

Complex Solutions for the Fisher Equation and the Benjamin-Bona-Mahony Equation

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Özet. Bu çalışmada Fisher ve Benjamin-Bona-Mahony denklemlerinin karmaşık çözümleri için direkt cebirsel metodu sunulur. Bu metot kullanılarak Fisher ve Benjamin-Bona-Mahony denklemlerinin bazı karmaşık çözümleri elde edilir.

Anahtar Kelimeler. Fisher denklemi, Benjamin-Bona-Mahony denklemi, direkt cebirsel metot, karmaşık çözümler, hareket eden dalga çözümler.

Abstract. In this article, we give direct algebraic method for the complex solutions of the Fisher equation and Benjamin-Bona-Mahony equation. We get some complex solutions of the Fisher equation and Benjamin-Bona-Mahony equation by this method.

Keywords. Fisher equation, Benjamin-Bona-Mahony equation, direct algebraic method, complex solutions, traveling wave solutions.

1. Introduction

The theory of nonlinear dispersive wave motion is an interesting area investigated in numerous articles in which it appears in relation to various subjects. We do not attempt to characterize the general form of nonlinear dispersive wave equations [1, 2]. These studies for nonlinear partial differential equations have attracted much attention in mathematical physics and play a crucial role in applied mathematics. Furthermore, when an original nonlinear equation is directly calculated, the solution will be in accord with the physical characteristics of the actual phenomena [3]. Explicit solutions to the nonlinear equations are of fundamental importance. Also different methods for acquiring explicit solutions to nonlinear evolution equations have been suggested. Many analytical and numerical methods have been established in [4-27]. We may list such examples as the generalized Miura transformation, Darboux transformation, Cole-Hopf transformation, Hirota's dependent variable transformation, the inverse scattering transform and the Bäcklund transformation, the

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tanh method, sine-cosine method, Painlevé method, homogeneous balance method, similarity reduction method, improved tanh method, etc. In [12], Parkes and Duffy have recently constructed an automated tanh-function method. The authors present a Mathematica package that is concerned with complicated algebra and outputs directly the required solutions for particular nonlinear equations. In this paper, our aim is to find exact solutions of nonlinear PDE's, especially complex solutions.

In this article, the first section presents the scope of the study as an introduction. The second section contains an analysis of the method given in [24]. In the third section, we apply the method given in [24] to the nonlinear Fisher equation and the Benjamin-Bona-Mahony equation. In the last section, we present the conclusion.

2. An Analysis of the Method and Applications

Firstly, we will give a simple description of the direct algebraic method [24]. For this, one can consider the general form of the nonlinear PDE in two variables

$$Q(u, u_t, u_x, u_{xx}, \dots) = 0, \quad (1)$$

and transform (1) with $u(x, t) = u(\xi)$, $\xi = ik(x - ct)$, where k and c are real constants. After transformation, we get a nonlinear ODE for $u(\xi)$

$$Q'(u, -ikcu', iku', -k^2u'', \dots) = 0, \quad (2)$$

where $u' = \frac{du}{d\xi}$.

The solution of (2) we are looking for is expressed in the form

$$u(\xi) = \sum_{m=0}^n a_m F^m(\xi), \quad (3)$$

where $\xi = ik(x - ct)$ (where k and c are real constants), n is a positive integer that can be determined by balancing the highest order derivative with the highest nonlinear terms in the equation, and a_m and ξ can be determined. Substituting (3) into (2) yields a set of algebraic equations for F^m , ($m = 0, 1, 2, \dots$), then all coefficients of F^m will vanish. After this separated algebraic equation, we find the coefficients a_0, a_m and ξ . $F(\xi)$ expresses the solution of the auxiliary ordinary differential equation

$$F'(\xi) = \alpha + F^2(\xi),$$

where $F' = \frac{dF}{d\xi}$ and α is a constant. Some solutions were given in [24].

In this work, we will consider complex solutions of the Fisher equation and the Benjamin-Bona-Mahony equation by using the direct algebraic method which is introduced by Zhang [24].

