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Research Article

Analysis of the Layer Behavior to the Parameterized Problem with Integral Boundary Condition

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ABSTRACT

A parameterized singularly perturbed first order quasilinear boundary value problem with integral boundary conditions is considered. Asymptotic estimates for the solution and its first derivative have been established. Given an example supports these theoretical results and indicate that the estimates are sharp. The estimates are obtained with the use of a mathematical technique that can also be applied in appropriate grid computations. 2010 Mathematical Subject Classification: 34K10, 34K26, 34B08

Keywords: Parameterized problem, Asymptotic estimate, Singular perturbation, Boundary layer, İntegral boundary conditions

İntegral Sınır Şartlı Parametreye Bağlı Problemin Sınır Katı Davranışının İncelenmesi

ÖZ

Bu çalışmada, integral sınır şartlı parametreye bağlı singüler pertürbe özellikli kuazi-lineer sınır-değer problemi ele alınmıştır. Problemin çözümü ve birinci türevleri için asimptotik değerlendirmeler elde edilmiştir. Bu teorik sonuçları destekleyen ve değerlendirmelerin kesin olduğunu gösteren bir örnek verilmiştir. Asimptotik değerlendirmelerin elde edilmesinde kullanılan yöntem uygun nümerik çözümlerin incelenmesinde kullanılabilir.

Anahtar kelimeler: Parametreli problem, Asimptotik değerlendirme, Singüler pertürbasyon, Sınır katı, İntegral sınır şartı

1. Introduction

In this paper we consider the following parameterized singular perturbation problem with integral boundary condition:

$$\varepsilon u'(t) + f(t,u(t),\lambda) = 0, \quad t \in \Omega = (0,T], \; T > 0, \tag{1}$$

$$u(0) + \int_0^T K(s,u)ds = A, \tag{2}$$

$$u(T) = B, \tag{3}$$

where $0 < \varepsilon \le 1$ is small and known as the singular perturbation parameter, λ known as the control parameter, A and B are given constants. $f(t,u,\lambda)$ and $K(t,u)$) are assumed to be sufficiently continuously differentiable for our purpose functions in $\bar{\Omega}\text{x}\mathbb{R}^2$ and $\bar{\Omega}\text{x}\mathbb{R}$ respectively ($\bar{\Omega} = \Omega \cup \{t = 0\}$) and moreover

$$0 < \alpha \le \frac{\partial f}{\partial u} \le a^* < \infty \quad , \quad m_1 \le \frac{\partial f}{\partial \lambda} \le M_1 < \infty ,$$

$$0 \le \frac{\partial K}{\partial u} \le K^* < \infty .$$

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By a solution of (1)-(3) we mean a $\{u(t),\lambda\}\in C^1\left(\bar{\Omega}\right)\mathrm{x}\mathbb{R}$ for which problem (1)-(3) is satisfied. For $\varepsilon << 1$ the function $u(t)$ has a boundary layer of thickness $O(\varepsilon)$ near $t=0$ (see, Section 2). Parameter dependent differential equations (such as (1)) occur naturally in various fields of science and engineering. Singular perturbation problems belong to the class of such problems in which a very small positive parameter is multiplied to the highest order derivative term in the differential equation. Such problem undergo rapid changes within very thin layers near the boundary or inside the problem domain, so most of the conventional methods fail when this small parameter approaches to zero. This kind of problems arise very frequently in the fields of applied mathematics and physics which include fluid dynamics, quantum mechanics, elasticity, chemical reactions, gas porous electrodes theory, the Navier-Stokes equations of fluid flow at high Reynolds number, oceanography, meteorology, reaction-diffusion processes etc. For more details on singular perturbation, one can refer to the books [Kevorkian and Cole, 1981; Miller et al., 2012; Nayfeh, 1993; O'Malley, 1991; Roos et al., 2008] and the references therein. Differential equations with integral boundary conditions constitute a very interesting and important class of problems. Note that the boundary condition (2) includes periodic, two-point, there-point, multipoint and initial conditions as special cases. Such types of problems have been considered for many years. For a discussion of existence and uniqueness results and for applications of problems with integral boundary conditions see, [Ashyralyev and Sharifov, 2013; Benchohra et al., 2010; 2011; Jankowski 2002; Samoilenko,1991] and the references therein. In [Ahmad, et al., 2005; Amiraliyev, et al., 2007; Cakir and Amiraliyev, 2007;. Jankowski 2003; Khan, 2003; Kudu and Amiraliyev, 2015], have been considered some approximating aspects of this kind of problems. This paper deals with an integral boundary value problem for a singularly perturbed first order quasilinear ordinary differential equation depending on a parameter. A priori asymptotic estimates for the solution and its first derivative are proved. Similar investigations for this type of problems, have been made by [Amiraliyev and Duru, 2005; Kudu, 2014; Kudu and Amirali 2016; Kudu et al., 2016; Lui and Mcare, 2001; Na, 1979; Pomentale, 1976], when the integral condition is linear. The estimates are obtained with the use of a mathematical technique that can also be used to justify the uniform convergence of various appropriate finite-difference schemes. Henceforth, C and c denote the generic positive constants independent of ε and of the mesh parameter. Such a subscripted constant is also independent of ε and mesh parameter, but whose value is fixed.

