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Research Article

Rough Ideal Convergence of Double Sequences of Fuzzy Numbers

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Abstract

In this paper, we introduce the concepts of rough ideal convergence, rough ideal limit set and rough ideal Cauchy sequence for double sequences of fuzzy numbers. We establish some properties of this convergence and obtain relation between rough ideal limit set and extreme ideal limit points of such sequences. Next, we explore the relation between ideal convergence and its rough analogue. Finally, we examine the connections between the set of cluster points and rough ideal limit set within double sequences of fuzzy numbers.

Keywords: Double sequence, Fuzzy numbers, Ideal convergence, Rough convergence

1. INTRODUCTION

The concept of ideal convergence of sequences in metric space was introduced by Kostyrko et al. [8] in 2000. This concept is a generalization of statistical convergence Fast [7] which holds significant importance in many fields such as summability theory, number theory and mathematical analysis. Therefore, most studies on statistical convergence have been extended to ideal convergence by many authors. Some notions like symmetry, monotonicity and Cauchy condition which known for statistical convergence were extended to ideal convergence by Salat et al. [16]. Also, extremal limit points and their basic properties were studied with respect to the ideal by Kostyrko et al. [8] and the concepts of limit superior and limit inferior were defined by Demirci [4]. Then, the statistical convergence of double sequences presented by Savaş and Mursaleen [12] was generalized to ideal convergence by Das et al [3]. In addition to these, applying fuzzy logic to functions, sequences and series in traditional mathematical analysis expands the scope and results of classical mathematical analysis. So, all these studies conducted for ideal convergence have been examined for fuzzy number sequences. We refer readers to [6,9,14] for more details.

In 2001, Phu [15] introduced rough convergence and investigated boundedness, convexity and closure of rough limit set. Also, he transformed some properties of classical convergence to rough convergence. Then, Akçay and Aytar [1] examined this convergence for sequences of fuzzy numbers and they showed some properties of rough limit set. After that, Babaarslan and Tuncer [2] defined rough convergence and rough Cauchy of double sequences of fuzzy numbers.

Recently, Dündar [5] studied on rough ideal convergence of double sequences. All of these studies have been pioneering for us and than, in this paper, we introduce the concepts of rough ideal convergence, rough ideal limit set and rough ideal Cauchy for double sequences of fuzzy numbers. We give some properties of these sets and a relationship between them by defining the rough ideal limit inferior and rough ideal limit superior. Next, we explore the correlation between ideal convergence and rough ideal convergence. Finally, we examine the connections between the set of cluster points and rough ideal limit set within double sequences of fuzzy numbers. So that, the concepts defined in the fuzzy number space that does not satisfy the properties of a vector space give us different results and these can provide important theoretical tools to examine the generalized convergence of sequences widely used in fuzzy information theory.

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2. MATERIAL AND METHODS

In this section, we briefly recall some basic concepts and notations of theory of fuzzy numbers and we refer to [11] for more details.

A fuzzy number X is a fuzzy subset of the real line $\mathbb{R}$, which is normal, fuzzy convex, upper semi-continuous and closure of the set $X^0 = \{x \in \mathbb{R}: X(x) > 0\}$ is compact. This properties imply that for each $\gamma \epsilon [0,1]$ the $\gamma-$level set of X defined by $X^\gamma = \{x \in \mathbb{R}: X(x) \geq \gamma\} = [\underline{X^\gamma}, \overline{X^\gamma}]$ is a nonempty, compact convex subset of $\mathbb{R}$.

$L(\mathbb{R})$ denotes the set of all fuzzy numbers on $\mathbb{R}$ and the supremum metric on this is defined by

$$\bar{d}(X,Y) = \sup_{\gamma \in [0,1]} d(X^\gamma, Y^\gamma),$$

where $d(X^\gamma, Y^\gamma) = maks\{|\underline{X^\gamma} - \underline{Y^\gamma}|, |\overline{X^\gamma} - \overline{Y^\gamma}|\}$.

Note that a_1 ($a \in \mathbb{R}$) defined by

$$a_1(x) = \begin{cases} 1, & if\ x = a \\ 0, & otherwise \end{cases}$$

is a fuzzy number.