3. Applications

Example 1. Consider Fisher equation,

$$u_t + u_{xx} - u + u^3 = 0. \tag{4}$$

For this example, we can use transformation with (1) and then (4) becomes

$$-ikcu' - k^2u'' - u + u^3 = 0. \tag{5}$$

Balancing u^3 with u'' gives $m = 1$. Therefore, we may choose

$$u = a_0 + a_1F. \tag{6}$$

Substituting (6) into (5) yields a set of algebraic equations for a_0, a_1, a_2, k, c and α . These systems are found to be

$$\begin{aligned} -a_0 + a_0^3 - ia_1ck\alpha &= 0, \\ -a_1 + 3a_0^2a_1 - 2a_1k^2\alpha &= 0, \\ 3a_0a_1^2 - ia_1ck &= 0, \\ a_1^3 - 2a_1k^2 &= 0. \end{aligned}$$

From the solutions of the system, we have the following two cases:

Case 1.

$$a_0 = -\frac{1}{2}, a_1 = -\frac{i}{2\sqrt{\alpha}}, c = -\frac{3i}{\sqrt{2}}, k = \frac{i}{2\sqrt{2}\sqrt{\alpha}}, \alpha \neq 0. \tag{7}$$

Case 2.

$$a_0 = \frac{1}{2}, a_1 = \frac{i}{2\sqrt{\alpha}}, c = \frac{3i}{\sqrt{2}}, k = \frac{i}{2\sqrt{2}\sqrt{\alpha}}, \alpha \neq 0. \tag{8}$$

By means of Mathematica, substituting (7) and (8) into (6), we have obtained the following exact complex traveling wave solutions of (4). These solutions are as follows:

Family 1.

$$u_1 = -\frac{1}{2} - \frac{i}{2\sqrt{\alpha}} \left(-\sqrt{-\alpha} \tanh \left(\sqrt{-\alpha} \left(-\frac{1}{2\sqrt{2}\sqrt{\alpha}} \right) \left(x + \frac{3i}{\sqrt{2}}t \right) \right) \right),$$

where $\alpha < 0$,

$$u_2 = -\frac{1}{2} - \frac{i}{2\sqrt{\alpha}} \left(-\sqrt{-\alpha} \coth \left(\sqrt{-\alpha} \left(-\frac{1}{2\sqrt{2}\sqrt{\alpha}} \right) \left(x + \frac{3i}{\sqrt{2}}t \right) \right) \right),$$

where $\alpha < 0$,

$$u_3 = -\frac{1}{2} - \frac{i}{2\sqrt{\alpha}} \left(\sqrt{\alpha} \tan \left(-\frac{1}{2\sqrt{2}} \left(x + \frac{3i}{\sqrt{2}}t \right) \right) \right),$$

where $\alpha > 0$,

$$u_4 = -\frac{1}{2} - \frac{i}{2\sqrt{\alpha}} \left(-\sqrt{\alpha} \cot \left(-\frac{1}{2\sqrt{2}} \left(x + \frac{3i}{\sqrt{2}}t \right) \right) \right),$$

where $\alpha > 0$.

Family 2.

$$u_5 = \frac{1}{2} + \frac{i}{2\sqrt{\alpha}} \left(-\sqrt{-\alpha} \tanh \left(\sqrt{-\alpha} \left(-\frac{1}{2\sqrt{2}\sqrt{\alpha}} \right) \left(x - \frac{3i}{\sqrt{2}}t \right) \right) \right),$$

where $\alpha < 0$,

$$u_6 = \frac{1}{2} + \frac{i}{2\sqrt{\alpha}} \left(\sqrt{-\alpha} \coth \left(\sqrt{-\alpha} \left(-\frac{1}{2\sqrt{2}\sqrt{\alpha}} \right) \left(x - \frac{3i}{\sqrt{2}}t \right) \right) \right),$$