2. Asymptotic estimates for the solution of (1)-(3)

Theorem 1. The solution $\{u(t),\lambda\}$ of the problem (1)-(3) satisfies the inequalities

$$|\lambda| \le c_0, \qquad (4)$$

$$\|u\|_\infty \le c_1, \qquad (5)$$

where

$$c_0 = m_1^{-1}\left\{\frac{\alpha|A|}{e^{\alpha T}-1} + \frac{|B|a^*(1-K^*T)}{1-e^{-a^*T}} + \|F\|_\infty\right\},$$
$F(t) = f(t,0,0)$,

$$c_1 = A + \alpha^{-1}(1+K^*T)\left(\|F\|_\infty + c_0 M_1\right)$$

and

$$|u'(x)| \le C\left\{1+\frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\right\},\ t\in[0,T] \qquad (6)$$

provided $|\partial f/\partial t| \le C$ for $t\in[0,T]$ and $|u|\le c_1$.

Proof: We rewrite Eq.(1) in the form

$$\varepsilon u'(t) + a(t)u(t) = F(t) + \lambda b(t), \qquad (7)$$

Where $a(t) = \frac{\partial f}{\partial u}(t,\tilde{u},\tilde{\lambda}),\ b(t) = -\frac{\partial f}{\partial \lambda}(t,\tilde{u},\tilde{\lambda});$
$\tilde{u} = \gamma u,\ \tilde{\lambda} = \gamma\lambda\,(0<\gamma<1)$ -intermediate values.

Integrating (7), (3) we have

$$u(t) = Be^{\frac{1}{\varepsilon}\int_t^T a(\tau)d\xi} - \frac{1}{\varepsilon}\int_t^T F(\tau)e^{\frac{1}{\varepsilon}\int_t^\tau a(\eta)d\eta}d\tau + \frac{\lambda}{\varepsilon}\int_t^T b(\tau)e^{\frac{1}{\varepsilon}\int_t^\tau a(\eta)d\eta}d\tau,$$

from which, after using the relation

$$K(t,u(t)) = K(t,0) + c(t)u(t),\ (c(t) = \frac{\partial K}{\partial u}(t,\tilde{u}))$$

and integral boundary condition (2), it follows that,

$$Be^{\frac{1}{\varepsilon}\int_0^T a(\tau)d\tau} - \frac{1}{\varepsilon}\int_0^T F(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau$$
$$+\frac{\lambda}{\varepsilon}\int_0^T b(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau + \int_0^T K(s,0)ds$$
$$+B\int_0^T c(s)e^{\frac{1}{\varepsilon}\int_s^T a(\tau)d\tau}ds - \frac{1}{\varepsilon}\int_0^T c(s)[\int_s^T F(\tau)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}d\tau]ds$$
$$+\frac{1}{\varepsilon}\int_0^T c(s)[\int_s^T b(\tau)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}d\tau]ds = A$$

and thereby

$$\lambda = \frac{A - \int_0^T K(s,0)ds}{\frac{1}{\varepsilon}\int_0^T b(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau + \frac{1}{\varepsilon}\int_0^T b(\tau)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}ds]d\tau}$$
$$-\frac{Be^{\frac{1}{\varepsilon}\int_0^T a(\tau)d\tau} + B\int_0^T c(s)e^{\frac{1}{\varepsilon}\int_s^T a(\tau)d\tau}ds}{\frac{1}{\varepsilon}\int_0^T b(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau + \frac{1}{\varepsilon}\int_0^T b(\tau)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}ds]d\tau}$$
$$+\frac{\frac{1}{\varepsilon}\int_0^T F(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau + \frac{1}{\varepsilon}\int_0^T F(\tau)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}ds]d\tau}{\frac{1}{\varepsilon}\int_0^T b(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau + \frac{1}{\varepsilon}\int_0^T b(\tau)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}ds]d\tau}.$$
(8)

