}_{(p^\infty)}$ is the Prüfer p-group for any prime p. Since $\mathrm{Soc}(\mathbb{Z}) = 0$ and $\mathrm{Soc}(\mathbb{Z}_{(p^\infty)}) \cong \mathbb{Z}/p\mathbb{Z}$, then $\mathrm{Soc}(M) \neq 0$. M is w-coatomic, because $p\mathbb{Z} \oplus \mathbb{Z}_{(p^\infty)}$ is maximal in M and every proper semisimple submodule of M is contained in $p\mathbb{Z} \oplus \mathbb{Z}_{(p^\infty)}$. But $M/\mathbb{Z} \cong \mathbb{Z}_{(p^\infty)}$ is not w-coatomic, since $\mathbb{Z}_{(p^\infty)}$ has no maximal submodule.

A ring R is a left V-ring if each simple left R-module is injective; see [5].

Proposition 2.5. *Let R be a left V-ring and let M be w-coatomic. Then M/N is w-coatomic for a simple submodule N of M.*

Proof. Let L/N be a proper semisimple submodule of M/N and let $\mathrm{Rad}(M/L) = M/L$. Obviously, $\mathrm{Soc}(L/N) = L/N$. Since R is a left V-ring and N is simple submodule of M, then N is an injective submodule in M, and it is a direct summand in L. By [6, Ch. 9, Exercise 12(a)], $\mathrm{Soc}(L/N) = (\mathrm{Soc}(L) + N)/N$, and it follows that $L/N = (\mathrm{Soc}(L) + N)/N$. Thus $L = \mathrm{Soc}(L) + N$. Then L is a semisimple submodule in M, because N is a simple submodule. By assumption $M = L$. □

Let M be an R-module and U, V submodules of M. We say that V is a *supplement* of U in M, if it is minimal with respect to the property $U + V = M$ or equivalently, $M = U + V$ and $U \cap V \ll V$; see [7, Ch. 8, Sec. 41]. M is called *supplemented*, if every submodule of M has a supplement in M; see [7, Ch. 8, Sec. 41].

Proposition 2.6. *Let M be an R-module, U be a proper semisimple submodule of M and V be a supplement of U in M. Then M is w-coatomic if and only if V is w-coatomic.*

Proof. ($\Rightarrow$) Let M be w-coatomic and U be a proper semisimple submodule of M. Let V be a supplement of U, then $M = U + V$ and $U \cap V \ll V$. Because U is semisimple in M, $U = (U \cap V) \oplus U'$ for some submodule U' of U. Then it follows that $M = U + V = (U \cap V) + U' + V$. Therefore $M = U' + V$. Since $0 = (U \cap V) \cap U' = V \cap U'$, then $M = U' \oplus V$. Let $\mathrm{Rad}(V/V') = V/V'$ for a semisimple submodule V' of V. Then $M/(U' \oplus V') = (U' \oplus V)/(U' \oplus V') \cong V/V'$ is a radical module, that is, $\mathrm{Rad}(M/(U' \oplus V')) = M/(U' \oplus V')$. Since U' and V' are semisimple submodules of M, so is $U' \oplus V'$. By assumption, $M/(U' \oplus V') = 0$, that is, $M = U' \oplus V'$. By minimality of V, $V = V'$. Hence V is w-coatomic.

($\Leftarrow$) Let V be a supplement of U. Then $M = U + V$ and $U \cap V \ll V$. Since U is semisimple submodule in M and V is w-coatomic, by Proposition 2.1, M is w-coatomic. $\square$

Lemma 2.7. *Let M be an R-module with* $\mathrm{Rad}(M)$ *w-coatomic. If every semisimple submodule has a supplement in M, then M is w-coatomic.*

Proof. Let U be a proper semisimple submodule of M and let $\mathrm{Rad}(M/U) = M/U$. By assumption, U has a supplement V in M, that is, $M = U + V$ and $U \cap V \ll V$. Since U is a semisimple submodule in M, $U = (U \cap V) \oplus U'$ for some submodule U' of U. It follows that $M = U + V = (U \cap V) + U' + V$ and so $M = U' + V$. Because $0 = (U \cap V) \cap U' = V \cap U'$, then $M = U' \oplus V$. By [8, Lemma 1.2], $\mathrm{Rad}(M/U) = (\mathrm{Rad}(M) + U)/U$. So $(\mathrm{Rad}(M) + U)/U = M/U$. It implies $M = \mathrm{Rad}(M) + U$. Since $\mathrm{Rad}(M)$ and U are w-coatomic, M is w-coatomic by Proposition 2.1. $\square$

The proof of the following corollary follows from Lemma 2.3 and Proposition 2.6.

Corollary 2.8. *Let M be an R-module, U be a proper semisimple submodule of M and V be a supplement of U in M. If M/U is w-coatomic, then V is w-coatomic.*

We need some lemmas in order to prove that M is a w-coatomic module if M is reduced and every proper semisimple submodule has a supplement in M.

