CONSTRUCTIVE MATHEMATICAL ANALYSIS
7 (2024), No. 2, pp. 69-76
http://dergipark.org.tr/en/pub/cma
ISSN 2651 - 2939

Research Article

Convergence estimates for some composition operators

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ABSTRACT. There are different methods available in literature to construct a new operator. One of the methods to construct an operator is the composition method. It is known that Baskakov operators can be achieved by composition of Post Widder P_n and Szász-Mirakjan S_n operators in that order, which is a discretely defined operator. But when we consider different order composition namely $S_n \circ P_n$, we get another different operator. Here we study such and we establish some convergence estimates for the composition operators $S_n \circ P_n$, along with difference with other operators. Finally, we found the difference between two compositions by considering numeric values.

Keywords: Szász operators, Post-Widder operators, moment generating function, convergence.

2020 Mathematics Subject Classification: 41A25, 41A30.

1. SZÁSZ-MIRAKJAN AND POST-WIDDER COMPOSITION

In the last few decades, many new operators have been introduced by the researchers using different methods, some were generalizations of existing operators while some using generating functions, we mention here some of the recent studies [2, 3, 4, 6, 7, 9, 14, 18, 22] etc. . Here, we discuss a composition method to achieve a new operator. The present article is continuation in series of earlier recent papers [1, 15, 16]. The composition of Post-Widder operators and the Szász operators, i.e. $(P_n \circ S_n)$ provide us the Baskakov operators V_n (see [17]) in that order. But, when we change the order of composition it is not necessary to have same operator. Here, we discuss reverse order composition. The Szász-Mirakjan operators are given as follows:

$$(S_n f)(x) = \sum_{k=0}^{\infty} s_k(nx) f\left(\frac{k}{n}\right), \quad x \geq 0$$

where $s_k(nx) = e^{-nx}\frac{(nx)^k}{k!}$. The Post Widder operator is defined as

$$(P_n f)(x) = \frac{n^n}{x^n \Gamma(n)} \int_0^{\infty} e^{-nt/x} t^{n-1} f(t)dt, \quad x > 0$$

and $(P_n f)(0) = f(0)$. Now composition operator $A_n = S_n \circ P_n$ is defined by

$$(S_n \circ P_n f)(x) = \sum_{k=1}^{\infty} s_k(nx) \frac{n^{2n}}{k^n \Gamma(n)} \int_0^{\infty} e^{-n^2 t/k} t^{n-1} f(t)dt.$$

Received: 27.04.2024; Accepted: 02.06.2024; Published Online: 04.06.2024
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DOI: 10.33205/cma.1474535

In above $k \geq 1$ as for $k = 0$ above is not defined. In order to satisfy normalizer condition, our operators take the following form:

$$\begin{aligned}(A_n f)(x) &= \sum_{k=1}^{\infty} s_k(nx) \frac{n^{2n}}{k^n \Gamma(n)} \int_0^{\infty} e^{-n^2 t/k} t^{n-1} f(t)dt + s_0(nx) f(0) \\ &= \sum_{k=1}^{\infty} s_k(nx) \int_0^{\infty} \frac{n^2}{k} s_{n-1}\left(\frac{n^2 t}{k}\right) f(t)dt + s_0(nx) f(0) \qquad (1.1)\end{aligned}$$

which is a new approximation operator. These operators preserve constant function. In this article, we discuss some approximation properties of the operators A_n.

2. Moment generating function and Moments

The moment generating functions with the notation $\exp_A(t) = e^{At}$ are given by

$$\begin{aligned}(S_n \exp_A)(x) =& e^{nx(e^{A/n}-1)}, \\ (P_n \exp_A)(x) =& \left(1 - \frac{Ax}{n}\right)^{-n}, \\ (V_n \exp_A)(x) =& (P_n \circ S_n \exp_A)(x) \\ =& (P_n \exp_{n(e^{A/n}-1)}) = \left(1 - xe^{\frac{A}{n}} + x\right)^{-n}\end{aligned}$$

which is the moment generating function of the Baskakov operators V_n. But when we take reverse order composition i.e. $S_n \circ P_n$, then moment generating is not achieved in the close form and we have the same in summation form

$$(A_n \exp_A)(x) = (S_n \circ P_n \exp_A)(x) = \sum_{k=0}^{\infty} s_k(nx) \left(1 - \frac{Ak}{n^2}\right)^{-n}.$$