In view of $c(t)\ge 0$, then after applying the mean value theorem for integrals, we deduce that,

$$\left|\frac{\frac{1}{\varepsilon}\int_0^T F(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau + \frac{1}{\varepsilon}\int_0^T F(\tau)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}ds]d\tau}{\frac{1}{\varepsilon}\int_0^T b(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau + \frac{1}{\varepsilon}\int_0^T b(\tau)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}ds]d\tau}\right|$$

$$\le m_1^{-1}\|F\|_\infty, \qquad (9)$$

and

$$\left|\frac{Be^{\frac{1}{\varepsilon}\int_0^T a(\tau)d\tau}+B\int_0^T c(s)e^{\frac{1}{\varepsilon}\int_s^T a(\tau)d\tau}ds}{\frac{1}{\varepsilon}\int_0^T b(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau+\frac{1}{\varepsilon}\int_0^T b(\tau)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}ds]d\tau}\right|$$

$$\le\frac{|B|(1+K^*T)e^{\frac{1}{\varepsilon}\int_0^T a(\xi)d\xi}}{m_1\varepsilon^{-1}\int_0^T e^{\frac{1}{\varepsilon}\int_\xi^T a(\eta)d\eta}d\xi}$$

$$\le\frac{|B|(1+K^*T)}{m_1(a^*)^{-1}(1-e^{\frac{a^*T}{\varepsilon}})}$$

$$\le\frac{|B|(1+K^*T)}{m_1(a^*)^{-1}(1-e^{a^*T})}. \quad (10)$$

Also, for the first term in the right side of (8) for $\varepsilon\le 1$ values, we get

$$\left|\frac{A-\int_0^T K(s,0)ds}{\frac{1}{\varepsilon}\int_0^T b(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau+\frac{1}{\varepsilon}\int_0^T b(\tau)[\int_s^T c(s)e^{\frac{1}{\varepsilon}\int_s^\tau a(\eta)d\eta}ds]d\tau}\right|$$

$$\le\frac{\left|A-\int_0^T K(s,0)ds\right|}{\left|\frac{1}{\varepsilon}\int_0^T b(\tau)e^{\frac{1}{\varepsilon}\int_0^\tau a(\eta)d\eta}d\tau\right|}$$

$$\le\frac{\alpha\left|A-\int_0^T K(s,0)ds\right|}{m_1(e^{\alpha T}-1)} \quad (11)$$

The relation (8), by taking into consideration here (9)-(11), immediately leads to (4). Further, by integrating (7), we have

$$u(t)=u(0)e^{-\frac{1}{\varepsilon}\int_0^t a(s)ds}+\frac{1}{\varepsilon}\int_0^t\Phi(\tau)e^{\frac{1}{\varepsilon}\int_\tau^t a(\eta)d\eta}ds,\ \Phi(t)=F(t)-\lambda b(t),$$

from which, by setting the boundary condition (2), we get

$$u(0)=\frac{A-\frac{1}{\varepsilon}\int_0^T c(s)[\int_0^s\Phi(\tau)e^{-\frac{1}{\varepsilon}\int_\tau^s a(\eta)d\eta}d\tau]ds}{1+\int_0^T c(s)e^{-\frac{1}{\varepsilon}\int_0^s a(\tau)d\tau}ds}.$$