Also, the partial ordering relation $\preccurlyeq$ is defined by $X \preccurlyeq Y \Leftrightarrow \overline{X^\gamma} \leq \overline{Y^\gamma}$ and $\underline{X^\gamma} \leq \underline{Y^\gamma}$ for each $\gamma \epsilon [0,1]$.

Definition 1. [17] A double sequence of $X = (X_{\eta\varsigma})$ of fuzzy numbers is a function $\mathbb{N} \times \mathbb{N}$ into $L(\mathbb{R})$. $X(\eta, \varsigma)$ denotes the value of the function at $(\eta, \varsigma) \in \mathbb{N} \times \mathbb{N}$. This sequence is said to be convergent if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\bar{d}(X_{\eta\varsigma}, \tilde{X}) < \varepsilon$ for all $\eta, \varsigma \geq N$ and it is denoted by $P - limX = \tilde{X}$.

Definition 2. [17] A double sequence of $X = (X_{\eta\varsigma})$ is said to be Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\bar{d}(X_{\eta\varsigma}, X_{jk}) < \varepsilon$ whenever $\eta \geq j \geq N$ and $\varsigma \geq k \geq N$.

Definition 3. [10] Let $X \neq \emptyset$. A class I of subsets of X is said to be an ideal in X provided that

- **i.** $\emptyset \in I$,
- **ii.** $N, S \in I$ implies $N \cup S \in I$,
- **iii.** $N \in I$ and $S \subseteq N$ implies $S \in I$.

If $X \notin I$, then it is called a nontrivial ideal. A nontrivial ideal in X is called admissible if $\{x\} \in I$ for each $x \in X$.

Definition 4. [13] Let $X \neq \emptyset$. A nonempty class of subsets of X is said to be a filter in X provided that

- **i.** $\emptyset \notin$ F,
- **ii.** $N, S \in$ F implies $N \cap S \in$ F,
- **iii.** $N \in$ F and $N \subseteq S$ için $S \in$ F.

If I is a nontrivial ideal in X, $X \neq \emptyset$, then the class $F(I) = \{M \subset X: (\exists N \in I)(M = X \backslash A)\}$ is a filter on X, called the filter associated with I, [8].

A nontrivial ideal I_2 of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible ideal if $\{n\} \times \mathbb{N}$ and ve $\mathbb{N} \times \{n\}$ belongs to I_2 for each $n \in \mathbb{N}$.

Throughout the paper, I_2 denotes a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$, r be a nonnegative real number and $X = (X_{\eta\varsigma})$ denotes a double sequence of fuzzy numbers.

Definition 5. [6] A sequence $X=(X_{\eta\varsigma})$ is said to be I_2 −convergent to a fuzzy number $\tilde{X}$, denoted by $I_2-\lim_{\eta,\varsigma\to\infty} X_{\eta\varsigma}=\tilde{X}$, if for every $\varepsilon>0$, there exists $N(\varepsilon):=\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X})\geq\varepsilon\}\in I_2$.

Definition 6. [2] A sequence $(X_{\eta\varsigma})$ is said to be rough convergent to a fuzzy number $\tilde{X}$, denoted by $X_{\eta\varsigma}\overset{r}{\to}\tilde{X}$, if for every $\varepsilon>0$, there exists $i_\varepsilon\in\mathbb{N}$ provided that $\bar{d}(X_{\eta\varsigma},\tilde{X})<r+\varepsilon$ whenever $\eta,\varsigma\geq i_\varepsilon$.

Here, r is called roughness degree and the set $LIM^r X_{\eta\varsigma}:=\{\tilde{X}\in L(\mathbb{R}): X_{\eta\varsigma}\overset{r}{\to}\tilde{X}\}$ is called r −limit set.

Definition 7. [2] A sequence $(X_{\eta\varsigma})$ is said to be rough Cauchy if for every $\varepsilon>0$, there exists $i_\varepsilon\in\mathbb{N}$ provided that $\bar{d}(X_{\eta\varsigma},X_{kl})<r+\varepsilon$ whenever $k\geq\eta\geq i_\varepsilon$, $l\geq\varsigma\geq i_\varepsilon$.

3. RESULTS AND DISCUSSION

In this section, we define rough ideal convergence and rough ideal Cauchy of double sequences of fuzzy numbers. Then we give some properties and we prove theorems about these new definitions.

