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A New Type of Extended Soft Set Operation: Complementary Extended Theta Operation

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Abstract: A thorough mathematical foundation for dealing with uncertainty is provided by the notion of soft sets introduced by Molodtsov in 1999. In contrast to classical set theory, soft sets allow elements to have parametrization, providing a more complex representation of uncertainty. Soft set operations are important concepts in soft set theory, as they provide new approaches to dealing with problems involving parametric data. In this paper, we introduce a new soft set operation which we call "complementary extended theta operation," to contribute to the existing theory. We thoroughly analyze the properties of the operation and investigate the relationship between the complementary extended theta operation and other soft set operations by obtaining the distribution laws in order to further study the algebraic structures of soft sets with respect to this new operation in the future studies. Since studying the algebraic structure of soft sets from the perspective of soft set operations provides a thorough understanding of their application as well as an appreciation of how soft sets can be applied to classical and non-classical logic, this paper also aims to contribute to the literature of soft sets in this regard.

Keywords: soft sets, conditional complements, soft set operations, complementary extended soft set operations

Yeni Bir Tip Genişletilmiş Esnek Küme İşlemi: Tümleyenli Genişletilmiş Teta İşlemi

Özet: Molodtsov tarafından 1999'da öne sürülen esnek kümeler kavramı, belirsizlikle başa çıkmak için sağlam bir matematiksel temel sağlar. Klasik küme teorisinin aksine, esnek kümeler elemanların parametrelendirilmesine izin verir ve bu da belirsizliğin daha karmaşık bir temsilini sağlar. Esnek küme işlemleri, parametrik verileri içeren problemleri ele almak için yeni yaklaşımlar sunar ve bu nedenle esenek küme teorisinde önemli kavramlardır. Bu çalışmada, mevcut teoriye katkıda bulunmak amacıyla yeni bir esnek küme işlemi olan tümleyenli genişletilmiş teta işlemi tanıtılmıştır. İşlemin özelliklerini kapsamlı bir şekilde analiz edilmiş ve tümleyenli genişletilmiş teta işlemi ile diğer esnek küme işlemleri arasındaki ilişkiyi araştırarak dağılma kuralları elde edilmiştir. Bu, gelecekteki çalışmalarda esnek kümelerin cebirsel yapılarının bu yeni işlemle ilgili daha fazla incelenmesine olanak tanır. Esnek kümelerin cebirsel yapısını esnek küme işlemleri perspektifinden incelemek, uygulamalarının kapsamlı bir şekilde anlaşılmasını sağlamanın yanı sıra esnek kümelerin klasik ve klasik olmayan mantığa nasıl uygulanabileceğini anlama açısından da önemlidir. Bu makale bu kapsamda esnek kümeler konusundaki literatüre katkıda bulunmayı amaçlamaktadır.

Anahtar kelimeler: esnek kümeler, koşullu tümleyenler, esnek küme işlemleri, tümleyenli genişletilmiş esnek küme işlemleri

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INTRODUCTION

In many real-world scenarios, uncertainty arising from imprecision, vagueness, and ambiguity poses a challengeto decision-making processes and data analysis tasks. While traditional set theory provides robust methods for organising and manipulating data, it struggles to capture and represent this inherent uncertainty. To overcome this limitation, soft set theory emerges as a promising solution for dealing with uncertain and indeterminate information. Soft set theory, introduced by Molodtsov (1999) represents a soft and intuitive extension of classical set theory by introducing the concept of parametrization. Unlike crisp sets, where elements are either fully included or excluded, soft sets allow for parametrization that reflects the uncertainty associated with their inclusion. This adaptability empowers soft sets to effectively model and manage uncertain information, making them suitable for a wide range of applications, including decision support systems, pattern recognition, and information fusion.

Since its introduction, soft set theory has been widely applied in both theoretical and practical domains, and has inspired many new studies in the literature. Maji et al. (2003) paved the way for new studies in soft set theory by defining the equality of two soft sets, subset and superset of a soft set, complement of a soft set, soft binary operations such as and/or and union and intersection operations for soft sets. Pei and Miao (2005) redefined the concepts of soft subset and intersection of two soft sets based on set theoretical concepts. Then, Ali et al. (2009) proposed some new soft set operations, and Sezgin and Atagün (2011) and Ali et al. (2011) analyzed these soft set operations in detail. Sezgin et al. (2019) and Stojanovic (2021) proposed extended difference and extended symmetric difference of soft sets, respectively and studied their properties in detail in relation to other soft set operations, respectively.

A review of the existing literature indicates that restricted and extended soft set operations are the two main groups into which soft set operations often fall. Eren and Çalışıcı (2019) defined the soft binary piecewise difference operation for soft sets and studied its properties, and Sezgin and Çalışıcı (2024) studied the properties of this operation in detail. Çağman (2021) proposed the definitions of inclusive complement and exclusive complement of sets as new concepts of set theory, and applied these concepts to group theory. Sezgin et al. (2023a) introduced new binary complement concepts similar to the binary complement operations in Çağman (2021). Motivated by the new set operations recently specified in this research, Aybek (2024) proposed many new restricted and extended soft set operations and analyzed their properties. Furthermore, by taking the complement of the image set in the first row, the soft binary piecewise operation form, of which Eren and Çağman (2019) were the pioneers, was modified somewhat. As a result, the complementary soft binary piecewise operation has been thoroughly investigated by a number of scholars (Sezgin and Aybek, 2023; Sezgin and Demirci, 2023; Sezgin and Sarıalioğlu, 2024; Sezgin and Yavuz, 2023a; Sezgin, Aybek, Sezgin and Atagün 2023; Sezgin et al. 2023b).

On the other hand, Akbulut (2024) and Sarıalioğlu (2024) changed the form of the existing extended soft set operations in the literature by taking the complement of the image set in the first and second rows and defining the complementary extended difference, lambda and union, plus and theta, respectively, and giving their algebraic properties and relations with other soft set operations. We refer to the following for more uses of soft sets in relation to algebraic structures: (Çağman et al., 2012; Sezer, 2014; Muştuoğlu et al., 2015; Sezer et al., 2015; Sezgin et al., 2017; Atagün and Sezgin, 2018; Sezgin, 2018; Mahmood et al., 2018; Jana et al., 2019; Özlü and Sezgin, 2020; Sezgin et al., 2022).

In this paper, in order to advance the theory of soft sets, this paper presents a novel soft set operation called "complementary extended theta" is introduced and t its properties are thoroughly investigated. In addition, an analysis is conducted to investigate how the complementary extended theta operation interacts with other types of soft set operations, with the aim ofestablishing its relationship with them. Since understanding the algebraic structures of soft sets' in relation to novel operations is essential for a thorough understanding of their applications, this study is vital in this framework.

PRELIMINARIES

2.1. Definition

Let U be the universal set, E be the parameter set, P(U) be the power set of U, and let D ⊆ E. A pair (F, D) is called a soft set on U. Here, F is a function given by $F: D \rightarrow P(U)$ (Molodtsov, 1999)

The notation of the soft set (F,D) is also shown as F_D, however, we prefer to use the notation of (F,D) as is used by Molodtsov (1999) and Maji et al. (2003). The definition of soft set, introduced by Molodtsov, was modified by Çağman and Enginoğlu (2010). Throughout this study, we use the definition of soft set proposed by Molodtsov (1999).

The set of all soft sets over U is denoted by $S_E(U)$. Let K be a fixed subset of E, then the set of all soft sets over U with the fixed parameter set K is denoted by $S_K(U)$. In other words, in the collection $S_K(U)$, only soft sets with the parameter set K are included, while in the collection $S_E(U)$, soft sets over U with any parameter set can be included.

2.2. Definition

Let (F,D) be a soft set over U. If for all ℵ∈D, F(ℵ)=∅, then the soft set (F,D) is called a null soft set with respect to D, denoted by $\emptyset_D$. Similarly, let (F,E) be a soft set over U. If for all ℵ∈E, F(e)=∅, then the soft set (F,E) is called a null soft set with respect to E, denoted by $\emptyset_E$ (Ali et al., 2009).

A soft set can be defined as F: ∅ ⟶ P(U), where U is a universal set. Such a soft set is called a null soft set and is denoted as $\emptyset_\emptyset$. Thus, $\emptyset_\emptyset$ is the only soft set with an empty parameter set (Ali et al., 2011).

2.3. Definition

Let (F,D) be a soft set over U. If for all ℵ∈D, F(ℵ)=U, then the soft set (F,D) is called an absolute soft set with respect to D, denoted by U_D. Similarly, let (F,E) be a soft set over U. If for all ℵ∈E, F(e)=U, then the soft set (F,D) is called an absolute soft set OVER U, denoted by U_E (Ali et al., 2009)

2.4. Definition

Let (F,D) and (G,Y) be soft sets over U. If D⊆Y and for all ℵ∈D, F(ℵ) ⊆G(ℵ), then (F,D) is said to be a soft subset of (G,Y), denoted by (F,D)$\tilde{\subseteq}$(G,Y). If (G,Y) is a soft subset of (F,D), then (F,D) is said to be a soft superset of (G,Y), denoted by (F,D)$\tilde{\supseteq}$(G,Y). If (F,D)$\tilde{\subseteq}$(G,Y) and (G,Y)$\tilde{\subseteq}$(F,D), then (F,D) and (G,Y) are called soft equal sets (Pei and Maio, 2005)

2.5. Definition

Let (F,D) be a soft set over U. The soft complement of (F,D), denoted by $(F,KD)^r = (F^r,D)$, is defined as follows: for all ℵ∈D, F^r(ℵ)=U-F(ℵ) (Ali et al., 2009)

Çağman (2021) introduced two new complements as novel concepts in set theory, termed as the inclusive complement and exclusive complement. For ease of representation, we denote these binary operations as + and θ, respectively. For two sets D and Y, these binary operations are defined as

D+Y=D'∪Y, DθY=D'∩Y'. Sezgin et al. (2023c) examined the relations between these two operations and also defined three new binary operations and analyzed their relations with each other. Let D and Y be two sets D*Y=D'∪Y', DγY= D'∩Y, DλY=D∪Y'.

We can categorize all types of soft set operations as follows: Let "$\circledast$" be used to represent the set operations (i.e., here $\circledast$ can be ∩, ∪, \, Δ, +, θ, *, λ, γ), then all types of soft set operations are defined as follows:

2.6. Definition

Let $(F,D),(G,Y) \in S_E(U)$. The restricted $\circledast$ operation of (F,D) and (G,Y) is the soft set (H,K), denoted to be $(F,D)\circledast_{\mathfrak{R}}(G,Y) = (H,K)$, where K=D ∩ Y $\neq \emptyset$ and for all ℵ ∈ K, H(ℵ) = F(ℵ)$\circledast$G(ℵ). Here, if K = D ∩ Y = $\emptyset$, then $(F,D)\circledast_R(G,Y)=\emptyset_\emptyset$ (Ali et al., 2009; Sezgin and Atagün, 2011; Aybek, 2024).

2.7. Definition

Let $(F,D),(G,Y) \in S_E(U)$. The extended $\circledast$ operation (F,D) and (G,Y) is the soft set (H,K), denoted by (F, D) $\circledast_\varepsilon$(G, Y) = (H, K), where $K = D \cup Y$ and for $\forall ℵ \in K$,

$$H(ℵ) = \begin{cases} F(ℵ), & ℵ \in D - Y \\ G(ℵ), & ℵ \in Y - D \\ F(ℵ)\circledast G(ℵ), & ℵ \in D \cap Y \end{cases}$$

(Maji et al, 2003; Ali et al, 2009; Sezgin et al, 2019; Stojanavic, 2021; Aybek, 2024).

2.8. Definition

Let $(F,D),(G,Y) \in S_E(U)$. The complementary extended $\circledast$ operation (F,D) and (G,Y) is the soft set (H,K), denoted by (F, D) $\overset{*}{\circledast}_\varepsilon$ (G, Y) = (H, K), where $K = D \cup Y$ and for $\forall ℵ \in K$,

$$H(ℵ) = \begin{cases} F'(ℵ), & ℵ \in D - Y \\ G'(ℵ), & ℵ \in Y - D \\ F(ℵ)\circledast G(ℵ), & ℵ \in D \cap Y \end{cases}$$

(Sarıalioğlu, 2024; Akbulut, 2024).

2.9. Definition

Let $(F,D),(G,Y) \in S_E(U)$. The soft binary piecewise $\circledast$ of (F,D) and (G,Y) is the soft set (H,D), denoted by $(F,D)\overset{\sim}{\circledast}(G,Y) = (H,D)$, where for all ℵ ∈ D

$$H(ℵ) = \begin{cases} F(ℵ), & ℵ \in D - Y \\ F(ℵ)\circledast G(ℵ), & ℵ \in D \cap Y \end{cases}$$

(Eren and Çalışıcı, 2019; Sezgin and Yavuz, 2023b; Sezgin and Çalışıcı, 2024; Yavuz, 2024).

