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Lacunary Statistically Convergence via Modulus Function Sequences

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Abstract

Modifying the definition of density functions is one method used to generalise statistical convergence. In the present study, we use sequences of modulus functions and order $\alpha \in (0, 1]$ to introduce a new density. Based on this density framework, we define strong (f_k)-lacunary summability of order α and (f_k)-lacunary statistical convergence of order α for a sequence of modulus functions (f_k). This concept holds an intermediate position between the usual convergence and the statistical convergence for lacunary sequences. We also establish inclusion theorems and relations between these two concepts in the study.

Keywords: *Lacunary statistical convergence, Lacunary summability, Modulus function, Weighted density*

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1. Introduction

The concept of statistical convergence was initially proposed by Zygmund [1] in his research. Independently, Steinhaus [2] and Fast [3] also introduced this idea. Subsequently, Schoenberg [4] and numerous other mathematicians further explored and analyzed this concept. Statistical convergence and some derived concepts were introduced and studied in a variety of sequences. Following the demonstration of statistical convergence, the subject has been approached from many angles and various extensions have been produced. In particular, using functions belonging to different classes and sequences belonging to some classes, classes of sequences with statistical convergence have been derived. Meanwhile, it has been established that there is a relationship between statistical convergence and Cesàro summability, and this relationship has been revealed. Since the pioneering studies of Salat [5] and Fridy [6], statistical convergence has become a highly active area of research within summability theory.

The concept of asymptotic (or natural) density is the fundamental tool in statistical convergence, and it is defined for a set $K \subseteq \mathbb{N}^+$ as $\delta(K) = \lim\limits_{n\to\infty} n^{-1} |\{k \leq n : k \in K\}|$ whenever the limit exists. Here, the vertical bars indicate the cardinality of the enclosed set. So, $\delta(A) = 0$ for the finite set A, $\delta(\mathbb{N}\setminus A) = 1 - \delta(A)$ and $\delta(A) \leq \delta(B)$ whenever $A \subseteq B$.

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Base on the concept of natural density, a sequence of numbers (x_k) is said to be statistical convergent to some number x if for each positive number $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-1} |\{k \leq n : |x_k - x| \geq \varepsilon\}| = 0$$

whenever limit exists. In that case, x is called statistical limit of (x_k) and is written as $S - \lim x_k = x$ or $x_k \to x\,(S)$.

In literature, there exists generalizations of statistical convergence. For instance, a sequence (x_k) is statistically convergent of order $\alpha \in (0, 1]$ to some number x if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{-\alpha} |\{k \leq n : |x_k - x| \geq \varepsilon\}| = 0.$$

whenever limit exists (see [7] and [8]).

All statistically convergent sequences and all statistically convergent sequences of order α will be denoted by S and S^α respectively.

The notions of lacunary summability and convergence with lacunary sequences were established by Fridy and Orhan ([9] and [10]). A lacunary sequence $\theta = (k_r)_{r\in\mathbb{N}}$ is an increasing sequence of integers such that $k_0 = 0$ and $\lim\limits_{n\to\infty}(k_r - k_{r-1}) = \infty$. For lacunary sequences, we use the notations $h_r = k_r - k_{r-1}$, $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. For the sake of brevity, the set of all lacunary sequences of integers will be denoted by $\mathcal{LS}(\mathbb{Z})$.

A sequence (x_k) is lacunary statistically convergent and, respectively, lacunary statistically convergent of order α to some number L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$$

and, respectively,

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} |\{k \in I_r : |x_k - L| \geq \varepsilon\}| = 0$$

whenever limit exists. All lacunary statistically convergent sequences and all lacunary statistically convergent sequences of order α are denoted by S_θ and S_θ^α, respectively. Lacunary statistically convergent, lacunary boundedness order α and strongly summable sequences of order α have been studied by Connor [11], Çolak [12], Şengül and Et in [13], [14]. Pehlivan and Fisher [15] introduce the concept of lacunary strong convergence with respect to a sequence of modulus functions in a Banach spaces.