Lemma 2.9. *Let M be a module and U be a proper semisimple submodule of M contained in* $\mathrm{Rad}(M)$. *Then U is small in M.*

Proof. Let U be a submodule of $\mathrm{Rad}(M)$ where U is semisimple in M. By [6, Corollary 9.1.5], $\mathrm{Soc}(U) \subseteq \mathrm{Soc}(\mathrm{Rad}(M))$. Since U is semisimple, then $\mathrm{Soc}(U) = U$. Thus $U \subseteq \mathrm{Soc}(\mathrm{Rad}(M))$. By [9, Ch. 1, Sec. 2.8(9)], $\mathrm{Soc}(\mathrm{Rad}(M)) \ll M$. Then it follows that $U \ll M$. $\square$

Lemma 2.10. *Let $M/\mathrm{Rad}(M)$ be semisimple, then every semisimple submodule of M has a supplement.*

Proof. Let N be a proper semisimple submodule of M. Then $(N+\mathrm{Rad}(M))/\mathrm{Rad}(M)$ is a semisimple submodule of $M/\mathrm{Rad}(M)$. By assumption, $(N+\mathrm{Rad}(M))/\mathrm{Rad}(M)$ is direct summand in $M/\mathrm{Rad}(M)$, that is,

$$\frac{M}{\mathrm{Rad}(M)} = \frac{N+\mathrm{Rad}(M)}{\mathrm{Rad}(M)} \oplus \frac{N'}{\mathrm{Rad}(M)}$$

for some submodule N' of M. Then $M = N + N'$ and $N \cap N' \subseteq \mathrm{Rad}(M)$. Since $N \cap N'$ is semisimple, by Lemma 2.9, $N \cap N' \ll M$. $N = (N \cap N') \oplus K$ for some submodule K of N because N is semisimple. Thus $M = N+N'$ implies $M = N'+K$. Then $N' \cap K = 0$ follows from $(N \cap N') \cap K = N' \cap K = 0$. That is, $M = N' \oplus K$. If $N' \cap N \ll M$, by [7, Ch. 3, Sec. 19.3], $N \cap N' \ll N'$. Hence N has a supplement N' in M. $\square$

We say that a module M is *reduced* if the only radical submodule of M is the zero module; see [8].

Proposition 2.11. *Let M be a reduced module. If every semisimple submodule of M has a supplement in M, then M is w-coatomic.*

Proof. Let N be a proper semisimple submodule of M and let $\mathrm{Rad}\,(M/N) = M/N$. By assumption, N has a supplement K in M. That is, $M = N+K$ and $N\cap K \ll K$. Then $M = N' \oplus K$ for some submodule N' of N because N is semisimple. By [7, Ch. 4, Sec. 21.6], $\mathrm{Rad}(M) = \mathrm{Rad}(K) \leq K$. By [8, Lemma 1.2], $\mathrm{Rad}\,(M/N) = (\mathrm{Rad}(M)+N)/N$, then $M/N = (\mathrm{Rad}(M)+N)/N$. It follows that $M = \mathrm{Rad}(M)+N$. By [7, Ch. 8, Sec. 41], $K = \mathrm{Rad}(M) = \mathrm{Rad}(K)$. Since M is reduced, $K = 0$. Thus $M = N$. $\square$

Proposition 2.12. *Let $\mathrm{Rad}(M) \ll M$. If every semisimple submodule of M has a supplement in M, then M is w-coatomic.*

Proof. Let N be a proper semisimple submodule of M and let $\mathrm{Rad}\,(M/N) = M/N$. By assumption, N has a supplement in M. Therefore $\mathrm{Rad}\,(M/N) =$

$(\mathrm{Rad}(M)+N)/N$, by [8, Lemma 1.2]. It implies $M/N=(\mathrm{Rad}(M)+N)/N$, that is, $M=\mathrm{Rad}(M)+N$. Then $M=N$ because $\mathrm{Rad}(M)\ll M$. □

We also have the following corollary by Lemma 2.10 and Proposition 2.11.

Corollary 2.13. *Let R be a discrete valuation ring and M be a reduced R-module. Then M is w-coatomic.*

In [1, Lemma 3], Güngöroğlu has proved that every submodule of a coatomic module is coatomic over a discrete valuation ring. We proved the same lemma under a weaker condition.

Lemma 2.14. *Let R be a discrete valuation ring. Every submodule of* $\mathrm{Rad}(M)$ *is w-coatomic if and only if every submodule U of M is w-coatomic.*

Proof. $(\Rightarrow)$ Let U be a submodule of M and N be a semisimple submodule of U. Let $\mathrm{Rad}\,(U/N)=U/N$. We have by the isomorphism theorem and the modular law:

$$\frac{U+\mathrm{Rad}(M)}{N+\mathrm{Rad}(M)}=\frac{U+(N+\mathrm{Rad}(M))}{N+\mathrm{Rad}(M)}\cong\frac{U}{U\cap(N+\mathrm{Rad}(M))}\cong\frac{U}{N+(U\cap\mathrm{Rad}(M))}.$$