2.10. Definition

Let $(F,D),(G,Y) \in S_E(U)$. The complementary soft binary piecewise $\circledast$ of (F,D) and (G,Y) is the soft set (H,D), denoted by $(F,D)\underset{\circledast}{\overset{*}{\sim}}(G,Y) = (H,D)$, where for all ℵ ∈ D

$$H(ℵ) = \begin{cases} F(ℵ), & ℵ \in D - Y \\ F(ℵ)\circledast G(ℵ), & ℵ \in D \cap Y \end{cases}$$

(Sezgin and Demirci, 2023; Sezgin and Aybek, 2023; Sezgin et al. 2023a, 2023b; Sezgin and Atagün, 2023; Sezgin and Yavuz, 2023a; Sezgin and Dagtoros, 2023; Sezgin and Çağman, 2024; Sezgin and Sarıalioğlu, 2024; Sezgin and Sarıalioğlu, 2024).

2.11. Definition

Let (S, ⍟) be an algebraic structure. An element s ∈S is called idempotent if s2=s. If for ∀s∈S, s2=s, then the algebraic structure (S,⍟) is said to be idempotent. An idempotent semigroup is called a band; an idempotent and commutative semigroup is called a semilattice; an idempotent and commutative monoid is called a bounded semilattice (Clifford, 1956).

In a monoid, although the identity element is unique, a semigroup/groupoid can have one or more left identities; however, if it has more than one left identity, it does not have a right identity element, thus it does not have an identity element. Similarly, a semigroup/groupoid can have one or more right identities; however, if it has more than one right identity, it does not have a left identity element, thus it does not have an identity element (Clifford, 1954).

Similarly, in a group, although each element has a unique inverse, in a monoid, an element can have one or more left inverses; however, if an element has more than one left inverse, it does not have a right inverse, thus it does not have an inverse. Similarly, in a monoid, an element can have one or more right inverses; however, if an element has more than one right inverse, it does not have a left inverse, thus it does not have an inverse (Clifford, 1954).

We refer to Pant et al. (2024) for the implications of network analysis and graph application for the possible implemetantation with respect to soft sets, which are determined by the divisibility of determinants.

COMPLEMENTARY EXTENDED THETA OPERATION

In this section, a new soft set operation called complementary extended theta operation of soft sets is introduced with its example and its full algebraic properties are analyzed.

Definition 3.1

Let (F, Ƶ) and (G, B) be soft sets over U. The complementary extended theta operation of (F,Ƶ) and (G, B) is the soft set (H, Ş), denoted by $(F,Ƶ) \overset{*}{\theta_\varepsilon} (G,B) = (H, Ş)$, where Ş= Ƶ ∪B and for ∀ℵ∈Ƶ ∪B;

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in B-Ƶ \\ F(\aleph)\theta G(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Here, $F(\aleph)\theta G(\aleph)= F'(\aleph)\cap G'(\aleph)$ for ∀ℵ∈Ƶ ∪B.

Example 3.2

Let $E=\{e_1,e_2,e_3,e_4\}$ be the parameter set, $Ƶ=\{e_1, e_3\}$ and $B=\{e_2, e_3, e_4\}$ be two subsets of E, and $U=\{h_1,h_2,h_3,h_4,h_5\}$ the universal set.

Assume that $(F,Ƶ)=\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$,$(G,B)=\{(e_2,\{h_1,h_4,h_5\}),(e_3,\{h_2,h_3,h_4\}),(e_4,\{h_3,h_5\})\}$ be two soft sets over U. Let $(F,Ƶ) \overset{*}{\theta_\varepsilon} (G,B)=(H,Ƶ\cup B)$, where ∀ℵ ∈Ƶ∪B;

$$H(\aleph)=\begin{cases} F'(\aleph) & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in B-Ƶ \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Since ƵUB={e_1,e_2,e_3,e_4}, Ƶ-B={e_1}, B-Ƶ={e_2,e_4}, Ƶ∩B={e_3} thus, $H(e_1)=F'(e_1)=\{h_1,h_3,h_4\}$, $H(e_2)=G'(e_2)=\{h_2,h_3\}$, $H(e_4)=G'(e_4)=\{h_1,h_2,h_4\}$ and $H(e_3)=F'(e_3)\cap G'(e_3)=\{h_3,h_4\}\cap\{h_1,h_4,h_5\}=\{h_4\}$. Hence, (F, Ƶ) $\overset{*}{\theta}_{\varepsilon}$(G,B)={(e_1,{h_1,h_3,h_4}), (e_2,{ h_2,h_3}),(e_3,{h_4}),(e_4,{h_1,h_2,h_4})}.

Theorem 3.3. *(Algebraic Properties of Operation)*

1) S_E (U) is closed under $\overset{*}{\theta}_{\varepsilon}$.

***Proof*:** It is clear that $\overset{*}{\theta}_{\varepsilon}$ is a binary operation in S_E(U). Indeed,

$$\overset{*}{\theta}_{\varepsilon} : S_E(U) \times S_E(U) \rightarrow S_E(U)$$

$$((F,Ƶ),(G,B)) \rightarrow (F,Ƶ)\overset{*}{\theta}_{\varepsilon}(G,B)=(H,Ƶ\cup B)$$

Similarly,

$$\overset{*}{\theta}_{\varepsilon} : S_Ƶ(U) \times S_Ƶ(U) \rightarrow S_Ƶ(U)$$

$$((F,Ƶ),(G,Ƶ)) \rightarrow (F,Ƶ)\overset{*}{\theta}_{\varepsilon}(G,Ƶ)=(T,Ƶ\cup Ƶ)=(T,Ƶ)$$

That is, when Ƶ is a fixed subset of the set E and (F, Ƶ) and (G, Ƶ) are elements of $S_Ƶ(U)$, then so is (F, Ƶ) $\overset{*}{\theta}_{\varepsilon}$ (G, Ƶ). Namely, $S_Ƶ(U)$ is closed under $\overset{*}{\theta}_{\varepsilon}$ either.

2) [(F, Ƶ) $\overset{*}{\theta}_{\varepsilon}$(G,B)] $\overset{*}{\theta}_{\varepsilon}$(H,Ş) ≠ (F, Ƶ) $\overset{*}{\theta}_{\varepsilon}$[(G,B) $\overset{*}{\theta}_{\varepsilon}$(H,Ş)].

Proof: Firstly, let's handle the left hand side (LHS). Let (F, Ƶ) $\overset{*}{\theta}_{\varepsilon}$(G,B)=(T,ƵUB), where, ∀ℵ∈ƵUB;

$$T(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in B-Ƶ \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Let (T,ƵUB) $\overset{*}{\theta}_{\varepsilon}$(H,Ş) = (M,ƵUBUŞ), where ∀ℵ∈ƵUBUŞ;

$$M(\aleph)=\begin{cases} T'(\aleph), & \aleph\in(Ƶ\cup B)-Ş \\ H'(\aleph), & \aleph\in Ş-(Ƶ\cup B) \\ T'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\cup B)\cap Ş \end{cases}$$

Thus,

$$M(\aleph)=\begin{cases} F(\aleph), & \aleph\in(Ƶ-B)-Ş=Ƶ\cap B'\cap Ş' \\ G(\aleph), & \aleph\in(B-Ƶ)-Ş=Ƶ'\cap B\cap Ş' \\ F(\aleph)\cup G(\aleph), & \aleph\in(Ƶ\cap B)-Ş=Ƶ\cap B\cap Ş' \\ H'(\aleph) & \aleph\in Ş-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş \\ F(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-B)\cap Ş=Ƶ\cap B'\cap Ş \\ G(\aleph)\cap H'(\aleph), & \aleph\in(B-Ƶ)\cap Ş=Ƶ'\cap B\cap Ş \\ \big(F(\aleph)\cup G(\aleph)\big)\cap H'(\aleph), & \aleph\in(Ƶ\cap B)\cap Ş=Ƶ\cap B\cap Ş \end{cases}$$

Now let's handle the right hand side (RHS) of the equation i.e. (F, Ƶ) $\overset{*}{\theta}_{\varepsilon}$[(G,B) $\overset{*}{\theta}_{\varepsilon}$(H,Ş)]. Let (G,B) $\overset{*}{\theta}_{\varepsilon}$(H,Ş)=(K,BUŞ), where ∀ℵ∈BUŞ;

$$K(\aleph)=\begin{cases} G'(\aleph), & \aleph\in B-Ş \\ H'(\aleph), & \aleph\in Ş-B \\ G'(\aleph)\cap H'(\aleph), & \aleph\in B\cap Ş \end{cases}$$

Let (F, Ƶ) $\overset{*}{\theta}_{\varepsilon}$(K,BUŞ) = (S,ƵUBUŞ), where ∀ℵ∈ƵUBUŞ;

$$S(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-(B\cup Ş) \\ K'(\aleph), & \aleph\in (B\cup Ş)-Ƶ \\ F'(\aleph)\cap K'(\aleph), & \aleph\in Ƶ\cap (B\cup Ş) \end{cases}$$

Thus,

$$S(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ G(\aleph), & \aleph\in (B-Ş)-Ƶ=Ƶ'\cap B\cap Ş' \\ H(\aleph) & \aleph\in (Ş-B)-Ƶ=Ƶ'\cap B'\cap Ş \\ G(\aleph)\cup H(\aleph), & \aleph\in (B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş \\ F'(\aleph)\cap G(\aleph), & \aleph\in Ƶ\cap (B-Ş)=Ƶ\cap B\cap Ş' \\ F'(\aleph)\cap H(\aleph), & \aleph\in Ƶ\cap (Ş-B)=Ƶ\cap B'\cap Ş \\ F'(\aleph)\cap (G(\aleph)\cup H(\aleph)), & \aleph\in Ƶ\cap (B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

It is seen that M≠ S. That is, in $S_E(U)$, $\overset{*}{\theta_\varepsilon}$ is not associative.

3) $[(F,Ƶ)\ \overset{*}{\theta_\varepsilon}(G,Ƶ)]\ \overset{*}{\theta_\varepsilon}(H,Ƶ) \neq (F,Ƶ)\ \overset{*}{\theta_\varepsilon}[(G,Ƶ)\ \overset{*}{\theta_\varepsilon}(H,Ƶ)]$.

Proof: Firstly, let's look at the LHS. Let $(F,Ƶ)\ \overset{*}{\theta_\varepsilon}(G,Ƶ)=(T,Ƶ\cup Ƶ)$, where $\forall\aleph\in Ƶ\cup Ƶ=Ƶ$;

$$T(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ G'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Let $(T,Ƶ)\ \overset{*}{\theta_\varepsilon}(H,Ƶ) = (M,Ƶ\cup Ƶ)$, where $\forall\aleph\in Ƶ$;

$$M(\aleph)=\begin{cases} T'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ H'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ T'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Thus,

$$M(\aleph)=\begin{cases} T'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ H'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ (F(\aleph)\cup G(\aleph))\cap H'(\aleph), & \aleph\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Now let's handle RHS. Let $(G,Ƶ)\ \overset{*}{\theta_\varepsilon}(H,Ƶ)=(L,Ƶ\cup Ƶ)$, where $\forall\aleph\in Ƶ$;

$$L(\aleph)=\begin{cases} G'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ H'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ G'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Let $(F,Ƶ)\ \overset{*}{\theta_\varepsilon}(L,Ƶ)=(N,Ƶ\cup Ƶ)$, where $\forall\aleph\in Ƶ$;

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ L'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ F'(\aleph)\cap L'(\aleph), & \aleph\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Hence,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ L'(\aleph), & \aleph\in Ƶ-Ƶ=\emptyset \\ F'(\aleph)\cap (G(\aleph)\cup H(\aleph)), & \aleph\in Ƶ\cap Ƶ=Ƶ \end{cases}$$

Thus, it is observed that M≠N. That is, $\overset{*}{\theta_\varepsilon}$ is not associative in $S_Ƶ(U)$, where $Ƶ \subseteq E$ is a fixed subset of E.

4) $(F,Ƶ)\ \overset{*}{\theta_\varepsilon}(G,B)=(G,B)\ \overset{*}{\theta_\varepsilon}(F,Ƶ)$.

Proof: Consider first the LHS. Let $(F,Ƶ)\ \overset{*}{\theta_\varepsilon}(G,B)=(H,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$;

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}B \\ G'(\aleph), & \aleph\in B\text{-}Z \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Z\cap B \end{cases}$$

Now let's handle the RHS. Assume that (G,B) $\overset{*}{\theta_\varepsilon}$(F,Z)=(T,B∪Z), where ∀ℵ∈B∪Z;

$$T(\aleph)=\begin{cases} G'(\aleph), & \aleph\in B\text{-}Z \\ F'(\aleph), & \aleph\in Z\text{-}B \\ G'(\aleph)\cap F'(\aleph), & \aleph\in B\cap Z \end{cases}$$

Thus, it is seen that H=T. Similarly, it is easily seen that (F, Z) $\overset{*}{\theta_\varepsilon}$(G,Z)=(G, Z) $\overset{*}{\theta_\varepsilon}$(F,Z). That is, $\overset{*}{\theta_\varepsilon}$ is commutative in both $S_E(U)$ and $S_Z(U)$.

5) (F, Z) $\overset{*}{\theta_\varepsilon}$(F,Z)=$(F,Z)^r$.

***Proof*:** Let (F, Z) $\overset{*}{\theta_\varepsilon}$(F,Z)=(H,Z∪Z), where ∀ℵ∈Z;

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ F'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ F'(\aleph)\cap F'(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Hence, ∀ℵ∈Z; H(ℵ)= F'(ℵ)∩F'(ℵ)=F'(ℵ) and (H,Z)=$(F,Z)^r$.