Since $c(t)$ is nonnegative, then

$$|u(0)|=\left|\frac{A-\frac{1}{\varepsilon}\int_0^T c(s)[\int_0^s\Phi(\tau)e^{-\frac{1}{\varepsilon}\int_\tau^s a(\eta)d\eta}d\tau]ds}{1+\int_0^T c(s)e^{-\frac{1}{\varepsilon}\int_0^s a(\tau)d\tau}ds}\right|.$$

$$\le|A|+\frac{1}{\varepsilon}\int_0^T c(s)[\int_0^s|\Phi(\tau)|e^{-\frac{1}{\varepsilon}\int_\tau^s a(\eta)d\eta}d\tau]ds$$

$$\le|A|+\frac{1}{\varepsilon}K^*\left(\|F\|_\infty+M_1c_0\right)\alpha^{-1}\varepsilon\int_0^T(1-e^{-\frac{\alpha s}{\varepsilon}})ds$$

$$\le|A|+\alpha^{-1}K^*T\left(\|F\|_\infty+M_1c_0\right). \quad (12)$$

Next, by virtue of maximum principle we have

$$\|u\|_\infty\le|u(0)|+\alpha^{-1}\left(\|F\|_\infty-b\lambda M_1\right)$$
$$\le|u(0)|+\alpha^{-1}\left(\|F\|_\infty+|\lambda|M_1\right),$$

which, after taking into account (4) and (12) leads to (5).

To prove (6), first we estimate $|u'(0)|$:

$$|u'(0)|\le\frac{|F(0)-a(0)u(0)-b(0)\lambda|}{\varepsilon}\le\frac{C}{\varepsilon}.$$

Differentiating, now the equation (7), we have

$$\varepsilon v'(t) + p(t)v(t) = g(t),$$

with

$v = u',\ p(t) = \frac{\partial f}{\partial u}(t, u(t), \lambda)$ and

$g(t) = \frac{\partial f}{\partial t}(t, u(t), \lambda).$

Since $p(t) \geq \alpha > 0$ and $|g(t)| \leq C,$ for $v(t)$ we obtain

$$v(t) = v(0)e^{-\frac{1}{\varepsilon}\int_0^t a(s)ds} + \frac{1}{\varepsilon}\int_0^t g(s)e^{\frac{1}{\varepsilon}\int_s^t a(\eta)d\eta} ds.$$

Hence it follows that

$$|v(t)| \leq \frac{C}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}} + \frac{C}{\varepsilon}\int_0^t e^{-\frac{\alpha(t-s)}{\varepsilon}} ds$$
$$\leq \frac{C}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}} + C(1 - e^{-\frac{\alpha t}{\varepsilon}}),$$

which implies validity of (6).

3. Example

Consider the particular problem with

$$f(t,u,\lambda) = u - e^{-u} + (t+\lambda)e^{-1/\varepsilon} + e^{te^{-1/\varepsilon} - e^{-t/\varepsilon}} + e^{\lambda} + \lambda - 1 = 0,$$
$$K(t,u) = 2u,$$
$$A = 1 + 2\varepsilon - (1+2\varepsilon)e^{-1/\varepsilon},\ B = 0.$$

The solution $\{u(t), \lambda\}$ has the form

$$u(t) = e^{-t/\varepsilon} - te^{-1/\varepsilon} \tag{13}$$

with control parameter λ satisfying

$$g(\lambda) \equiv (\varepsilon - \lambda)e^{-1/\varepsilon} + e^{\lambda} + \lambda - 1 = 0.$$

It is not difficult to see that, the functions $f(t,u,\lambda)$ and $K(t,u)$ satisfy requirements from Section 1. Therefore from (13) for the first derivative we have

$$|u'(t)| = \frac{1}{\varepsilon}e^{-t/\varepsilon} + e^{-1/\varepsilon} \leq 1 + \frac{1}{\varepsilon}e^{-t/\varepsilon}. \tag{14}$$

Since $g(0) = \varepsilon e^{-1/\varepsilon} > 0,$ $g(-1) = (\varepsilon+1)e^{-1/\varepsilon} + e^{-1} - 2 < 0$ and $g'(\lambda) = -e^{-1/\varepsilon} + e^{\lambda} + 1 > 0$ we confirm that $\lambda \in (-1, 0).$ Thereby the control parameter λ uniformly bounded in $\varepsilon.$ From (14) it is also clear that the first derivative of $u(t)$ is unbounded while ε values are tending to zero and $u(t)$ has an initial layer near $t = 0$ of thickness $O(\varepsilon).$ Therefore we observe here the accordance with our theoretical results described above.

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