Lemma 2.1. *The moments satisfy the representation*

$$\begin{aligned}(A_n e_r)(x) =& \sum_{k=1}^{\infty} s_k(nx) \frac{n^{2n}}{k^n \Gamma(n)} \int_0^{\infty} e^{-n^2 t/k} t^{n+r-1} dt \\ =& \frac{\Gamma(n+r)}{\Gamma(n) n^{2r}} \sum_{k=1}^{\infty} s_k(nx) k^r.\end{aligned}$$

In particular

$$\begin{aligned}(A_n e_1)(x) =& \sum_{k=1}^{\infty} s_k(x) \frac{k}{n} = x \\ (A_n e_2)(x) =& \frac{(n+1)}{n} \sum_{k=1}^{\infty} s_k(x) \frac{k^2}{n^2} = x^2 + \frac{x(1+x)}{n} + \frac{x}{n^2} \\ (A_n e_3)(x) =& \left(1 + \frac{3}{n} + \frac{2}{n^2}\right) \left[x^3 + \frac{3x^2}{n} + \frac{x}{n^2}\right] \\ (A_n e_4)(x) =& \left(1 + \frac{6}{n} + \frac{11}{n^2} + \frac{6}{n^3}\right) \left[x^4 + \frac{6x^3}{n} + \frac{7x^2}{n^2} + \frac{x}{n^3}\right].\end{aligned}$$

The proof of this lemma follows by using the moments of Szász operators, which can be obtained from $(S_n \exp_A)(x)$.

Lemma 2.2. *If the central moments are denoted by* $\mu_{n,r}(x) = (A_n(e_1 - xe_0)^r)(x)$*,* $r = 0, 1, 2, ...,$ *then*

$$\mu_{n,0}(x) = 1$$
$$\mu_{n,1}(x) = 0$$
$$\mu_{n,2}(x) = \frac{x(1+x)}{n} + \frac{x}{n^2}.$$

The proof follows by Lemma 2.1 and linearity of A_n.

3. Approximation Estimations

Let $\widetilde{C}[0, \infty)$ denotes the space of all real-valued bounded and uniformly continuous functions f on $[0, \infty)$ with the norm $||f|| = \sup_{x\in[0,\infty)} |f(x)|$.

Theorem 3.1. *For* $f' \in \widetilde{C}[0, \infty)$ *and* $x \in [0, \infty)$*, we have*

$$|(A_n f)(x) - f(x)| \leq 2\sqrt{\frac{x(1+x)}{n} + \frac{x}{n^2}}\omega\left(f', \sqrt{\frac{x(1+x)}{n} + \frac{x}{n^2}}\right),$$

where $\omega(f, \delta)$ *is the modulus of continuity of first-order.*

Proof. For $f' \in \widetilde{C}[0, \infty)$ and $x, t \in [0, \infty)$, we can write

$$(A_n(f(u) - f(x)))(x) = f'(x)(A_n(u - x))(x) + \left(A_n \int_x^u (f'(v) - f'(x))dv\right)(x).$$

Also, for $\delta > 0$, we have

$$\left|\int_x^u (f'(v) - f'(x))dv\right| \leq \omega(f', \delta)\left(\frac{(u-x)^2}{\delta} + |u - x|\right).$$

Thus using Schwarz inequality and Lemma 2.2, we get

$$|[(A_n f) - f](x)| \leq |f'(x)| \cdot |\mu_{n,1}(x)| + \omega(f', \delta)\left[\frac{\sqrt{\mu_{n,2}(x)}}{\delta} + 1\right]\sqrt{\mu_{n,2}(x)},$$

selecting $\delta = \sqrt{\mu_{n,2}(x)}$, the result follows at once. □

Theorem 3.2. *For* $f \in C_B[0, \infty)$ (*denoting the class of continuous and bounded function on the interval* $[0, \infty)$)*, there exists a positive constant* C*, such that*

$$|[(A_n f) - f](x)| \leq C\omega_2\left(f, \sqrt{\frac{x(1+x)}{n} + \frac{x}{n^2}}\right).$$