The operation $\overset{*}{\theta_\varepsilon}$ does not have the idempotent property in $S_E(U)$.

6) (F, Z) $\overset{*}{\theta_\varepsilon}$ $\emptyset_Z$=$\emptyset_Z$ $\overset{*}{\theta_\varepsilon}$(F,Z)=$(F,Z)^r$.

***Proof*:** Let $\emptyset_Z$=(S,Z). Thus ∀ℵ∈Z; S(ℵ)= ∅. Let (F, Z) $\overset{*}{\theta_\varepsilon}$(S,Z)=(H,Z∪Z), where ∀ℵ∈Z;

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ S'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ F'(\aleph)\cap S'(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Hence, ∀ℵ∈Z; H(ℵ)= F'(ℵ)∩S'(ℵ)= F'(ℵ)∩U= F'(ℵ) and (H,Z)=$(F,Z)^r$.

7) (F, Z) $\overset{*}{\theta_\varepsilon}$ $\emptyset_\emptyset$= $\emptyset_\emptyset$ $\overset{*}{\theta_\varepsilon}$(F,Z) = $(F,Z)^r$.

***Proof*:** Let $\emptyset_\emptyset$= (K, ∅) and (F, Z) $\overset{*}{\theta_\varepsilon}$(K, ∅) = (Q,Z∪∅) = (Q, Z), where ∀ℵ∈Z;

$$Q(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}\emptyset=Z \\ K'(\aleph), & \aleph\in \emptyset\text{-}Z=\emptyset \\ F'(\aleph)\cap K'(\aleph), & \aleph\in Z\cap\emptyset=\emptyset \end{cases}$$

Hence, ∀ℵ∈Z; Q(ℵ) = F'(ℵ) and (Q,Z) = $(F,Z)^r$.

8) (F, Z) $\overset{*}{\theta_\varepsilon}$ U_Z=U_Z $\overset{*}{\theta_\varepsilon}$(F,Z)= $\emptyset_Z$.

***Proof*:** Let U_Z=(T,Z), where ∀ℵ∈Z; T(ℵ)=U. Let (F, Z) $\overset{*}{\theta_\varepsilon}$(T, Z)=(H,Z∪Z), where ∀ℵ∈Z;

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ T'(\aleph), & \aleph\in Z\text{-}Z=\emptyset \\ F'(\aleph)\cap T'(\aleph), & \aleph\in Z\cap Z=Z \end{cases}$$

Hence ∀ℵ∈Z; H(ℵ)= F'(ℵ)∩T'(ℵ)= F'(ℵ)∩∅=∅ and (H,Z)= $\emptyset_Z$.

9) (F, Z) $\overset{*}{\theta_\varepsilon}$ U_E=U_E $\overset{*}{\theta_\varepsilon}$(F, Z) =$\emptyset_E$.

***Proof*:** Let U_E=(S,E), where ∀ℵ∈E; S(ℵ)=U. Let (F, Z) $\overset{*}{\theta_\varepsilon}$(S,E)=(H,Z∪E), where ∀ℵ∈Z∪E=E;

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Z\text{-}E=\emptyset \\ S'(\aleph), & \aleph\in E\text{-}Z=Z' \\ F'(\aleph)\cap S'(\aleph), & \aleph\in Z\cap E=Z \end{cases}$$

Here, $\forall ℵ \in Ƶ \cap E = Ƶ$, $F'(ℵ) \cap S'(ℵ) = F'(ℵ) \cap \emptyset = \emptyset$ and so,

$$H(ℵ)=\begin{cases} F'(ℵ), & ℵ \in Ƶ - E = \emptyset \\ \emptyset, & ℵ \in E - Ƶ = Ƶ' \\ \emptyset, & ℵ \in Ƶ \cap E = Ƶ \end{cases}$$

Thus, $(H,Ƶ) = \emptyset_E$.

10) $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (F,Ƶ)^r = (F,Ƶ)^r \overset{*}{\theta}_{\varepsilon} (F,Ƶ) = \emptyset_Ƶ$.

***Proof*:** Let $(F,Ƶ)^r = (H,Ƶ)$, where $\forall ℵ \in Ƶ$; $H(ℵ) = F'(ℵ)$. Let $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (H,Ƶ) = (T, Ƶ \cup Ƶ)$, where $\forall ℵ \in Ƶ$;

$$T(ℵ)=\begin{cases} F'(ℵ), & ℵ \in Ƶ - Ƶ = \emptyset \\ H'(ℵ), & ℵ \in Ƶ - Ƶ = \emptyset \\ F'(ℵ) \cap H'(ℵ), & ℵ \in Ƶ \cap Ƶ = Ƶ \end{cases}$$

Thus, $\forall ℵ \in Ƶ$; $T(ℵ) = F'(ℵ) \cap H'(ℵ) = F'(ℵ) \cap F(ℵ) = \emptyset$ and so $(T,Ƶ) = \emptyset_Ƶ$.

11) $[(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B)]^r = (F,Ƶ) \cup_{\varepsilon} (G,B)$.

***Proof*:** Let $(F,Ƶ) \overset{*}{\theta} (G,B) = (H, Ƶ \cup B)$, where $\forall ℵ \in Ƶ \cup B$;

$$H(ℵ)=\begin{cases} F'(ℵ), & ℵ \in Ƶ - B \\ G'(ℵ), & ℵ \in B - Ƶ \\ F'(ℵ) \cap G'(ℵ), & ℵ \in Ƶ \cap B \end{cases}$$

Let $(H, Ƶ \cup B)^r = (T, Ƶ \cup B)$, where $\forall ℵ \in Ƶ \cup B$;

$$T(ℵ)=\begin{cases} F(ℵ), & ℵ \in Ƶ - B \\ G(ℵ), & ℵ \in B - Ƶ \\ F(ℵ) \cup G(ℵ), & ℵ \in Ƶ \cap B \end{cases}$$

Hence, $(T, Ƶ \cup B) = (F,Ƶ) \cup_{\varepsilon} (G,B)$.

12) $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B) = U_{Ƶ \cup B} \Leftrightarrow (F,Ƶ) = \emptyset_Ƶ$ and $(G,B) = \emptyset_B$.

***Proof*:** Let $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B) = (T, Ƶ \cup B)$, where $\forall ℵ \in Ƶ \cup B$;

$$T(ℵ)=\begin{cases} F'(ℵ), & ℵ \in Ƶ - B \\ G'(ℵ), & ℵ \in B - Ƶ \\ F'(ℵ) \cap G'(ℵ), & ℵ \in Ƶ \cap B \end{cases}$$

Sie $(T, Ƶ \cup B) = U_{Ƶ \cup B}$, $T(ℵ) = U$ for $\forall ℵ \in Ƶ \cup B$. Hence, $\forall ℵ \in Ƶ - B$; $F'(ℵ) = U$, $\forall ℵ \in B - Ƶ$; $G'(ℵ) = U$ and $\forall ℵ \in Ƶ \cap B$; $F'(ℵ) \cap G'(ℵ) = U$. Thus, $\forall ℵ \in Ƶ - B$; $F(ℵ) = \emptyset$, $\forall ℵ \in B - Ƶ$; $G(ℵ) = \emptyset$, $\forall ℵ \in Ƶ \cap B$; $F'(ℵ) = U$ and $G'(ℵ) = U$. Thus $\forall ℵ \in Ƶ \cap B$; $F(ℵ) = \emptyset$, $G(ℵ) = \emptyset$. Thus, $\forall ℵ \in Ƶ$; $F(ℵ) = \emptyset$ and $\forall ℵ \in B$; $G(ℵ) = \emptyset$. So, $(F,Ƶ) = \emptyset_Ƶ$ and $(G,B) = \emptyset_B$.

13) $\emptyset_Ƶ \tilde{\subseteq} (F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B)$, $\emptyset_B \tilde{\subseteq} (F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B)$, $\emptyset_{Ƶ \cup B} \tilde{\subseteq} (F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B)$, $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B) \tilde{\subseteq} U_{Ƶ \cup B}$.

14) $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,Ƶ) \tilde{\subseteq} (F,Ƶ)^r$ and $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,Ƶ) \tilde{\subseteq} (G,Ƶ)^r$.

***Proof*:** Let $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B) = (H, Ƶ \cup Ƶ)$, where $\forall ℵ \in Ƶ$;

$$H(ℵ)=\begin{cases} F'(ℵ), & ℵ \in Ƶ - Ƶ = \emptyset \\ G'(ℵ), & ℵ \in Ƶ - Ƶ = \emptyset \\ F'(ℵ) \cap G'(ℵ), & ℵ \in Ƶ \cap Ƶ = Ƶ \end{cases}$$

Sinnce $\forall ℵ \in Ƶ$; $H(ℵ) = F'(ℵ) \cap G'(ℵ) \subseteq F'(ℵ)$. Thus, $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,Ƶ) \tilde{\subseteq} (F,Ƶ)^r$. Similarly, $\forall ℵ \in Ƶ$; $H(ℵ) = F'(ℵ) \cap G'(ℵ) \subseteq G'(ℵ)$. Thus, $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,Ƶ) \tilde{\subseteq} (G,Ƶ)^r$.

15) If $(F,Ƶ) \tilde{\subseteq} (G,B)$, then $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,B) = (G,B)^r$ and if $(F,Ƶ) \tilde{\subseteq} (G,Ƶ)$, then $(F,Ƶ) \overset{*}{\theta}_{\varepsilon} (G,Ƶ) = (G,Ƶ)^r$.

Proof: Let $(F,Ƶ)\ \tilde{\subseteq}\ (G,B)$. So $Ƶ\subseteq B$ and $\forall\aleph\in Ƶ$, $F(\aleph)\subseteq G(\aleph)$. Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)=(H,Ƶ\cup B=B)$, where $\forall\aleph\in Ƶ\cup B$;

$$H(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B=\emptyset \\ G'(\aleph), & \aleph\in B-Ƶ \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Ƶ\cap B=Ƶ \end{cases}$$

Since $\forall\aleph\in Ƶ$, $F(\aleph)\subseteq G(\aleph)$, thus $G'(\aleph)\subseteq F'(\aleph)$. Hence, $\forall\aleph\in Ƶ$; $F'(\aleph)\cap G'(\aleph)=G'(\aleph)$. So, $\forall\aleph\in B$; $H(\aleph)=G'(\aleph)$. Thus, $(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)=(G,B)^r$. Similarly, if $(F,Ƶ)\ \tilde{\subseteq}\ (G,Ƶ)$ then $(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,Ƶ)=(G,Ƶ)^r$ be shown.

16) If $(F,Ƶ)\ \tilde{\subseteq}(G,Ƶ)$, then $(G,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)\ \tilde{\subseteq}\ (F,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)$.

Proof: Let $(F,Ƶ)\ \tilde{\subseteq}\ (G,Ƶ)$. Thus, $\forall\aleph\in Ƶ$, $F(\aleph)\subseteq G(\aleph)$ and so $\forall\aleph\in Ƶ$, $G'\aleph)\subseteq F'(\aleph)$. Let $(G,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)=(W,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$,

$$W(\aleph)=\begin{cases} G'(\aleph), & \aleph\in Ƶ-B \\ H'(\aleph), & \aleph\in B-Ƶ \\ G'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)=(L,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$,

$$L(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ H'(\aleph), & \aleph\in B-Ƶ \\ F'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Thus, if $\forall\aleph\in Ƶ-B$; $W(\aleph)=G'(\aleph)\subseteq F'(\aleph)=L(\aleph)$, if $\forall\aleph\in B-Ƶ$; $W(\aleph)=H'(\aleph)\subseteq H'(\aleph)=L(\aleph)$ and if $\forall\aleph\in Ƶ\cap B$; $W(\aleph)=G'(\aleph)\cap H'(\aleph)\subseteq F'(\aleph)\cap H'(\aleph)=L(\aleph)$. Hence, $(G,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)\ \tilde{\subseteq}\ (F,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)$.

17) If $(G,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)\ \tilde{\subseteq}\ (F,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)$, then $(F,Ƶ)\ \tilde{\subseteq}(G,Ƶ)$ needs not have to be true. That is, the converse of Theorem 3.3. (16) is not true.

Proof: Let us give an example to show that the converse of Theorem 3.3 (16) is not true. Let $E=\{e_1,e_2,e_3,e_4,e_5\}$ be the parameter set, $Ƶ=\{e_1,e_3\}$ and $B=\{e_1,e_3,e_5\}$be two subsets of E, $U=\{h_1,h_2,h_3,h_4,h_5\}$ be the universal set.

Let $(F,Ƶ)=\{(e_1,\{h_2,h_5\}),(e_3,\{h_1,h_2,h_5\})\}$, $(G,Ƶ)=\{(e_1,\{h_2\}),(e_3,\{h_1,h_2\})\}$,$(H,B)=\{(e_1,U),(e_3,U),(e_5,\{h_2,h_5\})\}$ be soft sets over U. Let $(G,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)=(L,Ƶ\cup B)$, then $(L,Ƶ\cup B)=\{(e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_1,h_5\})\}$ and let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)=(K,Ƶ\cup B)$, then $(K,Ƶ\cup B)=\{(e_1,\emptyset),(e_3,\emptyset),(e_5,\{h_1,h_5\})\}$. Hence, $(G,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)\ \tilde{\subseteq}\ (F,Ƶ)\overset{*}{\theta_\varepsilon}(H,B)$ but $(F,Ƶ)$ is not a subset of $(G,Ƶ)$.