where $\alpha < 0$,

$$u_7 = \frac{1}{2} + \frac{i}{2\sqrt{\alpha}} \left(\sqrt{\alpha} \tan \left(-\frac{1}{2\sqrt{2}} \left(x - \frac{3i}{\sqrt{2}}t \right) \right) \right),$$

where $\alpha > 0$,

$$u_8 = \frac{1}{2} + \frac{i}{2\sqrt{\alpha}} \left(-\sqrt{\alpha} \cot \left(-\frac{1}{2\sqrt{2}} \left(x - \frac{3i}{\sqrt{2}}t \right) \right) \right),$$

where $\alpha > 0$.

Example 2. Consider the Benjamin-Bona-Mahony equation,

$$u_t + u_x + uu_x - u_{xxt} = 0. \quad (9)$$

For this example, if we use transformation with (1), then (9) becomes

$$-ikcu' + iku' + ikuu' + i^3k^3cu''' = 0,$$

or equivalently,

$$u'(1 - c) + uu' - k^2cu''' = 0. \quad (10)$$

Balancing uu' with u''' gives $m = 2$. Therefore, we may choose

$$u = a_0 + a_1F + a_2F^2. \tag{11}$$

Substituting (11) into (10) yields a set of algebraic equations for a_0, a_1, a_2, k, c and α . These systems are found to be

$$\begin{aligned} a_1 + a_0a_1 - a_1c - 2a_1ck^2\alpha &= 0, \\ a_1^2 + 2a_2 + 2a_0a_2 - 2a_2c - 16a_2ck^2\alpha &= 0, \\ a_1 + a_0a_1 - a_1c + 3a_1a_2\alpha - 8a_1ck^2\alpha &= 0, \\ a_1^2 + 2a_2 + 2a_0a_2 - 2a_2c + 2a_2^2\alpha - 40a_2ck^2\alpha &= 0, \\ 3a_1a_2 - 6a_1ck^2 &= 0, \\ 2a_2^2 - 24a_2ck^2 &= 0. \end{aligned}$$

From the solutions of the system, we have the following two cases:

Case 1.

$$a_0 = -1 + c + 8ck^2\alpha, \quad a_1 = 0, \quad a_2 = 12ck^2, \quad ck \neq 0, \quad \alpha \neq 0. \tag{12}$$

Case 2.

$$a_0 = -1 + c, \quad a_1 = 0, \quad a_2 = 12ck^2, \quad \alpha = 0. \tag{13}$$

By means of Mathematica, substituting (12), (13) into (11), we have obtained the following exact complex traveling wave solutions of (9). These solutions are as follows:

Family 1.

$$u_1 = -1 + c + 8ck^2\alpha + 12ck^2 \left(-\sqrt{-\alpha} \tanh(\sqrt{-\alpha}ik(x - ct)) \right)^2,$$

where $\alpha < 0$,

$$u_2 = -1 + c + 8ck^2\alpha + 12ck^2 \left(-\sqrt{-\alpha} \coth(\sqrt{-\alpha}ik(x - ct)) \right)^2,$$

where $\alpha < 0$,

$$u_3 = -1 + c + 8ck^2\alpha + 12ck^2 \left(\sqrt{\alpha} \tan(\sqrt{\alpha}ik(x - ct)) \right)^2,$$

where $\alpha > 0$,

$$u_4 = -1 + c + 8ck^2\alpha + 12ck^2 \left(-\sqrt{\alpha} \cot \left(\sqrt{\alpha} ik(x - ct) \right) \right)^2,$$

where $\alpha > 0$.

Family 2.

$$u_5 = -\frac{1}{ik(x - ct)},$$

where $\alpha = 0$.

4. Conclusion

In this paper, we implement a direct algebraic method [24] with symbolic computation to construct new exact complex solutions of the Fisher equation and the Benjamin-Bona-Mahony equation. The method can be used for many other nonlinear equations. In addition, this method is computerizable, which allows us to perform complicated and tedious algebraic calculations on a computer.

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