The modulus function is the other idea we employ in our research. A function $f : [0, \infty) \to [0, \infty)$ is referred modulus provided that the following conditions hold:

i. $f(v) = 0 \Leftrightarrow v = 0$

ii. $f(v_1 + v_2) \leq f(v_1) + f(v_2)$ for every $v_1, v_2 \in [0, \infty)$

iii. f is increasing

iv. f is continuous from the right at 0.

Although the continuity of any modulus function is obvious, a modulus function need not to be bounded. For instance, the modulus $f(v) = \log(v + 1)$ is unbounded, while $g(v) = \frac{v}{v+1}$ is a bounded modulus function. For any modulus f and for every $m \in \mathbb{N}^+$, the inequality $f(mv) \leq mf(v)$ and so that $f(m) \leq mf(1)$ holds from the condition (ii). The notion of modulus was first established by Nakano [16] and subsequently, Ruckle [17], established a new sequence spaces by a modulus function f and these sequence spaces were then used in many researches (for example see [18], [19], [20], [21]).

The space of sequences of unbounded modulus functions $F = (f_k)$ such that $\lim\limits_{u\to 0^+} \sup\limits_{k\in\mathbb{N}} f_k(u) = 0$ will be denoted by $\mathcal{M}^{ub}$.

Changing definition of the density function is one method used to distinguish the statistical convergence. Researchers have explored various generalizations of the concept of asymptotic density. One of these is the density $f-$ given by Aizpuru et al. [22], which is obtained by employing modulus functions.

Definition 1.1. [22] Let f be an unbounded modulus function. The f-density of a set $\mathbb{N}$ is defined by

$$d_f(A) = \lim_{n\to\infty} \frac{f(|A|)}{f(n)}$$

in case this limit exists.

Base on this density, Aizpuru et al. [22] defined f -statistical convergence in normed space as follows.

Definition 1.2. [22] Let f be an unbounded modulus function. The sequence (x_n) in the normed space X is called $f-$statistical convergence to $x \in X$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{f(|\{n \in \mathbb{N} : \|x_n - x\| > \varepsilon\}|)}{f(n)} = 0.$$

Obviously, if the modulus function is the identity function, f-statistical convergence coincides with statistical convergence, and since the f-density of a finite set is zero, topological convergence coincides with f-statistical convergence. Consequently, f-statistical convergence lies between ordinary convergence and statistical convergence. Recently, Bhardwaj and Dhawan [23] proposed f-statistical convergence of order α and strong Cesàro summability of order α with respect to a modulus f, using the f_α-density of a set $A \subseteq \mathbb{N}$. León-Saavedra [24] proved results related to a characterization of the modulus f for cases where f-strong Cesàro convergence coincides with f-statistical convergence and uniform integrability. In addition, İbrahim and Çolak [25] introduced strong lacunary summability of order α via a modulus function.

This paper aims to introduce and study the concept of lacunary statistical convergence and lacunary summability according to a sequence of modulus for number sequences, using f_α-density. This study is motivated by the work of Pehlivan and Fisher [15], Bhardwaj and Dhawan [23], and İbrahim and Çolak [25].

2. Main results

For each $\alpha \in \mathbb{R}$ such that $\alpha > 1$, lacunary statistical convergence is not well defined (see [14], [13]). Therefore, in the rest of article, we consider the case $\alpha \in (0, 1]$.

2.1 Lacunary summability using a sequence of modulus

We proposed a slight generalisation of strongly lacunary summability of order α by using a sequence of modulus functions. Depending on this definition, inclusion relations are given under certain conditions.