18) If $(F,Ƶ)\ \tilde{\subseteq}\ (G,Ƶ)$ and $(K,B)\ \tilde{\subseteq}\ (L,B)$, then $(G,Ƶ)\overset{*}{\theta_\varepsilon}(L,B)\ \tilde{\subseteq}(F,Ƶ)\overset{*}{\theta_\varepsilon}(K,B)$.

Proof: Let $(F,Ƶ)\ \tilde{\subseteq}\ (G,Ƶ)$ and $(K,B)\ \tilde{\subseteq}\ (L,B)$. Hence, $\forall\aleph\in Ƶ$, $F(\aleph)\subseteq G(\aleph)$ and so $G'(\aleph)\subseteq F'(\aleph)$ and $\forall\aleph\in B$, $K(\aleph)\subseteq L(\aleph)$ and thus $L'(\aleph)\subseteq K'(\aleph)$. Let $(G,Ƶ)\overset{*}{\theta_\varepsilon}(L,B)=(W,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$,

$$W(\aleph)=\begin{cases} G'(\aleph), & \aleph\in Ƶ-B \\ L'(\aleph), & \aleph\in B-Ƶ \\ G'(\aleph)\cap L'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(K,B)=(S,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$,

$$S(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ K'(\aleph), & \aleph\in B-Ƶ \\ F'(\aleph)\cap K'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Thus, if $\forall\aleph\in Ƶ-B$; $W(\aleph)=G'(\aleph)\subseteq F'(\aleph)=S(\aleph)$; if $\forall\aleph\in B-Ƶ$; $W(\aleph)=L'(\aleph)\subseteq K'(\aleph)=S(\aleph)$ and if $\forall\aleph\in Ƶ\cap B$; $W(\aleph)=G'(\aleph)\cap L'(\aleph)\subseteq F'(\aleph)\cap K'(\aleph)=S(\aleph)$. Thus, $(G,Ƶ)\overset{*}{\theta_\varepsilon}(L,B)\ \tilde{\subseteq}\ (F,Ƶ)\overset{*}{\theta_\varepsilon}(K,B)$.

Theorem 3.4.

The complementary extended theta operation has the following distributions over other soft set operations:

Theorem 3.4.1.

The complementary extended theta operation has the following distributions over restricted soft set operations:

i) LHS Distributions of the Complementary Extended Theta Operation on Restricted Soft Set Operations:

1) $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\cap_R(H,Ş)]=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)]\cup_R[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

Proof: Consider first the LHS. Let $(G,B)\cap_R(H,Ş)=(M,B\cap Ş)$, where $\forall ℵ\in B\cap Ş$; $M(ℵ)=G(ℵ)\cap H(ℵ)$. Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(M,B\cap Ş)=(N,Ƶ\cup(B\cap Ş))$, where $\forall ℵ\in Ƶ\cup(B\cap Ş)$;

$$N(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Ƶ-(B\cap Ş)\\ M'(ℵ), & ℵ\in(B\cap Ş)-Ƶ\\ F'(ℵ)\cap M'(ℵ), & ℵ\in Ƶ\cap(B\cap Ş)\end{cases}$$

Thus,

$$N(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Ƶ-(B\cap Ş)=Ƶ-(B\cap Ş)\\ G'(ℵ)\cup H'(ℵ), & ℵ\in(B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş\\ F'(ℵ)\cap(G'(ℵ)\cup H'(ℵ)) & ℵ\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş\end{cases}$$

Now lets handle the RHS i.e. $[(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)]\cup_R[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)=(V,Ƶ\cup B)$, where $\forall ℵ\in Ƶ\cup B$;

$$V(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Ƶ-B\\ G'(ℵ), & ℵ\in B-Ƶ\\ F'(ℵ)\cap G'(ℵ), & ℵ\in Ƶ\cap B\end{cases}$$

Assume that $(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)=(W,Ƶ\cup Ş)$, where $\forall ℵ\in Ƶ\cup Ş$;

$$W(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Ƶ-Ş\\ H'(ℵ), & ℵ\in Ş-Ƶ\\ F'(ℵ)\cap H'(ℵ), & ℵ\in Ƶ\cap Ş\end{cases}$$

Let $(V,Ƶ\cup B)\cup_R(W,Ƶ\cup Ş)=(T,(Ƶ\cup B)\cap(Ƶ\cup Ş))$, where $\forall ℵ\in Ƶ\cup(B\cap Ş)$; $T(ℵ)=V(ℵ)\cup W(ℵ)$. Hence;

$$T(ℵ)=\begin{cases} F'(ℵ)\cup F'(ℵ), & ℵ\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş'\\ F'(ℵ)\cup H'(ℵ), & ℵ\in(Ƶ-B)\cap(Ş-Ƶ)=\emptyset\\ F'(ℵ)\cup(F'(ℵ)\cap H'(ℵ)) & ℵ\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş\\ G'(ℵ)\cup F'(ℵ), & ℵ\in(B-Ƶ)\cap(Ƶ-Ş)=\emptyset\\ G'(ℵ)\cup H'(ℵ), & ℵ\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş\\ G'(ℵ)\cup\big(F'(ℵ)\cap H'(ℵ)\big), & ℵ\in(B-Ƶ)\cap(Ƶ\cap Ş)=\emptyset\\ (F'(ℵ)\cap G'(ℵ))\cup F'(ℵ), & ℵ\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş'\\ (F'(ℵ)\cap G'(ℵ))\cup H'(ℵ), & ℵ\in(Ƶ\cap B)\cap(Ş-Ƶ)=\emptyset\\ (F'(ℵ)\cap G'(ℵ))\cup(F'(ℵ)\cap H'(ℵ)), & ℵ\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş\end{cases}$$

Thus,

$$T(ℵ)=\begin{cases} F'(ℵ), & ℵ\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş'\\ F'(ℵ) & ℵ\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş\\ G'(ℵ)\cup H'(ℵ), & ℵ\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş\\ F'(ℵ) & ℵ\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş'\\ (F'(ℵ)\cap G'(ℵ))\cup(F'(ℵ)\cap H'(ℵ)), & ℵ\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş\end{cases}$$

Hence, N=T.

2) If Ƶ'∩B∩Ş= Ƶ∩B∩Ş=∅, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\cup_R(H,Ş)]=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)]\cup_R[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

3) If Ƶ'∩B∩Ş= Ƶ∩B∩Ş=∅, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)*_R(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(G,B)\cup_R[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(H,Ş)]$.

4) If Ƶ'∩B∩Ş=∅, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\theta_R(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(G,B)]\cup_R[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(H,Ş)]$.

ii) RHS Distribution of Complementary Extended Theta Operation on Restricted Soft Set Operations

1) If (ƵΔB)∩ Ş=∅, then $[(F,Ƶ)\cup_R(G,B)]\overset{*}{\theta_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\cap_R[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

Proof: Consider first LHS. Let $(F,Ƶ)\cup_R(G,B)=(M,Ƶ\cap B)$, where ∀ℵ∈Ƶ∩B; M(ℵ)=F(ℵ)∪G(ℵ). Let $(M,Ƶ\cap B)\overset{*}{\theta_\varepsilon}(H,Ş)=(N,(Ƶ\cap B)\cup Ş))$, where ∀ℵ∈(Ƶ∩B)∪Ş;

$$N(\aleph)=\begin{cases} M'(\aleph), & \aleph\in(Ƶ\cap B)-Ş \\ H'(\aleph), & \aleph\in Ş-(Ƶ\cap B) \\ M'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap(B\cap Ş) \end{cases}$$

Thus,

$$N(\aleph)=\begin{cases} F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ\cap B)-Ş=Ƶ\cap B\cap Ş' \\ H'(\aleph), & \aleph\in Ş-(Ƶ\cap B)=Ş-(Ƶ\cap B) \\ (F'(\aleph)\cap G'(\aleph))\cap H'(\aleph), & \aleph\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Now consider RHS. i.e. $[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\cap_R[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)=(V,Ƶ\cup Ş)$, where ∀ℵ∈Ƶ∪Ş;

$$V(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ş \\ H'(\aleph), & \aleph\in Ş-Ƶ \\ F'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap Ş \end{cases}$$

Now, let $(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)=(W,B\cup Ş)$, where ∀ℵ∈B∪Ş;

$$W(\aleph)=\begin{cases} G'(\aleph), & \aleph\in B-Ş \\ H'(\aleph), & \aleph\in Ş-B \\ G'(\aleph)\cap H'(\aleph), & \aleph\in B\cap Ş \end{cases}$$

Let $(V,Ƶ\cup Ş)\cap_R(W,B\cup Ş)=(T,(Ƶ\cup Ş)\cap(B\cup Ş))$, where ∀ℵ∈(Ƶ∩B)∪S; T(ℵ)=V(ℵ)∩W(ℵ). Thus,

$$T(\aleph)=\begin{cases} F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ-Ş)\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-Ş)\cap(Ş-B)=\emptyset \\ F'(\aleph)\cap\big(G'(\aleph)\cap H'(\aleph)\big), & \aleph\in(Ƶ-Ş)\cap(B\cap Ş)=\emptyset \\ H'(\aleph)\cap G'(\aleph), & \aleph\in(Ş-Ƶ)\cap(B-Ş)=\emptyset \\ H'(\aleph)\cap H'(\aleph), & \aleph\in(Ş-Ƶ)\cap(Ş-B)=Ƶ'\cap B'\cap Ş \\ H'(\aleph)\cap(G'(\aleph)\cap H'(\aleph)), & \aleph\in(Ş-Ƶ)\cap(B\cap Ş)=Ƶ'\cap B\cap Ş \\ \big(F'(\aleph)\cap H'(\aleph)\big)\cap G'(\aleph), & \aleph\in(Ƶ\cap Ş)\cap(B-Ş)=\emptyset \\ \big(F'(\aleph)\cap H'(\aleph)\big)\cap H'(\aleph) & \aleph\in(Ƶ\cap Ş)\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ (F'(\aleph)\cap H'(\aleph))\cap(G'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Thus,

$$T(\aleph)=\begin{cases} F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ-Ş)\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ H'(\aleph) & \aleph\in(Ş-Ƶ)\cap(Ş-B)=Ƶ'\cap B'\cap Ş \\ G'(\aleph)\cap H'(\aleph), & \aleph\in(Ş-Ƶ)\cap(B\cap Ş)=Ƶ'\cap B\cap Ş \\ \big(F'(\aleph)\cap H'(\aleph)\big), & \aleph\in(Ƶ\cap Ş)\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ (F'(\aleph)\cap H(\aleph))\cap(G'(\aleph)\cap H(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Here, when considering Ƶ-(B∩Ş) in the function N, since Ƶ-(B∩Ş)=Ƶ-(B∩Ş)', then if an element is in the complement of (B∩Ş), it is either in B-Ş, in Ş-B, or in (B ∪ Ş)′. Thus, if ℵ ∈ Ƶ −(B∩Ş), then ℵ ∈ Ƶ ∩B∩Ş' or

$\aleph \in Ƶ \cap B' \cap Ş$ or $\aleph \in Ƶ \cap B' \cap Ş'$. Hence, N=T is satisfied under the condition $Z' \cap B \cap Ş = Z \cap B' \cap Ş = \emptyset$. The condition $Z' \cap B \cap Ş = Z \cap B' \cap Ş = \emptyset$ implies that $(Z \Delta B) \cap Ş = \emptyset$ is obvious.

2)$[(F,Ƶ) \cap_R (G,B)] \overset{*}{\theta_\varepsilon} (H,Ş) = [(F,Ƶ) \overset{*}{\theta_\varepsilon} (H,Ş)] \cup_R [(G,B) \overset{*}{\theta_\varepsilon} (H,Ş)]$.

3) If $Ƶ \cap B \cap Ş' = Ƶ \cap B \cap Ş = \emptyset$, then $(F,Ƶ) *_R (G,B)] \overset{*}{\theta_\varepsilon} (H,Ş) = [(F,Ƶ) \overset{*}{\backslash_\varepsilon} (H,Ş)] \cup_R [(G,B) \overset{*}{\backslash_\varepsilon} (H,Ş)]$.

4) If $Ƶ \cap B \cap Ş' = Ƶ \cap B \cap Ş = \emptyset$, then $(F,Ƶ) \theta_R (G,B)] \overset{*}{\theta_\varepsilon} (H,Ş) = [(F,Ƶ) \overset{*}{\backslash_\varepsilon} (H,Ş)] \cup_R [(G,B) \overset{*}{\backslash_\varepsilon} (H,Ş)]$.

Theorem 3.4.2.

The following distributions of the complementary extended theta operation over extended soft set operations hold:

i)LHS Distributions of the Complementary Extended Theta Operation over Extended Soft Set Operations

1) If $Ƶ' \cap B \cap Ş = Ƶ \cap B \cap Ş = \emptyset$, then $(F,Ƶ) \overset{*}{\theta_\varepsilon} [(G,B) \cap_\varepsilon (H,Ş)] = [(F,Ƶ) \overset{*}{\theta_\varepsilon} (G,B)] \cap_\varepsilon [(F,Ƶ) \overset{*}{\theta_\varepsilon} (H,Ş)]$.