Definition 8. For $r\geq 0$, a sequence $(X_{\eta\varsigma})$ is said to be $r-I_2$ −convergent to a fuzzy number $\tilde{X}$ with the roughness degree r, denoted by $X_{\eta\varsigma}\xrightarrow{r-I_2}\tilde{X}$ if $A(\varepsilon):=\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X})\geq r+\varepsilon\}\in I_2$ for every $\varepsilon>0$.

If we take $r=0$, then we obtain I_2 −convergence of these sequences. In general, the $r-I_2$ −limit may not be unique for $r>0$. Therefore, rough I_2 −limit set of $(X_{\eta\varsigma})$ is defined as $I_2-LIM^r(X_{\eta\varsigma})=\left\{\tilde{X}\in L(\mathbb{R}): X_{\eta\varsigma}\xrightarrow{r-I_2}\tilde{X}\right\}$.

A sequence $(X_{\eta\varsigma})$ is said to be $r-I_2$ −convergent, if $I_2-LIM^r(X_{\eta\varsigma})\neq\emptyset$.

Theorem 1. Let $X=(X_{\eta\varsigma})$ and $Y=(Y_{\eta\varsigma})$ be two double sequences of fuzzy numbers. Then,

i. $r-lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} X_{\eta\varsigma}=\tilde{X}$ implies $r-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} X_{\eta\varsigma}=\tilde{X}$,

ii. $\frac{r}{|c|}-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} X_{\eta\varsigma}=\tilde{X}$ implies $r-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} c\,X_{\eta\varsigma}=c\tilde{X}$, for any $c\in\mathbb{R}\backslash\{0\}$,

iii. if $r_1-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} X_{\eta\varsigma}=\tilde{X}$ and $r_2-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} Y_{\eta\varsigma}=\tilde{Y}$, then $r_1+r_2-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}}(X_{\eta\varsigma}+Y_{\eta\varsigma})=\tilde{X}+\tilde{Y}$.

Proof.

i. Let $r>0$ and $r-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} X_{\eta\varsigma}=\tilde{X}$. Then, for each $\varepsilon>0$, there exists $q=q(\varepsilon)\in\mathbb{N}$ such that $\overline{d}(X_{\eta\varsigma},\tilde{X})<r+\varepsilon$ for every $\eta,\varsigma>q$. For $\varepsilon>0$, let $A(\varepsilon)=\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}: \overline{d}(X_{\eta\varsigma},\tilde{X})\geq r+\varepsilon\}\subset(\mathbb{N}\times\{1,2,\dots(q-1)\}\cup\{1,2,\dots(q-1)\}\times\mathbb{N})$. Since I_2 is a strongly admissible ideal, then $A(\varepsilon)$ belongs to I_2. Hence, we get $r-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} X_{\eta\varsigma}=\tilde{X}$.

ii. Let $\gamma\in[0,1]$ and c be any real number. Let $X_{\eta\varsigma}^{\gamma},\tilde{X}^{\gamma}$ be the γ −level set of $X_{\eta\varsigma}$ and $\tilde{X}$, respectively. So, we assume that $c\neq 0$. Let $\varepsilon>0$ be given. Since $d(cX_{\eta\varsigma}^{\gamma},c\tilde{X}^{\gamma})=|c|d(X_{\eta\varsigma}^{\gamma},\tilde{X}^{\gamma})$, we have $\overline{d}(cX_{\eta\varsigma},c\tilde{X})=|c|\overline{d}(X_{\eta\varsigma},\tilde{X})$. From our assumption, since $\frac{r}{|c|}-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} X_{\eta\varsigma}=\tilde{X}$, we write $\left\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}: \overline{d}(X_{\eta\varsigma},\tilde{X})\geq\frac{r}{|c|}+\frac{\varepsilon}{|c|}\right\}\in I_2$. Then, we have

$$\left\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}: \overline{d}(cX_{\eta\varsigma},c\tilde{X})\geq r+\varepsilon\right\}=\left\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}: |c|\overline{d}(X_{\eta\varsigma},\tilde{X})\geq r+\varepsilon\right\}$$

$$=\left\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}: \overline{d}(X_{\eta\varsigma},\tilde{X})\geq\frac{r}{|c|}+\frac{\varepsilon}{|c|}\right\}\in I_2.$$