Proof. The operators A_n preserve linear functions. By Taylor's expansion, for $g \in C_B^2[0, \infty)$ and $x, t \in [0, \infty)$, we have

$$|[(A_n g) - g](x)| = \left|A_n\left(\int_x^t (t-u)g''(u)du, x\right)\right|.$$

Also, we have $|\int_x^t (t-u)g''(u)du| \leq (t-x)^2||g''||$. Therefore by Lemma 2.2, we have

$$\left|A_n\left(\int_x^t (t-u)g''(u)du, x\right)\right| \leq ||g''||\left(\frac{x(1+x)}{n} + \frac{x}{n^2}\right).$$

Next

$$|(A_n f)(x)| = \sum_{k=1}^{\infty} s_k(nx)\int_0^{\infty} \frac{n^2}{k} s_{n-1}\left(\frac{n^2 t}{k}\right)|f(t)|dt + s_0(nx)|f(0)| \leq ||f||.$$

Thus, we have

$$\begin{aligned}|(A_nf)(x)-f(x)|=&|[(A_n(f-g))-(f-g)](x)|+|[(A_ng)-g](x)|\\ \leq&2\|f-g\|+\left(\frac{x(1+x)}{n}+\frac{x}{n^2}\right)\|g''\|.\end{aligned}$$

Taking the infimum over all $g\in C_B^2[0,\infty)$ and using the inequality

$$C\omega_2(f,\sqrt{\eta})\geq K_2(f,\eta),\eta>0$$

(see [10]), we get the required result. □

If we denote

$$B_2[0,\infty)=\{g:|g(x)|\leq c_g(1+x^2),\forall x\in[0,\infty)\},$$

where c_g is certain absolute constant that depends on g, but free from x. Let $C_2[0,\infty)=C[0,\infty)\cap B_2[0,\infty)$. For each $g\in C_2[0,\infty)$, the weighted modulus of continuity (see [23]) is defined as

$$\Omega(g,\delta)=\sup_{|h|<\delta,x\in R^+}\frac{|g(x+h)-g(x)|}{(1+h^2)(1+x^2)}.$$

Also, $C_2^*[0,\infty)$ denotes the subspace of continuous functions $g\in B_2[0,\infty)$ for which

$$\lim_{x\to\infty}|g(x)|(1+x^2)^{-1}<\infty.$$

We consider the norm by

$$||g||_2=\sup_{0\leq x<\infty}\frac{|g(x)|}{(1+x^2)}.$$

Following Gadjiev [13], we have:

Theorem 3.3. *If $f\in C_2^*[0,\infty)$ satisfying*

$$\lim_{n\to\infty}\|(A_ne_i)-e_i\|_2=0,\quad i=0,1,2,$$

then we have

$$\lim_{n\to\infty}\|(A_nf)-f\|_2=0.$$

Proof. To prove the result, we use Lemma 2.1, as the operators preserve constant and linear functions, the result is true for $i=0,1$. Next

$$\lim_{n\to\infty}\|(A_ne_2(x)-e_2\|_2=\lim_{n\to\infty}\frac{1}{(1+x^2)}\left[\frac{x(1+x)}{n}+\frac{x}{n^2}\right]=0.$$

The proof is complete. □

Theorem 3.4. *If $f''\in C_2^*[0,\infty)$, then for $x\in[0,\infty)$, we have*

$$\begin{aligned}&\left|(A_nf)(x)-f(x)-\left(\frac{x(1+x)}{n}+\frac{x}{n^2}\right)f''(x)\right|\\ \leq&8(1+x^2)O(n^{-1})\Omega(f'',1/\sqrt{n}).\end{aligned}$$