***Proof*:** Consider first LHS. Let $(G,B) \cap_\varepsilon (H,Ş) = (M, B \cup Ş)$, where $\forall \aleph \in B \cup Ş$;

$$M(\aleph) = \begin{cases} G(\aleph), & \aleph \in B - Ş \\ H(\aleph), & \aleph \in Ş - B \\ G(\aleph) \cap H(\aleph), & \aleph \in B \cap Ş \end{cases}$$

Let $(F,Ƶ) \overset{*}{\theta_\varepsilon} (M, B \cup Ş) = (N, Ƶ \cup (B \cup Ş))$, where $\forall \aleph \in Ƶ \cup B \cup Ş$;

$$N(\aleph) = \begin{cases} F'(\aleph), & \aleph \in Ƶ - (B \cup Ş) \\ M'(\aleph), & \aleph \in (B \cup Ş) - Ƶ \\ F'(\aleph) \cap M'(\aleph), & \aleph \in Ƶ \cap (B \cup Ş) \end{cases}$$

Thus,

$$N(\aleph) = \begin{cases} F'(\aleph), & \aleph \in Ƶ - (B \cup Ş) = Ƶ \cap B' \cap Ş' \\ G'(\aleph), & \aleph \in (B - Ş) - Ƶ = Ƶ' \cap B \cap Ş' \\ H'(\aleph), & \aleph \in (Ş - B) - Ƶ = Ƶ' \cap B' \cap Ş \\ G'(\aleph) \cup H'(\aleph), & \aleph \in (B \cap Ş) - Ƶ = Ƶ' \cap B \cap Ş \\ F'(\aleph) \cap G'(\aleph), & \aleph \in Ƶ \cap (B - Ş) = Ƶ \cap B \cap Ş' \\ F'(\aleph) \cap H'(\aleph), & \aleph \in Ƶ \cap (Ş - B) = Ƶ \cap B' \cap Ş \\ F'(\aleph) \cap (G'(\aleph) \cup H'(\aleph)), & \aleph \in Ƶ \cap (B \cap Ş) = Ƶ \cap B \cap Ş \end{cases}$$

Now consider the RHS, i.e. $[(F,Ƶ) \overset{*}{\theta_\varepsilon} (G,B)] \cap_\varepsilon [(F,Ƶ) \overset{*}{\theta_\varepsilon} (H,Ş)]$. Let $(F,Ƶ) \overset{*}{\theta_\varepsilon} (G,B) = (V, Ƶ \cup B)$, where $\forall \aleph \in Ƶ \cup B$;

$$V(\aleph) = \begin{cases} F'(\aleph), & \aleph \in Ƶ - B \\ G'(\aleph), & \aleph \in B - Ƶ \\ F'(\aleph) \cap G'(\aleph), & \aleph \in Ƶ \cap B \end{cases}$$

Let $(F,Ƶ) \overset{*}{\theta_\varepsilon} (H,Ş) = (W, Ƶ \cup Ş)$, where $\forall \aleph \in Ƶ \cup Ş$;

$$W(\aleph) = \begin{cases} F'(\aleph), & \aleph \in Ƶ - Ş \\ H'(\aleph), & \aleph \in Ş - Ƶ \\ F'(\aleph) \cap H'(\aleph), & \aleph \in Ƶ \cap Ş \end{cases}$$

Let $(V, Ƶ \cup B) \cap_\varepsilon (W, Ƶ \cup Ş) = (T, (Ƶ \cup B) \cup Ş)$, where $\forall \aleph \in Ƶ \cup B \cup Ş$;

$$T(\aleph) = \begin{cases} V(\aleph), & \aleph \in (Ƶ \cup B) - (Ƶ \cup Ş) \\ W(\aleph), & \aleph \in (Ƶ \cup Ş) - (Ƶ \cup B) \\ V(\aleph) \cap W(\aleph), & \aleph \in (Ƶ \cup B) \cap (Ƶ \cup Ş) \end{cases}$$

Thus,

$$
T(\aleph)=\begin{cases}
F'(\aleph), & \aleph\in(Ƶ-B)-(Ƶ\cup Ş)=\emptyset \\
G'(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\
F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ\cap B)-(Ƶ\cup Ş)=\emptyset \\
F'(\aleph), & \aleph\in(Ƶ-Ş)-(Ƶ\cup B)=\emptyset \\
H'(\aleph), & \aleph\in(Ş-Ƶ)-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş \\
F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\cap Ş)-(Ƶ\cup B)=\emptyset \\
F'(\aleph)\cap F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş' \\
F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-B)\cap(Ş-Ƶ)=\emptyset \\
F'(\aleph)\cap(F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\
G'(\aleph)\cap F'(\aleph), & \aleph\in(B-Ƶ)\cap(Ƶ-Ş)=\emptyset \\
G'(\aleph)\cap H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş \\
G'(\aleph)\cap(F'(\aleph)\cap H'(\aleph)), & \aleph\in(B-Ƶ)\cap(Ƶ\cap Ş)=\emptyset \\
(F'(\aleph)\cap G'(\aleph))\cap F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş' \\
(F'(\aleph)\cap G'(\aleph))\cap H'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ş-Ƶ)=\emptyset \\
(F'(\aleph)\cap G'(\aleph))\cap(F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş
\end{cases}
$$

Thus,

$$
T(\aleph)=\begin{cases}
G'(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\
H'(\aleph), & \aleph\in(Ş-Ƶ)-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş \\
F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş' \\
F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\
G'(\aleph)\cap H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş \\
F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş' \\
(F'(\aleph)\cap G'(\aleph))\cap(F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş
\end{cases}
$$

It is seen that N=T is satisfied under the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B\cap Ş=\emptyset$.

2) $(F,Ƶ)\overset{*}{\theta}_{\varepsilon}[(G,B)\cup_{\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(G,B)]\cap_{\varepsilon}[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(H,Ş)]$.

3) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\theta}_{\varepsilon}[(G,B)*_{\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma}_{\varepsilon}(G,B)]\cap_{\varepsilon}[(F,Ƶ)\overset{*}{\gamma}_{\varepsilon}(H,Ş)]$

4) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\theta}_{\varepsilon}[(G,B)\theta_{\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma}_{\varepsilon}(G,B)]\cup_{\varepsilon}[(F,Ƶ)\overset{*}{\gamma}_{\varepsilon}(H,Ş)]$

ii) RHS Distributions of Complementary Extended Theta Operation over Extended Soft Set Operations

1) $([(F,Ƶ)\cup_{\varepsilon}(G,B)]\overset{*}{\theta}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(H,Ş)]\cap_{\varepsilon}[(G,B)\overset{*}{\theta}_{\varepsilon}(H,Ş)]$.

Proof: Consider first the LHS. Let $(F,Ƶ)\cup_{\varepsilon}(G,B)=(M,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$;

$$
M(\aleph)=\begin{cases}
F(\aleph), & \aleph\in Ƶ-B \\
G(\aleph), & \aleph\in B-Ƶ \\
F(\aleph)\cup G(\aleph), & \aleph\in Ƶ\cap B
\end{cases}
$$

Let $(M,Ƶ\cup B)\overset{*}{\theta}_{\varepsilon}(H,Ş)=(N,(Ƶ\cup B)\cup Ş$, where $\forall\aleph\in Ƶ\cup B\cup Ş$;

$$
N(\aleph)=\begin{cases}
M'(\aleph), & \aleph\in(Ƶ\cup B)-Ş \\
H'(\aleph), & \aleph\in Ş-(Ƶ\cup B) \\
M'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\cup B)\cap Ş
\end{cases}
$$

Thus,

$$N(ℵ)=\begin{cases} F'(ℵ), & ℵ\in(Ƶ-B)-Ş=Ƶ\cap B'\cap Ş' \\ G'(ℵ), & ℵ\in(B-Ƶ)-Ş=Ƶ'\cap B\cap Ş' \\ F'(ℵ)\cap G'(ℵ), & ℵ\in(Ƶ\cap B)-Ş=Ƶ\cap B\cap Ş' \\ H'(ℵ), & ℵ\in Ş-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş \\ F'(ℵ)\cap H'(ℵ), & ℵ\in(Ƶ-B)\cap Ş=Ƶ\cap B'\cap Ş \\ G'(ℵ)\cap H'(ℵ), & ℵ\in(B-Ƶ)\cap Ş=Ƶ'\cap B\cap Ş \\ (F'(ℵ)\cap G'(ℵ))\cap H'(ℵ), & ℵ\in(Ƶ\cap B)\cap Ş=Ƶ\cap B\cap Ş \end{cases}$$

Now consider the RHS, i.e. $[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\cap_\varepsilon[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)=(V,Ƶ\cup Ş)$, where $\forall ℵ\in Ƶ\cup Ş$;

$$V(ℵ)=\begin{cases} F'(ℵ), & ℵ\in Ƶ-Ş \\ H'(ℵ), & ℵ\in Ş-Ƶ \\ F'(ℵ)\cap H'(ℵ), & ℵ\in Ƶ\cap Ş \end{cases}$$

Let $(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)=(W,B\cup Ş)$, where $\forall ℵ\in B\cup Ş$;

$$W(ℵ)=\begin{cases} G'(ℵ), & ℵ\in B-Ş \\ H'(ℵ), & ℵ\in Ş-B \\ G'(ℵ)\cap H'(ℵ), & ℵ\in B\cap Ş \end{cases}$$

Let $(V,Ƶ\cup Ş)\cap_\varepsilon(W,B\cup Ş)=(T,Ƶ\cup B\cup Ş)$, where $\forall ℵ\in Ƶ\cup B\cup Ş$;

$$T(ℵ)=\begin{cases} V(ℵ), & ℵ\in(Ƶ\cup Ş)-(B\cup Ş) \\ W(ℵ), & ℵ\in(B\cup Ş)-(Ƶ\cup Ş) \\ V(ℵ)\cap W(ℵ), & ℵ\in(Ƶ\cup Ş)\cap(B\cup Ş) \end{cases}$$

Thus,

$$T(ℵ)=\begin{cases} F'(ℵ), & ℵ\in(Ƶ-Ş)-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ H'(ℵ), & ℵ\in(Ş-Ƶ)-(B\cup Ş)=\emptyset \\ F'(ℵ)\cap H'(ℵ), & ℵ\in(Ƶ\cap Ş)-(B\cup Ş)=\emptyset \\ G'(ℵ), & ℵ\in(B-Ş)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ H'(ℵ), & ℵ\in(Ş-B)-(Ƶ\cup Ş)=\emptyset \\ G'(ℵ)\cap H'(ℵ), & ℵ\in(B\cap Ş)-(Ƶ\cup Ş)=\emptyset \\ F'(ℵ)\cap G'(ℵ), & ℵ\in(Ƶ-Ş)\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ F'(ℵ)\cap H'(ℵ), & ℵ\in(Ƶ-Ş)\cap(Ş-B)=\emptyset \\ F'(ℵ)\cap(G'(ℵ)\cap H'(ℵ)), & ℵ\in(Ƶ-Ş)\cap(B\cap Ş)=\emptyset \\ H'(ℵ)\cap G'(ℵ), & ℵ\in(Ş-Ƶ)\cap(B-Ş)=\emptyset \\ H'(ℵ)\cap H'(ℵ), & ℵ\in(Ş-Ƶ)\cap(Ş-B)=Ƶ'\cap B'\cap Ş \\ H'(ℵ)\cap\big(G'(ℵ)\cap H'(ℵ)\big) & ℵ\in(Ş-Ƶ)\cap(B\cap Ş)=Ƶ'\cap B\cap Ş \\ (F'(ℵ)\cap H'(ℵ))\cap G'(ℵ), & ℵ\in(Ƶ\cap Ş)\cap(B-Ş)=\emptyset \\ (F'(ℵ)\cap H'(ℵ))\cap H'(ℵ) & ℵ\in(Ƶ\cap Ş)\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ (F'(ℵ)\cap H'(ℵ))\cap(G'(ℵ)\cap H'(ℵ)) & ℵ\in(Ƶ\cap Ş)\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Thus,

$$T(ℵ)=\begin{cases} F'(ℵ), & ℵ\in(Ƶ-Ş)-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ G'(ℵ), & ℵ\in(B-Ş)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F'(ℵ)\cap G'(ℵ), & ℵ\in(Ƶ-Ş)\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ H'(ℵ), & ℵ\in(Ş-Ƶ)\cap(Ş-B)=Ƶ'\cap B'\cap Ş \\ G'(ℵ)\cap H'(ℵ) & ℵ\in(Ş-Ƶ)\cap(B\cap Ş)=Ƶ'\cap B\cap Ş \\ F'(ℵ)\cap H'(ℵ) & ℵ\in(Ƶ\cap Ş)\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ (F'(ℵ)\cap H'(ℵ))\cap(G'(ℵ)\cap H'(ℵ)) & ℵ\in(Ƶ\cap Ş)\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Hence, N=T.

2) If $Ƶ\cap B\cap Ş'=Ƶ\cap B\cap Ş=\emptyset$, then $[(F,Ƶ)\cap_\varepsilon(G,B)]\overset{*}{\theta_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\cap_\varepsilon[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

3) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\theta_\varepsilon(G,B)]\overset{*}{\theta_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\setminus_\varepsilon}(H,Ş)]\cup_\varepsilon[(G,B)\overset{*}{\setminus_\varepsilon}(H,Ş)]$.

4) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\ast_{\varepsilon}(G,B)]\overset{*}{\theta_{\varepsilon}}(H,Ş)=[(F,Ƶ)\overset{*}{\setminus_{\varepsilon}}(H,Ş)]\cap_{\varepsilon}[(G,B)\overset{*}{\setminus_{\varepsilon}}(H,Ş)]$.

Theorem 3.4.3.

The following distributions of the complementary extended theta operation over complementary extended operations hold:

i) LHS Distributions of Complementary Extended Theta Operations over Complementary Extended Soft Set Operations

1) If $Ƶ\cap(B\Delta Ş)=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_{\varepsilon}}[(G,B)\overset{*}{\cap_{\varepsilon}}(H,Ş)]=[(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(G,B)]\overset{*}{\cup_{\varepsilon}}[(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(H,Ş)]$.

Proof: Consider first LHS. Let $(G,B)\overset{*}{\cap_{\varepsilon}}((H,Ş)=(M,B\cup Ş)$, where $\forall\aleph\in B\cup Ş$;

$$M(\aleph)=\begin{cases} G'(\aleph), & \aleph\in B-Ş \\ H'(\aleph), & \aleph\in Ş-B \\ G(\aleph)\cap H(\aleph), & \aleph\in B\cap Ş \end{cases}$$

Let $(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(M,B\cup Ş)=(N,Ƶ\cup(B\cup Ş))$, where $\forall\aleph\in Ƶ\cup B\cup Ş$;

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-(B\cup Ş) \\ M'(\aleph), & \aleph\in(B\cup Ş)-Ƶ \\ F'(\aleph)\cap M'(\aleph), & \aleph\in Ƶ\cap(B\cup Ş) \end{cases}$$

Thus,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ G(\aleph), & \aleph\in(B-Ş)-Ƶ=Ƶ'\cap B\cap Ş' \\ H(\aleph), & \aleph\in(Ş-B)-Ƶ=A'\cap B'\cap Ş \\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş \\ F'(\aleph)\cap G(\aleph), & \aleph\in Ƶ\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ F'(\aleph)\cap H(\aleph), & \aleph\in Ƶ\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ F'(\aleph)\cap(G'(\aleph)\cup H'(\aleph)), & \aleph\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Now consider the RHS, i.e. $[(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(G,B)]\overset{*}{\cup_{\varepsilon}}[(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(G,B)=(V,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$;

$$V(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in B-Ƶ \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Let $(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(H,Ş)=(W,Ƶ\cup Ş)$, where $\forall\aleph\in Ƶ\cup Ş$;

$$W(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ş \\ H'(\aleph), & \aleph\in Ş-Ƶ \\ F'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap Ş \end{cases}$$

Let $(V,Ƶ\cup B)\overset{*}{\cup_{\varepsilon}}(W,Ƶ\cup Ş)=(T,(Ƶ\cup B)\cup Ş)$, where $\forall\aleph\in Ƶ\cup B\cup Ş$;

$$T(\aleph)=\begin{cases} V'(\aleph), & \aleph\in(Ƶ\cup B)-(Ƶ\cup Ş) \\ W'(\aleph), & \aleph\in(Ƶ\cup Ş)-(Ƶ\cup B) \\ V(\aleph)\cup W(\aleph), & \aleph\in(Ƶ\cup B)\cap(Ƶ\cup Ş) \end{cases}$$

Thus,

$$
T(\aleph)=\begin{cases}
F(\aleph), & \aleph\in(Ƶ-B)-(Ƶ\cup Ş)=\emptyset\\
G(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş'\\
F(\aleph)\cup G(\aleph), & \aleph\in(Ƶ\cap B)-(Ƶ\cup Ş)=\emptyset\\
F(\aleph), & \aleph\in(Ƶ-Ş)-(Ƶ\cup B)=\emptyset\\
H(\aleph), & \aleph\in(Ş-Ƶ)-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş\\
F(\aleph)\cup H(\aleph), & \aleph\in(Ƶ\cap Ş)-(Ƶ\cup B)=\emptyset\\
F'(\aleph)\cup F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş'\\
F'(\aleph)\cup H'(\aleph), & \aleph\in(Ƶ-B)\cap(Ş-Ƶ)=\emptyset\\
F'(\aleph)\cup(F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş\\
G'(\aleph)\cup F'(\aleph), & \aleph\in(B-Ƶ)\cap(Ƶ-Ş)=\emptyset\\
G'(\aleph)\cup H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş\\
G'(\aleph)\cup(F'(\aleph)\cap H'(\aleph)), & \aleph\in(B-Ƶ)\cap(Ƶ\cap Ş)=\emptyset\\
(F'(\aleph)\cap G'(\aleph))\cup F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş'\\
(F'(\aleph)\cap G'(\aleph))\cup H'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ş-Ƶ)=\emptyset\\
(F'(\aleph)\cap G'(\aleph))\cup(F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş
\end{cases}
$$

Hence,

$$
T(\aleph)=\begin{cases}
G(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş'\\
H(\aleph), & \aleph\in(Ş-Ƶ)-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş\\
F'(\aleph) & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş'\\
F'(\aleph) & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş\\
G'(\aleph)\cup H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş\\
F'(\aleph) & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş'\\
(F'(\aleph)\cap G'(\aleph))\cup(F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş
\end{cases}
$$

N=T is satisfied under the condition $Z'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$. It is obvious that the condition $Z'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$ is equivalent to $(Z\Delta B)\cap Ş=\emptyset$.

2) If $Ƶ\cap(B\Delta Ş)=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\overset{*}{\cup_\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)]\overset{*}{\cap_\varepsilon}[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

3) If $Ƶ'\cap B\cap Ş=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\overset{*}{*_\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(G,B)]\overset{*}{\cap_\varepsilon}[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(H,Ş)]$.

4) If $Ƶ'\cap B\cap Ş=Ƶ\cap B\cap Ş=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(G,B)]\overset{*}{\cap_\varepsilon}[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(H,Ş)]$.

ii) RHS Distributions of Complementary Extended Theta Operation over Complementary Extended Operations

1) If $Ƶ\cap(B\Delta Ş)=\emptyset$ then $[(F,Ƶ)\overset{*}{\cup_\varepsilon}(G,B)]\overset{*}{\theta_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\overset{*}{\cap_\varepsilon}[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

Proof: Consider first LHS. Let $(F,Ƶ)\overset{*}{\cup_\varepsilon}(G,B)=(M,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$;

$$
M(\aleph)=\begin{cases}
F'(\aleph), & \aleph\in Ƶ-B\\
G'(\aleph), & \aleph\in B-Ƶ\\
F(\aleph)\cup G(\aleph), & \aleph\in Ƶ\cap B
\end{cases}
$$

Let $(M,Ƶ\cup B)\overset{*}{\theta_\varepsilon}(H,Ş)=(N,(Ƶ\cup B)\cup Ş)$, where $\forall\aleph\in Ƶ\cup B\cup Ş$;

$$
N(\aleph)=\begin{cases}
M'(\aleph), & \aleph\in(Ƶ\cup B)-Ş\\
H'(\aleph), & \aleph\in Ş-(Ƶ\cup B)\\
M'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ\cup B)\cap Ş
\end{cases}
$$

Thus,

$$N(\aleph)=\begin{cases} F(\aleph), & \aleph\in(Ƶ-B)-Ş=Ƶ\cap B'\cap Ş' \\ G(\aleph), & \aleph\in(B-Ƶ)-Ş=Ƶ'\cap B\cap Ş' \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ\cap B)-Ş=Ƶ\cap B\cap Ş' \\ H'(\aleph), & \aleph\in Ş-(Ƶ\cup B)=Ƶ'\cap B'\cap Ş \\ F(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-B)\cap Ş=Ƶ\cap B'\cap Ş \\ G(\aleph)\cap H'(\aleph), & \aleph\in(B-Ƶ)\cap Ş=Ƶ'\cap B\cap Ş \\ (F'(\aleph)\cap G'(\aleph))\cap H'(\aleph), & \aleph\in(Ƶ\cap B)\cap Ş=Ƶ\cap B\cap Ş \end{cases}$$

Now consider the RHS, i.e. $[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\overset{*}{\cap_\varepsilon}[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)=(V,Ƶ\cup Ş)$, where $\forall\aleph\in Ƶ\cup Ş$;

$$V(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ş \\ H'(\aleph), & \aleph\in Ş-Ƶ \\ F'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap Ş \end{cases}$$

Let $(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)=(W,B\cup Ş)$, where $\forall\aleph\in B\cup Ş$;

$$W(\aleph)=\begin{cases} G'(\aleph), & \aleph\in B-Ş \\ H'(\aleph), & \aleph\in Ş-B \\ G'(\aleph)\cap H'(\aleph), & \aleph\in B\cap Ş \end{cases}$$

Let $(V,Ƶ\cup Ş)\overset{*}{\cap_\varepsilon}(W,B\cup Ş)=(T,Ƶ\cup B\cup Ş)$, where $\forall\aleph\in Ƶ\cup B\cup Ş$;

$$T(\aleph)=\begin{cases} V'(\aleph), & \aleph\in(Ƶ\cup Ş)-(B\cup Ş) \\ W'(\aleph), & \aleph\in(B\cup Ş)-(Ƶ\cup Ş) \\ V(\aleph)\cap W(\aleph), & \aleph\in(Ƶ\cup Ş)\cap(B\cup Ş) \end{cases}$$

Thus,

$$T(\aleph)=\begin{cases} F(\aleph), & \aleph\in(Ƶ-Ş)-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ H(\aleph), & \aleph\in(Ş-Ƶ)-(B\cup Ş)=\emptyset \\ F(\aleph)\cup H(\aleph), & \aleph\in(Ƶ\cap Ş)-(B\cup Ş)=\emptyset \\ G(\aleph), & \aleph\in(B-Ş)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ H(\aleph), & \aleph\in(Ş-B)-(Ƶ\cup Ş)=\emptyset \\ G(\aleph)\cup H(\aleph), & \aleph\in(B\cap Ş)-(Ƶ\cup Ş)=\emptyset \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ-Ş)\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-Ş)\cap(Ş-B)=\emptyset \\ F'(\aleph)\cap(G'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ-Ş)\cap(B\cap Ş)=\emptyset \\ H'(\aleph)\cap G'(\aleph), & \aleph\in(Ş-Ƶ)\cap(B-Ş)=\emptyset \\ H'(\aleph)\cap H'(\aleph), & \aleph\in(Ş-Ƶ)\cap(Ş-B)=Ƶ'\cap B'\cap Ş \\ H'(\aleph)\cap\big(G'(\aleph)\cap H'(\aleph)\big) & \aleph\in(Ş-Ƶ)\cap(B\cap Ş)=Ƶ'\cap B\cap Ş \\ (F'(\aleph)\cap H'(\aleph))\cap G'(\aleph), & \aleph\in(Ƶ\cap Ş)\cap(B-Ş)=\emptyset \\ \big(F'(\aleph)\cap H'(\aleph)\big)\cap H'(\aleph) & \aleph\in(Ƶ\cap Ş)\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ (F'(\aleph)\cap H'(\aleph))\cap(G'(\aleph)\cap H'(\aleph)) & \aleph\in(Ƶ\cap Ş)\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Thus,

$$T(\aleph)=\begin{cases} F(\aleph), & \aleph\in(Ƶ-Ş)-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ G(\aleph), & \aleph\in(B-Ş)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ-Ş)\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ H'(\aleph), & \aleph\in(Ş-Ƶ)\cap(Ş-B)=Ƶ'\cap B'\cap Ş \\ G'(\aleph)\cap H'(\aleph), & \aleph\in(Ş-Ƶ)\cap(B\cap Ş)=Ƶ'\cap B\cap Ş \\ F'(\aleph)\cap H'(\aleph) & \aleph\in(Ƶ\cap Ş)\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ (F'(\aleph)\cap H'(\aleph))\cap(G'(\aleph)\cap H'(\aleph)) & \aleph\in(Ƶ\cap Ş)\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

It is seen that N=T under the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$. It is obvious that the condition $Ƶ'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$ is equivalent to the condition $(Ƶ\Delta B)\cap Ş=\emptyset$.

2) If $(Ƶ\Delta B)\cap Ş=\emptyset$, then $[(F,Ƶ)\overset{*}{\cap_\varepsilon}(G,B)]\overset{*}{\theta_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\overset{*}{\cup_\varepsilon}[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

3) If $Ƶ\cap B\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(G,B)]\overset{*}{\theta}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\backslash}_{\varepsilon}(H,Ş)]\overset{*}{\cap}_{\varepsilon}[(G,B)\overset{*}{\backslash}_{\varepsilon}(H,Ş)]$.

4) If $Ƶ\cap B\cap Ş'=\emptyset$, then $[(F,Ƶ)\overset{*}{*}_{\varepsilon}(G,B)]\overset{*}{\theta}_{\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\backslash}_{\varepsilon}(H,Ş)]\overset{*}{\cap}_{\varepsilon}[(G,B)\overset{*}{\backslash}_{\varepsilon}(H,Ş)]$.

Theorem 3.4.4.

The following distributions of the complementary extended theta operation over soft binary piecewise operations hold:

i) LHS Distributions of the Complementary Extended Theta Operation on Soft Binary Pievewise Operations

1) If $Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\theta}_{\varepsilon}[(G,B)\overset{\sim}{\cap}(H,Ş)]=[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(G,B)]\overset{\sim}{\cup}[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(H,Ş)]$.