Hence we get $\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:\overline{d}(cX_{\eta\varsigma},c\tilde{X})\geq r+\varepsilon\}\in I_2$, that is, $r-I_2-\lim_{\substack{\eta\to\infty\\ \varsigma\to\infty}} c\,X_{\eta\varsigma}=c\tilde{X}$.

iii. For $\gamma\in[0,1]$, let $X_{\eta\varsigma}^{\gamma}, Y_{\eta\varsigma}^{\gamma}, \tilde{X}^{\gamma}, \tilde{Y}^{\gamma}$ be γ −level sets of $X_{\eta\varsigma}$, $Y_{\eta\varsigma}$, $\tilde{X}$ and $\tilde{Y}$, respectively. Since $d(X_{\eta\varsigma}^{\gamma}+Y_{\eta\varsigma}^{\gamma},\tilde{X}^{\gamma}+\tilde{Y}^{\gamma})\leq d(X_{\eta\varsigma}^{\gamma},\tilde{X}^{\gamma})+d(Y_{\eta\varsigma}^{\gamma},\tilde{Y}^{\gamma})$, we have $\overline{d}(X_{\eta\varsigma}+Y_{\eta\varsigma},\tilde{X}+\tilde{Y})\leq\overline{d}(X_{\eta\varsigma},\tilde{X})+\overline{d}(Y_{\eta\varsigma},\tilde{Y})$. Let $\varepsilon>0$. Take

$$A=\left\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:\overline{d}(X_{\eta\varsigma},\tilde{X})\geq r_1+\frac{\varepsilon}{2}\right\},$$

$$B=\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:\overline{d}(Y_{\eta\varsigma},\tilde{Y})\geq r_2+\frac{\varepsilon}{2}\}$$

and

$$C=\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:\overline{d}(X_{\eta\varsigma}+Y_{\eta\varsigma},\tilde{X}+\tilde{Y})\geq r_1+r_2+\varepsilon\}.$$

Since $C\subset A\cup B$ and from our assumptions, then we have C belongs to I_2.

Definition 9. A sequence $X=(X_{\eta\varsigma})$ is said to be rough I_2 −Cauchy with roughnesss degree r or $r-I_2-$ Cauchy if for every $\varepsilon>0$, there exists $i_\varepsilon\in\mathbb{N}$ such that $\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:\bar{d}(X_{\eta\varsigma},X_{pq})\geq r+\varepsilon\}\in I_2$ whenever $\eta\geq p\geq i_\varepsilon$ and $\varsigma\geq q\geq i_\varepsilon$.

Theorem 2. If $X=(X_{\eta\varsigma})$ is $r-I_2$ −convergent to a fuzzy number $\tilde{X}$, then it is $2r-I_2$ −Cauchy.

Proof. Let $\varepsilon>0$. Since $X_{\eta\varsigma}\xrightarrow{r-I_2}\tilde{X}$, we write

$$A:=\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:\bar{d}(X_{\eta\varsigma},\tilde{X})\geq r+\tfrac{\varepsilon}{2}\}\in I_2,$$

that is,

$$A^C:=\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:\bar{d}(X_{\eta\varsigma},\tilde{X})<r+\tfrac{\varepsilon}{2}\}\in F(I_2).$$

Since $A^C\neq\emptyset$, we can choose $(p,q)\in\mathbb{N}\times\mathbb{N}$ such that $(p,q)\notin A$, then we have $\bar{d}(X_{pq},\tilde{X})<r+\frac{\varepsilon}{2}$.

Let $B=\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:\bar{d}(X_{\eta\varsigma},X_{pq})\geq 2r+\varepsilon\}$. We show that $B\subset A$. Take $(\eta,\varsigma)\in B$. Then, $2r+\varepsilon\leq\bar{d}(X_{\eta\varsigma},X_{pq})\leq\bar{d}(X_{\eta\varsigma},\tilde{X})+\bar{d}(\tilde{X},X_{pq})<\bar{d}(X_{\eta\varsigma},\tilde{X})+\ r+\frac{\varepsilon}{2}$. This implies that $r+\frac{\varepsilon}{2}<\bar{d}(X_{\eta\varsigma},\tilde{X})$ and therefore $(\eta,\varsigma)\in A$. As $B\subset A$ and $A\in I_2$, hence it is $2r-I_2$ −Cauchy.