Proof. By applying Taylor's formula, with $h(t,x)$ a continuous function defined by $h(t,x):=\frac{1}{2}(f''(\xi)-f''(x)),x<\xi<t$, on the operators $(A_nf)(x)$, we obtain

$$(A_nf)(x)-f(x)=\mu_{n,1}(x)f'(x)+\frac{\mu_{n,2}(x)}{2}f''(x)+(A_nh(t,x)(t-x)^2)(x),$$

where $h(t,x)$ vanishes when $t \to x$. Now applying Lemma 2.2, we have

$$\left|(A_n f)(x) - f(x) - \mu_{n,1}(x)f'(x) + \frac{\mu_{n,2}(x)}{2}f''(x)\right| \leq (A_n h(t,x)(t-x)^2)(x).$$

Following [19, Thm. 2.1] the remainder term for A_n has the form:

$$|(A_n h(t,x)(t-x)^2)(x)| \leq 8(1+x^2)O(n^{-1})\Omega(f'', 1/\sqrt{n}).$$

The proof of the theorem is complete. □

Corollary 3.1. *If $f'' \in C_2^*[0,\infty)$, then we have*

$$\lim_{n\to\infty} n\,[[(A_n f) - f](x)] = \frac{x(1+x)}{2}f''(x).$$

The moduli of continuity with weights (see [24]) is considered:

$$\omega_\psi(f,\delta) = \sup\{|f(u) - f(v)| : |u-v| \leq \delta\psi\left((u+v)/2\right); u, v \geq 0\},$$

where $\psi(u) = \sqrt{u}/(1+u^m), m = 2, 3, 4, ...$

Following [20], suppose $W_\psi[0,\infty)$ denotes the subspace of all real-valued functions such that $f \circ e_2$ and $f \circ e_{2/(2m+1)}$ are uniformly continuous in the intervals $[0,1]$ and $[1,\infty)$, respectively.

Following [20, Th. 6.3] and references therein below quantitative estimate of error holds:

Theorem 3.5. *Let $f \in C_2[0,\infty) \cap E$, where E is the subspace of positive real axis also if $f'' \in W_\psi[0,\infty)$, then we have*

$$\begin{aligned}&\left|(A_n f)(x) - f(x) - \left(\frac{x(1+x)}{n} + \frac{x}{n^2}\right) f''(x)\right| \\ &\leq \left(\frac{x(1+x)}{n} + \frac{x}{n^2}\right)\left[1 + \frac{1}{\sqrt{2x}}C_{n,r,2}(x)\right]\omega_\psi(f'', \delta^{1/2}),\end{aligned}$$

where

$$C_{n,r,2}(x) = 1 + \frac{1}{(A_n|t-x|^3)(x)}\sum_{s=0}^{r}\binom{r}{s}x^{r-s}\frac{(A_n|t-x|^{r+s})(x)}{2^s}$$

and $\delta := \mu_{n,4}(x)/\mu_{n,2}(x)$, where the moments are given in Lemma 2.2.

For proof of above theorem, we use Lemma 2.2 and follow the steps as in [21].

Below we find the difference between our new composition operator A_n and the Szász-Mirakjan operators.

Theorem 3.6. *If $n \in N$ and $f \in C_B[0,\infty)$, then we get*

$$|(A_n f)(x) - (S_n f)(x)| \leq 2\omega\left(f, \left(\frac{x^2}{n} + \frac{x}{n^2}\right)^{-1/2}\right).$$

Proof. We prove the first inequality as follows

$$\begin{aligned}|(A_n f)(x) - (S_n f)(x)| &= |(S_n \circ P_n f)(x) - (S_n f)(x)| \\ &\leq \sum_{k\geq 0} s_k(nx)\left|(P_n f)\left(\frac{k}{n}\right) - f\left(\frac{k}{n}\right)\right| dt.\end{aligned}$$

In the following inequality using $(P_n(e_1 - xe_0)^2)(x) = \frac{x^2}{n}$, we can write

$$| (P_n f) (x) - f(x)| \leq \left(1 + \frac{(P_n(e_1 - xe_0)^2)(x)}{\delta^2}\right) \omega(f, \delta)$$
$$= \left(1 + \frac{x^2}{n\delta^2}\right) \omega(f, \delta).$$