Proof: Consider first the LHS. Let $(G,B)\overset{\sim}{\cap}(H,Ş)=(M,B)$. Hence $\forall\aleph\in B$;

$$M(\aleph)=\begin{cases} G(\aleph), & \aleph\in B-Ş \\ G(\aleph)\cap H(\aleph), & \aleph\in B\cap Ş \end{cases}$$

Let $(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(M,B)=(N,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$;

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ M'(\aleph), & \aleph\in B-Ƶ \\ F'(\aleph)\cap M'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Thus,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in(B-Ş)-Ƶ=Ƶ'\cap B\cap Ş' \\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Ƶ\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ F'(\aleph)\cap(G'(\aleph)\cup H('\aleph)), & \aleph\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Now consider the RHS, i.e. $[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(G,B)]\overset{\sim}{\cup}[(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(G,B)=(V,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$;

$$V(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-B \\ G'(\aleph), & \aleph\in B-Ƶ \\ F'(\aleph)\cap G'(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Let $(F,Ƶ)\overset{*}{\theta}_{\varepsilon}(H,Ş)=(W,Ƶ\cup Ş)$, where $\forall\aleph\in Ƶ\cup Ş$;

$$W(\aleph)=\begin{cases} F'(\aleph), & \aleph\in Ƶ-Ş \\ H'(\aleph), & \aleph\in Ş-Ƶ \\ F'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap Ş \end{cases}$$

Let $(V,Ƶ\cup B)\overset{\sim}{\cup}(W,Ƶ\cup Ş)=(T,(Ƶ\cup B))$, where $\forall\aleph\in Ƶ\cup B$;

$$T(\aleph)=\begin{cases} V(\aleph), & \aleph\in(Ƶ\cup B)-(Ƶ\cup Ş) \\ V(\aleph)\cup W(\aleph), & \aleph\in(Ƶ\cup B)\cap(Ƶ\cup Ş) \end{cases}$$

Thus,

$$T(\aleph)=\begin{cases} F'(\aleph), & \aleph\in(Ƶ-B)-(Ƶ\cup Ş)=\emptyset \\ G'(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ\cap B)-(Ƶ\cup Ş)=\emptyset \\ F'(\aleph)\cup F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş' \\ F'(\aleph)\cup H'(\aleph), & \aleph\in(Ƶ-B)\cap(Ş-Ƶ)=\emptyset \\ F'(\aleph)\cup(F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(\aleph)\cup F'(\aleph), & \aleph\in(B-Ƶ)\cap(Ƶ-Ş)=\emptyset \\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş \\ G'(\aleph)\cup(F'(\aleph)\cap H'(\aleph)), & \aleph\in(B-Ƶ)\cap(Ƶ\cap Ş)=\emptyset \\ (F'(\aleph)\cap G'(\aleph))\cup F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş' \\ (F'(\aleph)\cap G'(\aleph))\cup H'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ş-Ƶ)=\emptyset \\ (F'(\aleph)\cap G'(\aleph))\cup(F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Therefore,

$$T(\aleph)=\begin{cases} G'(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş' \\ F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş' \\ F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş \\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş \\ F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş' \\ (F'(\aleph)\cap G'(\aleph))\cup (F'(\aleph)\cap H'(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

It is seen that N=T is satisfied under the condition $Ƶ\cap B\cap Ş'=\emptyset$.

2) If $Ƶ\cap B'\cap Ş=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\,\tilde{\cup}\,(H,Ş)]=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)]\,\tilde{\cap}\,[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

3) If $(Ƶ\Delta Ş)\cap B=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\,\tilde{*}\,(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(G,B)]\,\tilde{\cup}\,[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(H,Ş)]$.

4) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\,\tilde{\theta}\,(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(G,B)]\,\tilde{\cup}\,[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(H,Ş)]$.

ii) RHS Distributions of the Complementary Extended Theta Operation over Soft Binary Piecewise Operations

1) If $Ƶ'\cap B\cap Ş=\emptyset$, then $(F,Ƶ)\tilde{\cup}(G,B)]\overset{*}{\theta_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\tilde{\cap}[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

Proof: Consider first LHS. Let $(F,Ƶ)\tilde{\cup}(G,B)=(M,Ƶ)$, where $\forall\aleph\in Ƶ$,

$$M(\aleph)=\begin{cases} F(\aleph), & \aleph\in Ƶ-B \\ F(\aleph)\cup G(\aleph), & \aleph\in Ƶ\cap B \end{cases}$$

Let $(M,Ƶ)\overset{*}{\theta}(H,Ş)=(N,Ƶ\cup Ş)$, where $\forall\aleph\in Ƶ\cup Ş$;

$$N(\aleph)=\begin{cases} M'(\aleph), & \aleph\in Ƶ-Ş \\ H'(\aleph), & \aleph\in Ş-Ƶ \\ M'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap Ş \end{cases}$$

Thus,

$$N(\aleph)=\begin{cases} F'(\aleph), & \aleph\in(Ƶ-B)-Ş=Ƶ\cap B'\cap Ş' \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ\cap B)-Ş=Ƶ\cap B\cap Ş' \\ H'(\aleph), & \aleph\in Ş-Ƶ \\ F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-B)\cap Ş=Ƶ\cap B'\cap Ş \\ (F'(\aleph)\cap G'(\aleph))\cap H'(\aleph), & \aleph\in(Ƶ\cap B)\cap Ş=Ƶ\cap B\cap Ş \end{cases}$$

Now consider the RHS, that is, $[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\tilde{\cap}[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)=(V,Ƶ\cup Ş)$, where $\forall\aleph\in Ƶ\cup Ş$;

$$V(ℵ)=\begin{cases}F'(ℵ), & ℵ∈Ƶ-Ş\\ H'(ℵ), & ℵ∈Ş-Ƶ\\ F'(ℵ)∩H'(ℵ), & ℵ∈Ƶ∩Ş\end{cases}$$

Now let $(G,B) \overset{*}{\theta_\varepsilon} (H,Ş)=(W,B∪Ş)$, where $∀ℵ∈B∪Ş$;

$$W(ℵ)=\begin{cases}G'(ℵ), & ℵ∈B-Ş\\ H'(ℵ), & ℵ∈Ş-B\\ G'(ℵ)∩H'(ℵ), & ℵ∈B∩Ş\end{cases}$$

Let $(V,Ƶ∪Ş) \overset{\sim}{∩} (W,B∪Ş)=(T,(Ƶ∪Ş))$, where $∀ℵ∈Ƶ∪Ş$;

$$T(ℵ)=\begin{cases}V(ℵ), & ℵ∈(Ƶ∪Ş)-(B∪Ş)\\ V(ℵ)∩W(ℵ), & ℵ∈(Ƶ∪Ş)∩(B∪Ş)\end{cases}$$

Thus,

$$T(ℵ)=\begin{cases}F'(ℵ), & ℵ∈(Ƶ-Ş)-(B∪Ş)=Ƶ∩B'∩Ş'\\ H'(ℵ), & ℵ∈(Ş-Ƶ)-(B∪Ş)=\emptyset\\ F'(ℵ)∩H'(ℵ), & ℵ∈(Ƶ∩Ş)-(B∪Ş)=\emptyset\\ F'(ℵ)∩G'(ℵ), & ℵ∈(Ƶ-Ş)∩(B-Ş)=Ƶ∩B∩Ş'\\ F'(ℵ)∩H'(ℵ), & ℵ∈(Ƶ-Ş)∩(Ş-B)=\emptyset\\ F'(ℵ)∩(G'(ℵ)∩H'(ℵ)), & ℵ∈(Ƶ-Ş)∩(B∩Ş)=\emptyset\\ H'(ℵ)∩G'(ℵ), & ℵ∈(Ş-Ƶ)∩(B-Ş)=\emptyset\\ H'(ℵ)∩H'(ℵ), & ℵ∈(Ş-Ƶ)∩(Ş-B)=Ƶ'∩B'∩Ş\\ H'(ℵ)∩(G'(ℵ)∩H'(ℵ)) & ℵ∈(Ş-Ƶ)∩(B∩Ş)=Ƶ'∩B∩Ş\\ (F'(ℵ)∩H'(ℵ))∩G'(ℵ), & ℵ∈(Ƶ∩Ş)∩(B-Ş)=\emptyset\\ (F'(ℵ)∩H'(ℵ))∩H'(ℵ) & ℵ∈(Ƶ∩Ş)∩(Ş-B)=Ƶ∩B'∩Ş\\ (F'(ℵ)∩H'(ℵ))∩(G'(ℵ)∩H'(ℵ)), & ℵ∈(Ƶ∩Ş)∩(B∩Ş)=Ƶ∩B∩Ş\end{cases}$$

Hence,

$$T(ℵ)=\begin{cases}F'(ℵ), & ℵ∈(Ƶ-Ş)-(B∪Ş)=Ƶ∩B'∩Ş'\\ F'(ℵ)∩G'(ℵ), & ℵ∈(Ƶ-Ş)∩(B-Ş)=Ƶ∩B∩Ş'\\ H'(ℵ), & ℵ∈(Ş-Ƶ)∩(Ş-B)=Ƶ'∩B'∩Ş\\ G'(ℵ)∩H'(ℵ), & ℵ∈(Ş-Ƶ)∩(B∩Ş)=Ƶ'∩B∩Ş\\ F'(ℵ)∩H'(ℵ) & ℵ∈(Ƶ∩Ş)∩(Ş-B)=Ƶ∩B'∩Ş\\ (F'(ℵ)∩H'(ℵ))∩(G'(ℵ)∩H'(ℵ)), & ℵ∈(Ƶ∩Ş)∩(B∩Ş)=Ƶ∩B∩Ş\end{cases}$$

It is seen that N=T is satisfied under the condition $Ƶ'∩B∩Ş=\emptyset$.

2) If $(Ƶ\Delta B)∩Ş=\emptyset$, then $(F,Ƶ)\overset{\sim}{∩}(G,B)] \overset{*}{\theta} (H,Ş) =[(F,Ƶ) \overset{*}{\theta_\varepsilon} (H,Ş)]\overset{\sim}{∪}[(G,B) \overset{*}{\theta_\varepsilon} (H,Ş)]$.

3) If $Ƶ∩(B\Delta Ş)=\emptyset$, then $[(F,Ƶ)\overset{\sim}{\theta} (G,B)] \overset{*}{\theta_\varepsilon} (H,Ş) =[(F,Ƶ) \overset{*}{\backslash_\varepsilon} (H,Ş)]\overset{\sim}{∪}[(G,B) \overset{*}{\backslash_\varepsilon} (H,Ş)]$.

4) If $(Ƶ\Delta B)∩Ş=Ƶ∩B∩Ş'=\emptyset$, then $[(F,Ƶ) \overset{\sim}{*} (G,B)] \overset{*}{\theta_\varepsilon}(H,Ş) =[(F,Ƶ) \overset{*}{\backslash_\varepsilon} (H,Ş)]\overset{\sim}{∩}[(G,B) \overset{*}{\backslash_\varepsilon} (H,Ş)]$.

Theorem 3.4.5.

The following distributions of the complementary extended theta operation over the complementary soft binary piecewise operations exist:

i) LHS Distribution of the Complementary Extended Theta Operation on Complementary Soft Binary Piecewise Operations

1) If $Ƶ∩B∩Ş'=\emptyset$, then $(F,Ƶ) \overset{*}{\theta_\varepsilon} [(G,B) \overset{*}{\underset{∩}{\sim}} (H,Ş)]=[(F,Ƶ) \overset{*}{\theta_\varepsilon} (G,B)] \overset{*}{\underset{∪}{\sim}} [(F,Ƶ) \overset{*}{\theta_\varepsilon} (H,Ş)]$.