Definition 10. I_2 −limit inferior and I_2 −limit superior of $(X_{\eta\varsigma})$ are defined as follows respectively:

$$I_2-liminf X_{\eta\varsigma}\coloneqq inf A_x,$$

and

$$I_2-limsup X_{\eta\varsigma}\coloneqq sup B_x,$$

where

$$A_X=\{\psi\in L(\mathbb{R}):\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:X_{\eta\varsigma}<\psi\}\notin I_2\}$$

and

$$B_X=\{\psi\in L(\mathbb{R}):\{(\eta,\varsigma)\in\mathbb{N}\times\mathbb{N}:X_{\eta\varsigma}>\psi\}\notin I_2\}.$$

Lemma 1. If $\tilde{X} \in I_2 - LIM^r(X_{\eta\varsigma})$, then $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(I_2 - limsupX_{\eta\varsigma}, \tilde{X}) > r\} \in I_2$ and $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(I_2 - liminfX_{\eta\varsigma}, \tilde{X}) > r\} \in I_2$.

Proof. We assume that $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(I_2 - limsupX_{\eta\varsigma}, \tilde{X}) > r\} \notin I_2$. Then, let we define $\acute{\varepsilon} > 0$ such that $\acute{\varepsilon} \coloneqq \frac{\bar{d}(I_2 - limsupX_{\eta\varsigma}, \tilde{X}) - r}{2}$. By the definition of I_2 −limit superior, we have $\bar{d}(I_2 - limsupX_{\eta\varsigma}, X_{\eta\varsigma}) < \acute{\varepsilon}$ for each $(\eta,\varsigma) \in K_1$ where $K_1 \notin I_2$. On the other hand, since $\tilde{X} \in I_2 - LIM^r(X_{\eta\varsigma})$, we have $\bar{d}(X_{\eta\varsigma}, \tilde{X}) \geq r + \acute{\varepsilon}$ for each $(\eta,\varsigma) \in K_2$ where $K_2 \in I_2$.

Let $K \coloneqq K_1 \cap K_2^C$. Then, $K \notin I_2$. Therefore, $\bar{d}(I_2 - limsupX_{\eta\varsigma}, \tilde{X}) \leq \bar{d}(I_2 - limsupX_{\eta\varsigma}, X_{\eta\varsigma}) + \bar{d}(X_{\eta\varsigma}, \tilde{X}) < r + 2\acute{\varepsilon} = r + \bar{d}(I_2 - limsupX_{\eta\varsigma}, \tilde{X}) - r = \bar{d}(I_2 - limsupX_{\eta\varsigma}, \tilde{X})$ for each $(\eta,\varsigma) \in K$. This is a contradiction.

That the set $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(I_2 - liminfX_{\eta\varsigma}, \tilde{X}) > r\}$ belongs to I_2 can be proved by using the similar way.

Theorem 3. If $I_2 - LIM^r(X_{\eta\varsigma}) \neq \emptyset$, then we have $I_2 - LIM^r(X_{\eta\varsigma}) \subseteq [I_2 - limsupX_{\eta\varsigma} - r, I_2 - liminfX_{\eta\varsigma} + r]$.

Proof. Let $\varepsilon > 0$ and $\tilde{X} \in I_2 - LIM^r(X_{\eta\varsigma})$. Then, $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma}, \tilde{X}) \geq r + \varepsilon\} \in I_2$. That is, there exists $(\eta,\varsigma) \in K_1$ such that $K_1 \notin \mathrm{I}_2$ and $\sup_{0\leq\gamma\leq1} max\left\{\left|\underline{X}_{\eta\varsigma}^{\gamma} - \underline{\tilde{X}}^{\gamma}\right|, \left|\overline{X}_{\eta\varsigma}^{\gamma} - \overline{\tilde{X}}^{\gamma}\right|\right\} < r + \varepsilon$

or for every $\gamma \in [0,1]$,

$$\left|\underline{X}_{\eta\varsigma}^{\gamma} - \underline{\tilde{X}}^{\gamma}\right| < r + \varepsilon \text{ and } \left|\overline{X}_{\eta\varsigma}^{\gamma} - \overline{\tilde{X}}^{\gamma}\right| < r + \varepsilon.$$