Thus using the fact that of $(S_n e_2)(x) = x^2 + \frac{x}{n}$, we have

$$|(S_n \circ P_n f)(x) - (S_n f)(x)| \leq \sum_{k \geq 0} s_k(nx) \left(1 + \frac{k^2}{n^3\delta^2}\right) \omega(f, \delta) = \left[1 + \frac{1}{n\delta^2}\left(x^2 + \frac{x}{n}\right)\right] \omega(f, \delta).$$

Choosing $\delta = \left(\frac{x^2}{n} + \frac{x}{n^2}\right)^{-1/2}$, the result follows. □

The Post-Widder operator P_n can be written as

$$(P_n f)(x) = \frac{1}{\Gamma(n)} \int_0^\infty e^{-u} u^{n-1} f\left(\frac{xu}{n}\right) du, \ u \geq 0.$$

It is easy to observe that,

$$(P_n f)(x) = E\left[f\left(\frac{xU(n)}{n}\right)\right], \ x \geq 0,$$

where $\{U(n) : n > 0\}$ is gamma process.

Proposition 3.1. *For $f \in C[0, \infty)$, $\omega(f, \delta) < \infty$ and $\delta \geq 0$, we have*

$$\omega(P_n f, \delta) \leq 2\omega(f, \delta).$$

Proof. Following the notations of [5], since $E\left[\frac{xU(n)}{n}\right] = (P_n e_1)(x) = x$, therefore

$$a_1(\delta, n) = \sup_{x, x+\delta \in [0,\infty)} E\left|\frac{(x+\delta)U(n)}{n} - \frac{xU(n)}{n}\right| = \delta,$$

and since $U(n)$ has zero density at origin, therefore

$$b(\delta, n) = \sup_{x, x+\delta \in [0,\infty)} P\left(\left|\frac{(x+\delta)U(n)}{n} - \frac{xU(n)}{n}\right| > 0\right)$$
$$= 1.$$

Following [5, Corollary 2], we have

$$\omega(P_n f, \delta) \leq \left(\frac{a_1(\delta, n)}{\delta} + b(\delta, n)\right) \omega(f, \delta).$$

Substituting above values, the result is immediate. □

Theorem 3.7. *If $n \in N$ and $f \in C_B[0, \infty)$, then we get*

$$|(A_n f)(x) - (P_n f)(x)| \leq 4\omega\left(f, \sqrt{\frac{x}{n}}\right).$$

Proof. We prove the first inequality by considering $g = P_n f$ as follows

$$\begin{aligned}|(A_n f)(x) - (P_n f)(x)| &= |(S_n \circ g)(x) - g(x)| \\ &\leq \left(1 + \frac{(S_n(e_1 - xe_0)^2)(x)}{\eta^2}\right)\omega(g,\eta) \\ &= \left(1 + \frac{x}{n\eta^2}\right)\omega(g,\eta).\end{aligned}$$

Choosing $\eta = \left(\frac{x}{n}\right)^{-1/2}$ and applying Proposition 3.1, the result follows. □

4. Comparison

The operator $(S_n \circ P_n f)$ provide a discrete operator namely Baskakov operator V_n and the composition $(P_n \circ S_n f)$ provide a summation-integral type operator A_n. Both have the different moments but their asymptotic formula are same and given by

$$\lim_{n\to\infty} n[(S_n \circ P_n f) - f(x)] = \lim_{n\to\infty} n[(P_n \circ S_n f) - f(x)] = \frac{x(1+x)}{2} f''(x).$$

In the following table, we give the error for the two compositions of operators.

Table 1. Upper bound for error between the two composition operators A_n and V_n

n \ Operator	A_n $(x \in [0,2])$	V_n $(x \in [0,2])$	A_n $(x \in [0,9])$	V_n $(x \in [0,9])$
5	1.28	1.2	18.36	18
10	0.62	0.6	9.09	9.0
50	0.1208	0.12	1.8036	1.8
100	0.0602	0.06	0.9009	0.90
1000	0.006002	0.006	0.090009	0.09

We observe here from the above table that the error is less in case we consider the discrete operator viz. $V_n := S_n \circ P_n$ and it increases slightly by taking the reverse order composition $A_n := P_n \circ S_n$.

One may study the composition of Mihesan and BBH operators discussed in [8], [11] and also the King type approach of our operators along the lines of [12]. We may discuss them elsewhere.

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