Proof: Consider first LHS. Let $(G,B)\overset{*}{\underset{\cap}{\sim}}(H,Ş)=(M,B)$. Hence $\forall\aleph\in B$;

$$M(\aleph)=\begin{cases}G'(\aleph), & \aleph\in B-Ş\\ G(\aleph)\cap H(\aleph), & \aleph\in B\cap Ş\end{cases}$$

Let $(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(M,B)=(N,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$;

$$N(\aleph)=\begin{cases}F'(\aleph), & \aleph\in Ƶ-B\\ M'(\aleph), & \aleph\in B-Ƶ\\ F'(\aleph)\cap M'(\aleph), & \aleph\in Ƶ\cap B\end{cases}$$

Thus,

$$N(\aleph)=\begin{cases}F'(\aleph), & \aleph\in Ƶ-B\\ G(\aleph), & \aleph\in(B-Ş)-Ƶ=Ƶ'\cap B\cap Ş'\\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B\cap Ş)-Ƶ=Ƶ'\cap B\cap Ş\\ F'(\aleph)\cap G(\aleph), & \aleph\in Ƶ\cap(B-Ş)=Ƶ\cap B\cap Ş'\\ F'(\aleph)\cap(G'(\aleph)\cup H'(\aleph)), & \aleph\in Ƶ\cap(B\cap Ş)=Ƶ\cap B\cap Ş\end{cases}$$

Now consider RHS, i.e. $[(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(G,B)]\overset{*}{\underset{\cup}{\sim}}[(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(G,B)=(V,Ƶ\cup B)$, where $\forall\aleph\in Ƶ\cup B$;

$$V(\aleph)=\begin{cases}F'(\aleph), & \aleph\in Ƶ-B\\ G'(\aleph), & \aleph\in B-Ƶ\\ F'(\aleph)\cap G'(\aleph), & \aleph\in Ƶ\cap B\end{cases}$$

Let $(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(H,Ş)=(W,Ƶ\cup Ş)$, where $\forall\aleph\in Ƶ\cup Ş$;

$$W(\aleph)=\begin{cases}F'(\aleph), & \aleph\in Ƶ-Ş\\ H'(\aleph), & \aleph\in Ş-Ƶ\\ F'(\aleph)\cap H'(\aleph), & \aleph\in Ƶ\cap Ş\end{cases}$$

Let $(V,Ƶ\cup B)\overset{*}{\underset{\cup}{\sim}}(W,Ƶ\cup Ş)=(T,(Ƶ\cup B))$, where $\forall\aleph\in Ƶ\cup B$;

$$T(\aleph)=\begin{cases}V'(\aleph), & \aleph\in(Ƶ\cup B)-(Ƶ\cup Ş)\\ V(\aleph)\cup W(\aleph), & \aleph\in(Ƶ\cup B)\cap(Ƶ\cup Ş)\end{cases}$$

Thus,

$$T(\aleph)=\begin{cases}F(\aleph), & \aleph\in(Ƶ-B)-(Ƶ\cup Ş)=\emptyset\\ G(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş'\\ F(\aleph)\cup G'(\aleph), & \aleph\in(Ƶ\cap B)-(Ƶ\cup Ş)=\emptyset\\ F'(\aleph)\cup F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş'\\ F'(\aleph)\cup H'(\aleph), & \aleph\in(Ƶ-B)\cap(Ş-Ƶ)=\emptyset\\ F'(\aleph)\cup(F'(\aleph)\cap H(\aleph)), & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş\\ G'(\aleph)\cup F'(\aleph), & \aleph\in(B-Ƶ)\cap(Ƶ-Ş)=\emptyset\\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş\\ G'(\aleph)\cup(F'(\aleph)\cap H(\aleph)), & \aleph\in(B-Ƶ)\cap(Ƶ\cap Ş)=\emptyset\\ (F'(\aleph)\cap G(\aleph))\cup F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş'\\ (F'(\aleph)\cap G(\aleph))\cup H'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ş-Ƶ)=\emptyset\\ (F'(\aleph)\cap G(\aleph))\cup(F'(\aleph)\cap H(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş\end{cases}$$

Hence,

$$T(\aleph)=\begin{cases}G(\aleph), & \aleph\in(B-Ƶ)-(Ƶ\cup Ş)=Ƶ'\cap B\cap Ş'\\ F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ-Ş)=Ƶ\cap B'\cap Ş'\\ F'(\aleph), & \aleph\in(Ƶ-B)\cap(Ƶ\cap Ş)=Ƶ\cap B'\cap Ş\\ G'(\aleph)\cup H'(\aleph), & \aleph\in(B-Ƶ)\cap(Ş-Ƶ)=Ƶ'\cap B\cap Ş\\ F'(\aleph), & \aleph\in(Ƶ\cap B)\cap(Ƶ-Ş)=Ƶ\cap B\cap Ş'\\ (F'(\aleph)\cap G(\aleph))\cup(F'(\aleph)\cap H(\aleph)), & \aleph\in(Ƶ\cap B)\cap(Ƶ\cap Ş)=Ƶ\cap B\cap Ş\end{cases}$$

It is seen that N=T is satisfied under the condition $Ƶ\cap B\cap Ş'=\emptyset$.

2) If $Ƶ\cap(B\Delta Ş)=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\underset{\cup}{\overset{*}{\sim}}(H,Ş)]=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(G,B)]\underset{\cap}{\overset{*}{\sim}}[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

3) If $(Ƶ\Delta B)\cap Ş=\emptyset$, then $(F,Ƶ)\overset{*}{\theta_\varepsilon}[(G,B)\underset{*}{\overset{*}{\sim}}(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(G,B)]\underset{\cap}{\overset{*}{\sim}}[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(H,Ş)]$.

4) If $(Ƶ\Delta B)\cap Ş=Ƶ\cap B\cap Ş'=\emptyset$, then $(F,Ƶ)\overset{*}{\theta}[(G,B)\underset{\theta}{\overset{*}{\sim}}(H,Ş)]=[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(G,B)]\underset{\cup}{\overset{*}{\sim}}[(F,Ƶ)\overset{*}{\gamma_\varepsilon}(H,Ş)]$.

ii) RHS Distributions of Complementary Extended Theta Operation over Complementary Soft Binary Piecewise Operations

1) If $(Ƶ\Delta B)\cap Ş=\emptyset$, then $(F,Ƶ)\underset{\cup}{\overset{*}{\sim}}(G,B)]\overset{*}{\theta_\varepsilon}(H,Ş)=[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\underset{\cap}{\overset{*}{\sim}}[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$.

Proof: Consider first LHS. Let $(F,Ƶ)\underset{\cup}{\overset{*}{\sim}}(G,B)=(M,Ƶ)$, where $\forall ℵ\in Ƶ$,

$$M(ℵ)=\begin{cases}F'(ℵ), & ℵ\in Ƶ-B\\ F(ℵ)\cup G(ℵ), & ℵ\in Ƶ\cap B\end{cases}$$

Let $(M,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)=(N,Ƶ\cup Ş)$, where $\forall ℵ\in Ƶ\cup Ş$;

$$N(ℵ)=\begin{cases}M'(ℵ), & ℵ\in Ƶ-Ş\\ H'(ℵ), & ℵ\in Ş-Ƶ\\ M'(ℵ)\cap H'(ℵ), & ℵ\in Ƶ\cap Ş\end{cases}$$

Thus,

$$N(ℵ)=\begin{cases}F(ℵ), & ℵ\in(Ƶ-B)-Ş=Ƶ\cap B'\cap Ş'\\ F'(ℵ)\cap G'(ℵ), & ℵ\in(Ƶ\cap B)-Ş=Ƶ\cap B\cap Ş'\\ H'(ℵ), & ℵ\in Ş-Ƶ\\ F(ℵ)\cap H'(ℵ), & ℵ\in(Ƶ-B)\cap Ş=Ƶ\cap B'\cap Ş\\ (F'(ℵ)\cap G'(ℵ))\cap H'(ℵ), & ℵ\in(Ƶ\cap B)\cap Ş=Ƶ\cap B\cap Ş\end{cases}$$

Now consider the RHS, that is, $[(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)]\underset{\cap}{\overset{*}{\sim}}[(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)]$. Let $(F,Ƶ)\overset{*}{\theta_\varepsilon}(H,Ş)=(V,Ƶ\cup Ş)$, where $\forall ℵ\in Ƶ\cup Ş$;

$$V(ℵ)=\begin{cases}F'(ℵ), & ℵ\in Ƶ-Ş\\ H'(ℵ), & ℵ\in Ş-Ƶ\\ F'(ℵ)\cap H'(ℵ), & ℵ\in Ƶ\cap Ş\end{cases}$$

Now let $(G,B)\overset{*}{\theta_\varepsilon}(H,Ş)=(W,B\cup Ş)$, where $\forall ℵ\in B\cup Ş$;

$$W(ℵ)=\begin{cases}G'(ℵ), & ℵ\in B-Ş\\ H'(ℵ), & ℵ\in Ş-B\\ G'(ℵ)\cap H'(ℵ), & ℵ\in B\cap Ş\end{cases}$$

Let $(V,Ƶ\cup Ş)\underset{\cap}{\overset{*}{\sim}}(W,B\cup Ş)=(T,(Ƶ\cup Ş))$, where $\forall ℵ\in Ƶ\cup Ş$;

$$T(ℵ)=\begin{cases}V'(ℵ), & ℵ\in(Ƶ\cup Ş)-(B\cup Ş)\\ V(ℵ)\cap W(ℵ), & ℵ\in(Ƶ\cup Ş)\cap(B\cup Ş)\end{cases}$$

Thus,

$$T(\aleph)=\begin{cases} F(\aleph), & \aleph\in(Ƶ-Ş)-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ H(\aleph), & \aleph\in(Ş-Ƶ)-(B\cup Ş)=\emptyset \\ F(\aleph)\cup H(\aleph), & \aleph\in(Ƶ\cap Ş)-(B\cup Ş)=\emptyset \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ-Ş)\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ F'(\aleph)\cap H'(\aleph), & \aleph\in(Ƶ-Ş)\cap(Ş-B)=\emptyset \\ F'(\aleph)\cap\big(G'(\aleph)\cap H'(\aleph)\big), & \aleph\in(Ƶ-Ş)\cap(B\cap Ş)=\emptyset \\ H'(\aleph)\cap G'(\aleph), & \aleph\in(Ş-Ƶ)\cap(B-Ş)=\emptyset \\ H'(\aleph)\cap H'(\aleph), & \aleph\in(Ş-Ƶ)\cap(Ş-B)=Ƶ'\cap B'\cap Ş \\ H'(\aleph)\cap\big(G'(\aleph)\cap H'(\aleph)\big) & \aleph\in(Ş-Ƶ)\cap(B\cap Ş)=Ƶ'\cap B\cap Ş \\ \big(F'(\aleph)\cap H'(\aleph)\big)\cap G'(\aleph), & \aleph\in(Ƶ\cap Ş)\cap(B-Ş)=\emptyset \\ \big(F'(\aleph)\cap H'(\aleph)\big)\cap H'(\aleph) & \aleph\in(Ƶ\cap Ş)\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ \big(F'(\aleph)\cap H'(\aleph)\big)\cap\big(G'(\aleph)\cap H'(\aleph)\big), & \aleph\in(Ƶ\cap Ş)\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Therefore,

$$T(\aleph)=\begin{cases} F(\aleph), & \aleph\in(Ƶ-Ş)-(B\cup Ş)=Ƶ\cap B'\cap Ş' \\ F'(\aleph)\cap G'(\aleph), & \aleph\in(Ƶ-Ş)\cap(B-Ş)=Ƶ\cap B\cap Ş' \\ H'(\aleph), & \aleph\in(Ş-Ƶ)\cap(Ş-B)=Ƶ'\cap B'\cap Ş \\ G'(\aleph)\cap H'(\aleph) & \aleph\in(Ş-Ƶ)\cap(B\cap Ş)=Ƶ'\cap B\cap Ş \\ F'(\aleph)\cap H'(\aleph) & \aleph\in(Ƶ\cap Ş)\cap(Ş-B)=Ƶ\cap B'\cap Ş \\ \big(F'(\aleph)\cap H'(\aleph)\big)\cap\big(G'(\aleph)\cap H'(\aleph)\big), & \aleph\in(Ƶ\cap Ş)\cap(B\cap Ş)=Ƶ\cap B\cap Ş \end{cases}$$

Under the condition $Z'\cap B\cap Ş=Ƶ\cap B'\cap Ş=\emptyset$, N=T is satisfied. It is obvious that the condition $Z'\cap B\cap Ş= Ƶ\cap B'\cap Ş=\emptyset$ is equivalent to $(Z\Delta B)\cap Ş=\emptyset$.

2) If $(Ƶ\Delta B)\cap Ş=\emptyset$, then $(F,Ƶ)\underset{\cap}{\overset{*}{\sim}}(G,B)]\overset{*}{\theta_{\varepsilon}}(H,Ş)=[(F,Ƶ)\overset{*}{\theta_{\varepsilon}}(H,Ş)]\underset{\cup}{\overset{*}{\sim}}[(G,B)\overset{*}{\theta_{\varepsilon}}(H,Ş)]$.

3) If $Ƶ\cap(B\Delta Ş)=\emptyset$, then $[(F,Ƶ)\underset{\theta}{\overset{*}{\sim}}(G,B)]\overset{*}{\theta_{\varepsilon}}(H,Ş)=[(F,Ƶ)\overset{*}{\backslash_{\varepsilon}}(H,Ş)]\underset{\cup}{\overset{*}{\sim}}[(G,B)\overset{*}{\backslash_{\varepsilon}}(H,Ş)]$.

4) If $(Ƶ\Delta Ş)\cap B=\emptyset$, then $[(F,Ƶ)\underset{*}{\overset{*}{\sim}}(G,B)]\overset{*}{\theta_{\varepsilon}}(H,Ş)=[(F,Ƶ)\overset{*}{\backslash_{\varepsilon}}(H,Ş)]\underset{\cap}{\overset{*}{\sim}}[(G,B)\overset{*}{\backslash_{\varepsilon}}(H,Ş)]$.

CONCLUSION

Soft set operations are crucial in soft set theory, providing a versatile framework for dealing with uncertainty in data analysis and decision-making processes. In this paper, a new soft set operation called, complementary extended theta is proposed and its algebraic properties are investigated. We treat the distributions of complementary extended theta over other different types of operations on soft sets. A complete knowledge of the applications of soft sets requires an understanding of their algebraic structures in connection with innovative operations; within this framework, the novel soft set operations play an equally important role. We hope that this study will be a guiding framework for future research on soft set operations. In order to determine what algebraic structures form in the collection of soft sets together with the complementary extended theta operation of soft sets, future research may look at different types of complementary extended soft set operations and their distributions and properties.

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