Hence, for each $(\eta,\varsigma) \in K_1$ such that $K_1 \notin I_2$, we have

$$\tilde{X} - (r + \varepsilon)_1 < X_{\eta\varsigma} < \tilde{X} + (r + \varepsilon)_1.$$

Here, two situations are obtained:

First, $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: X_{\eta\varsigma} < \tilde{X} + (r + \varepsilon)_1\} \notin I_2$ that is $\tilde{X} + (r + \varepsilon)_1 \in B_X$ and from Lemma 1., we write

$$I_2 - limsupX_{\eta\varsigma} \coloneqq supB_X = \varphi \leq \tilde{X} + (r + \varepsilon)_1.$$

Secondly, $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: X_{\eta\varsigma} > \tilde{X} - (r + \varepsilon)_1\} \notin I_2$ that is $\tilde{X} - (r + \varepsilon)_1 \in A_X$ and from Lemma 1., we write

$$I_2 - liminfX_{\eta\varsigma} \coloneqq infA_X = \varphi \geq \tilde{X} - (r + \varepsilon)_1$$

Hence we get $\tilde{X} \in [I_2 - limsupX_{\eta\varsigma} - r, I_2 - liminfX_{\eta\varsigma} + r]$.

Theorem 4. For a sequence $X = (X_{\eta\varsigma})$, the diameter of r−I_2 −limit set is not greater than $2r$.

Proof. Let us assume the opposite. Then there exists $Y, Z \in I_2 - LIM^r(X_{\eta\varsigma})$ such that $l = \bar{d}(Y, Z) > 2r$.

Let $\varepsilon \in (0, \frac{l}{2} - r)$. Since $Y, Z \in I_2 - LIM^r(X_{\eta\varsigma})$, we have $A_1 \in I_2$ and $A_2 \in I_2$, where

$$A_1 := \{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma}, Y) \geq r + \varepsilon\},$$

and

$$A_2 := \{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma}, Z) \geq r + \varepsilon\}.$$

Thus, for all $(\eta,\varsigma) \in A = A_1^c \cap A_2^c$

$$\bar{d}(Y,Z) \leq \bar{d}(X_{\eta\varsigma},Y) + \bar{d}(X_{\eta\varsigma},Z) < 2(r+\varepsilon) < 2\left(r+\frac{l}{2}-r\right) = l = \bar{d}(Y,Z)$$

which is a contradiction.

Theorem 5. The $r-I_2$ limit set of $(X_{\eta\varsigma})$ is closed.

Proof. If $I_2 - LIM^r(X_{\eta\varsigma}) = \emptyset$, it is obvious. Assume that $I_2 - LIM^r(X_{\eta\varsigma}) \neq \emptyset$. We take an arbitrary sequence $(Y_{\eta\varsigma})$ in $I_2 - LIM^r(X_{\eta\varsigma})$ such that it convergent to a fuzzy number $\tilde{Y}$. Let $\varepsilon > 0$. Since $Y_{\eta\varsigma} \to \tilde{Y}$, for each $(\eta,\varsigma) > (\eta,\varsigma)_{\frac{\varepsilon}{2}}$, there exists $(\eta,\varsigma)_{\frac{\varepsilon}{2}} \in \mathbb{N} \times \mathbb{N}$ such that $\bar{d}(Y_{\eta\varsigma},\tilde{Y}) < \frac{\varepsilon}{2}$. In the form " $(\eta,\varsigma) > (\eta,\varsigma)_{\frac{\varepsilon}{2}}$", the "<" relation used for indices is the coordinatewise order. Then, we choose $(\eta_0,\varsigma_0) \in \mathbb{N} \times \mathbb{N}$ provided that $(\eta_0,\varsigma_0) > (\eta,\varsigma)_{\frac{\varepsilon}{2}}$. So, we write $\bar{d}(Y_{\eta_0\varsigma_0},\tilde{Y}) < \frac{\varepsilon}{2}$. On the other hand, since $(Y_{\eta\varsigma}) \subseteq I_2 - LIM^r(X_{\eta\varsigma})$, we have $Y_{\eta_0\varsigma_0} \in I_2 - LIM^r(X_{\eta\varsigma})$. Hence $\left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},Y_{\eta_0\varsigma_0}) \geq r+\frac{\varepsilon}{2}\right\} \in I_2$. If we choose an arbitrary $(p,q) \in \left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},Y_{\eta_0\varsigma_0}) < r+\frac{\varepsilon}{2}\right\}$, then we have

$$\bar{d}(X_{pq},\tilde{Y}) \leq \bar{d}(X_{pq},Y_{\eta_0\varsigma_0}) + \bar{d}(Y_{\eta_0\varsigma_0},\tilde{Y}) < r + \varepsilon.$$