We claim that $(U+\mathrm{Rad}(M))/(N+\mathrm{Rad}(M))$ does not have a maximal submodule. Suppose that $(U+\mathrm{Rad}(M))/(N+\mathrm{Rad}(M))$ has a maximal submodule. By the above isomorphisms, there is a maximal submodule K of U containing $N+(U\cap\mathrm{Rad}(M))$. Then $K/(N+(U\cap\mathrm{Rad}(M)))$ is a maximal submodule in $U/(N+(U\cap\mathrm{Rad}(M)))$. So K/N is a maximal submodule in U/N. This is impossible since $\mathrm{Rad}\,(U/N)=U/N$, a contradiction. So, for $U_1=U+\mathrm{Rad}(M)$ and $N_1=N+\mathrm{Rad}(M)$, we have $\mathrm{Rad}\,(U_1/N_1)=U_1/N_1$. Since R is a discrete valuation ring, then $M/\mathrm{Rad}(M)$ is semisimple. So $N_1/\mathrm{Rad}(M)$ is a direct summand in $M/\mathrm{Rad}(M)$, that is, for submodule K_1 of M,

$$\frac{M}{\mathrm{Rad}(M)}=\frac{N_1}{\mathrm{Rad}(M)}\oplus\frac{K_1}{\mathrm{Rad}(M)}.$$

Then $M=N_1+K_1$ and $N_1\cap K_1=\mathrm{Rad}(M)$. Since

$$\frac{U_1}{N_1}=\mathrm{Rad}\left(\frac{U_1}{N_1}\right)\subseteq\mathrm{Rad}\left(\frac{M}{N_1}\right)\cong\mathrm{Rad}\left(\frac{K_1}{\mathrm{Rad}(M)}\right)\subseteq\mathrm{Rad}\left(\frac{M}{\mathrm{Rad}(M)}\right)=0,$$

then $U_1=N_1$. Hence $U+\mathrm{Rad}(M)=N+\mathrm{Rad}(M)$ and so $U=N+(U\cap\mathrm{Rad}(M))$. From

$$\mathrm{Rad}\left(\frac{U}{N}\right)=\frac{U}{N}=\frac{N+(U\cap\mathrm{Rad}(M))}{N}\cong\frac{U\cap\mathrm{Rad}(M)}{N\cap\mathrm{Rad}(M)},$$

it follows that

$$\frac{U \cap \operatorname{Rad}(M)}{N \cap \operatorname{Rad}(M)} = \operatorname{Rad}\left(\frac{U \cap \operatorname{Rad}(M)}{N \cap \operatorname{Rad}(M)}\right).$$

Since $U \cap \operatorname{Rad}(M)$ is a submodule of $\operatorname{Rad}(M)$, by assumption $U \cap \operatorname{Rad}(M)$ is w-coatomic. Since $N \cap \operatorname{Rad}(M)$ is a submodule of N, then $N \cap \operatorname{Rad}(M)$ is a semisimple submodule of $U \cap \operatorname{Rad}(M)$. Therefore $U \cap \operatorname{Rad}(M) = N \cap \operatorname{Rad}(M)$. Then $U = N$. Thus U is w-coatomic.

($\Leftarrow$) Let U be a submodule of $\operatorname{Rad}(M)$. Then U is also a submodule of M. By assumption, U is w-coatomic. $\square$

3. W-Local and W-Coatomic Modules

In [10], the authors defined w-local module as follows: A module M is called *w-local* if it has a unique maximal submodule. It is clear that a module is w-local if and only if its radical is maximal.

An example is given in order to show that not every w-local module is w-coatomic and vice versa.

Example 3.1. Let $M = \mathbb{Q} \oplus (\mathbb{Z}/p\mathbb{Z})$ be an abelian group for any prime p. Then $J = \mathbb{Q} \oplus 0$ is the unique maximal submodule of M. Thus M is w-local. $K = 0 \oplus (\mathbb{Z}/p\mathbb{Z})$ is a proper semisimple submodule of M and since K is not contained in J, then M is not w-coatomic. Conversely, let's consider the abelian group $M = \mathbb{Z} \oplus \mathbb{Z}_{(p^\infty)}$ for any prime p. By Example 2.4, M is w-coatomic. $p\mathbb{Z} \oplus \mathbb{Z}_{(p^\infty)}$ is the maximal submodule of M. But $p\mathbb{Z} \oplus \mathbb{Z}_{(p^\infty)}$ is not unique in M. So M is not w-local.

We need an extra property to give the relationship between w-local and w-coatomic modules.

Proposition 3.2. *If M is w-local and reduced, then M is w-coatomic.*

Proof. Let N be a proper semisimple submodule of M. Since M is w-local, then its radical is maximal. So $M/\operatorname{Rad}(M)$ is semisimple. By Lemma 2.10, every semisimple submodule of M has a supplement in M, then by Proposition 2.11, M is w-coatomic, because M is reduced. $\square$

Proposition 3.3. *Let M be a w-local module. If $\operatorname{Rad}(M) \ll M$, then M is w-coatomic.*

Proof. If M is w-local, then $M/\operatorname{Rad}(M)$ is semisimple. By Lemma 2.10 and Proposition 2.12, M is w-coatomic. $\square$

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