Then, we get

$$\left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{Y}) < r+\varepsilon\right\} \supseteq \left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},Y_{\eta_0\varsigma_0}) < r+\frac{\varepsilon}{2}\right\}.$$

Since we have

$$\left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},Y_{\eta_0\varsigma_0}) < r+\frac{\varepsilon}{2}\right\} \in F(I_2),$$

then we get $\left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{Y}) < r+\varepsilon\right\} \in F(I_2)$. This shows that $\tilde{Y} \in \mathrm{I}_2 - LIM^r(X_{\eta\varsigma})$.

Theorem 6. If $(X_{\eta\varsigma})$ is I_2 −convergent to $\tilde{X}$ and for each $(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}$, there exists $\bar{d}(X_{\eta\varsigma},Y_{\eta\varsigma}) \leq r$, then $(Y_{\eta\varsigma})$ r$-I_2$ −convergent to $\tilde{X}$.

Proof. From our assumption for every $\varepsilon > 0$, we have $A(\varepsilon) = \left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X}) \geq \varepsilon\right\} \in I_2$. Since $\bar{d}(X_{\eta\varsigma},Y_{\eta\varsigma}) \leq r$ for all $(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}$, we get

$$\left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(Y_{\eta\varsigma},\tilde{X}) \geq r+\varepsilon\right\} \subseteq \left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(Y_{\eta\varsigma},\tilde{X}) \geq \varepsilon\right\}.$$

Hence we write $A(\varepsilon) = \left\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(Y_{\eta\varsigma},\tilde{X}) \geq r+\varepsilon\right\} \in I_2$.

Theorem 7. If $X = (X_{\eta\varsigma})$ is I_2 −convergent to a fuzzy numbers $\tilde{X}$, then $I_2 - LIM^r(X_{\eta\varsigma}) = \overline{B_r}(\tilde{X}) \coloneqq \{Y \in L(\mathbb{R}): \bar{d}(Y,\tilde{X}) \leq r\}$.

Proof. Let $A_1(\varepsilon) := \{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X}) \geq \varepsilon\}$. From our assumption, we write $A_1(\varepsilon) \in I_2$. Let we take $Y \in \overline{B_r}(\tilde{X})$. For $(\eta,\varsigma) \in A_1^C(\varepsilon)$, since we have

$$\bar{d}(X_{\eta\varsigma},Y) \leq \bar{d}(X_{\eta\varsigma},\tilde{X}) + \bar{d}(\tilde{X},Y) < \varepsilon + r,$$

where $A_1^C(\varepsilon) = \{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X}) < \varepsilon\}$, we get $Y \in I_2 - LIM^r(X_{\eta\varsigma})$.

Let $A_2(\varepsilon) := \{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},Y) \geq r+\varepsilon\}$. For $Y \in I_2 - LIM^r(X_{\eta\varsigma})$, we write $A_2(\varepsilon) \in I_2$. Since $X = (X_{\eta\varsigma})$ I_2 −convergent to a fuzzy number $\tilde{X}$, then $A_1(\varepsilon) \in I_2$. Since $\bar{d}(Y,\tilde{X}) \leq \bar{d}(X_{\eta\varsigma},Y) + \bar{d}(X_{\eta\varsigma},\tilde{X}) < 2\varepsilon + r$ for $(\eta,\varsigma) \in A_1^C \cap A_2^C$, we get $Y \in \overline{B_r}(\tilde{X})$.

Definition 11. A fuzzy number $\tilde{X}$ is said to be ideal cluster point of $(X_{\eta\varsigma})$ provided that for each $\varepsilon > 0$, the set

$$\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X}) < \varepsilon\} \notin I_2.$$

We denote the set of all ideal cluster points of $(X_{\eta\varsigma})$ by $I_2(\Gamma(X))$.

Lemma 2. For $C \in I_2(\Gamma(X))$ and $\tilde{X} \in I_2 - LIM^r(X_{\eta\varsigma})$, there exists $\bar{d}(\tilde{X}, C) \le r$.

Proof. Let $\bar{d}(\tilde{X}, C) > r$ and $\tilde{X} \in I_2 - LIM^r(X_{\eta\varsigma})$ such that $C \in I_2(\Gamma(X))$. Then, we write

$$\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X}) \ge r+\varepsilon\} \supseteq \{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},C) < \varepsilon\}$$

for $\varepsilon := \frac{\bar{d}(\tilde{X},C)-r}{3}$. Since $C \in I_2(\Gamma(X))$, we get

$$\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},C) < \varepsilon\} \notin I_2.$$

Hence, we have

$$\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X}) \ge r+\varepsilon\} \notin I_2$$

This contradicts with $\tilde{X} \in I_2 - LIM^r(X_{\eta\varsigma})$.

Theorem 8.

i. If $C \in I_2(\Gamma(X))$, then $I_2 - LIM^r(X_{\eta\varsigma}) \subseteq \overline{B_r}(C)$,

ii. There exists $I_2 - LIM^r(X_{\eta\varsigma}) = \bigcap_{C\in I_2(\Gamma(X))} \overline{B_r}(C)=\{\tilde{X} \in L(\mathbb{R}): I_2(\Gamma(X)) \subseteq \overline{B_r}(\tilde{X})\}$.

Proof.

i. Let $\tilde{X} \in I_2 - LIM^r(X_{\eta\varsigma})$ and $C \in I_2(\Gamma(X))$. From Lemma 2., we have $\bar{d}(\tilde{X}, C) \le r$. Otherwise for $\varepsilon := \frac{\bar{d}(\tilde{X},C)-r}{3}$, we get $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},\tilde{X}) \ge r+\varepsilon\} \notin I_2$. This contradicts with $\tilde{X} \in I_2 - LIM^r(X_{\eta\varsigma})$.

ii. From (i), for every $C \in I_2(\Gamma(X))$ $I_2 - LIM^r(X_{\eta\varsigma}) \subseteq \overline{B_r}(C)$ is obvious. If we show that $\overline{B_r}(C) \subseteq I_2 - LIM^r(X_{\eta\varsigma})$ for every $C \in I_2(\Gamma(X))$, then we prove the theorem. Let $Y \in \bigcap_{C\in I_2(\Gamma(X))} \overline{B_r}(C)$. Then, for every $C \in I_2(\Gamma(X))$, we have $\bar{d}(Y, C) \le r$. This shows that $I_2(\Gamma(X)) \subseteq \overline{B_r}(C)$, that is $\bigcap_{C\in I_2(\Gamma(X))} \overline{B_r}(C) = \{\tilde{X} \in L(\mathbb{R})): I_2(\Gamma(X)) \subseteq \overline{B_r}(\tilde{X})\}$. Now, let $Y \notin I_2 - LIM^r(X_{\eta\varsigma})$. Hence, there exist $\varepsilon > 0$ such that $\{(\eta,\varsigma) \in \mathbb{N} \times \mathbb{N}: \bar{d}(X_{\eta\varsigma},Y) \ge r+\varepsilon\} \notin I_2$. This shows that $C \in I_2(\Gamma(X))$ provided that $\bar{d}(Y, C) > r$. Then we write $I_2(\Gamma(X)) \nsubseteq \overline{B_r}(\tilde{X})$ and $Y \notin \{\tilde{X} \in L(\mathbb{R}): I_2(\Gamma(X)) \subseteq \overline{B_r}(\tilde{X})\}$. Thus, we get $\{\tilde{X} \in L(\mathbb{R}): I_2(\Gamma(X)) \subseteq \overline{B_r}(Y)\} \subseteq I_2 - LIM^r(X_{\eta\varsigma})$.

AUTHOR'S CONTRIBUTIONS

The authors contributed equally.

CONFLICTS OF INTEREST

The authors have declared that there is no conflict of interest.

RESEARCH AND PUBLICATION ETHICS

The authors declare that this study complies with Research and Publication Ethics.

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