Definition 2.1. Suppose $F = (f_k) \subset \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$. The sequence $(z_k) \subset \mathbb{C}$ is strongly $F^\alpha-$lacunary summable (briefly $N_\theta^\alpha(F)-$summable) to some $L \in \mathbb{C}$ provided that

$$\lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} f_k(|z_k - L|) = 0.$$

holds and this is denoted by $z_k \to L(N_\theta^\alpha(F))$ or $N_\theta^\alpha(F) - \lim_k z_k = L$. The set of all $N_\theta^\alpha(F)$-summable sequences is denoted by $N_\theta^\alpha(F)$, i.e.

$$N_\theta^\alpha(F) = \{(z_k) : \lim_{r\to\infty} \frac{1}{h_r^\alpha} \sum_{k\in I_r} f_k(|z_k - L|) = 0 \text{ for some } L \in \mathbb{C}\}.$$

Remark 2.1. Note that in this definition, the modulus functions f_k are not required to be unbounded. On the other hand, for a sequence of modulus functions $F = (f_k)$,

i. $N_\theta^\alpha(F)$- summability is reduced to $N_\theta-$ summability in the particular case $\alpha = 1$ and $f_k(v) = v$ for all $k \in \mathbb{N}$ (see [26]).

ii. $N_\theta^\alpha(F)$- summability is reduced to $N_\theta^\alpha-$ summability in the particular case $f_k(v) = v$ for all $k \in N$ (see [7]).

iii. $N_\theta^\alpha(F)$- summability is reduced to $N_\theta^\alpha(f)-$summability in the particular case $\alpha = 1$ and $f_k = f$ for all $k \in N$ and for a modulus function f (see [15]).

Theorem 2.1. *Suppose $F = (f_k) \subset \mathcal{M}^{ub}$ and $G = (g_k) \subset \mathcal{M}^{ub}$, $\alpha_1, \alpha_2 \in (0, 1]$ such that $\alpha_1 \leq \alpha_2$ and $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$.*

i. *If* $\sup\limits_{u,k}\frac{f_k(u)}{g_k(u)} < \infty$ *holds then* $N_\theta^{\alpha_1}(G) \subset N_\theta^{\alpha_2}(F)$.

ii. *If* $\inf\limits_{u,k}\frac{f_k(u)}{g_k(u)} > 0$ *holds then* $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}(G)$.

iii. *If* $0 < \inf\limits_{u,k}\frac{f_k(u)}{g_k(u)} \leq \sup\limits_{u,k}\frac{f_k(u)}{g_k(u)} < \infty$ *holds then* $N_\theta^{\alpha_1}(F) = N_\theta^{\alpha_1}(G)$.

Note that infimum and supremum are taken over all $u \in (0,\infty)$ *and* $k \in \mathbb{N}$.

Proof. Choose $z = (z_k) \in N_\theta^{\alpha_1}(G)$. If $p = \sup\limits_{u,k}\frac{f_k(u)}{g_k(u)} < \infty$ holds then $0 < \frac{f_k(u)}{g_k(u)} \leq p$ and hence $f_k(u) \leq pg_k(u)$ holds for all $k \in \mathbb{N}$ and for any $u \in \mathbb{R}^+ \cup \{0\}$. On the other hand, since $0 < \alpha_1 \leq \alpha_2 \leq 1$, we have the following inequalities:

$$\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r} f_k(|z_k - l|) \leq \frac{1}{h_r^{\alpha_1}}\sum_{k\in I_r} f_k(|z_k - l|) \leq \frac{1}{h_r^{\alpha_1}}\sum_{k\in I_r} pg_k(|z_k - l|)$$

Taking limit as $r \to \infty$, strongly $N_\theta^{\alpha_1}(G)$ −summability to $l \in \mathbb{C}$ of (z_k) implies that $z = (z_k) \in N_\theta^{\alpha_1}(F)$.

In the proof of (ii), *if* $q = \inf\limits_{u,k}\frac{f_k(u)}{g_k(u)} > 0$ holds then $g_k(u) \leq \frac{1}{q} f_k(u)$ for every $u \in \mathbb{R}^+ \cup \{0\}$ and for all $k \in \mathbb{N}$. Thus, the rest of the proof is exactly similar to (i). Moreover, (iii) is a consequence of (i) and (ii). □

Remark 2.2. Let us choose $\alpha_1 = \alpha_2 = 1$ and $F = (f_k) \subset \mathcal{M}^{ub}$ and $G = (g_k) \subset \mathcal{M}^{ub}$ such that $f_k(u) = \frac{ku}{u+1}$ and $g_k(u) = 2^k u$ for all $k \in \mathbb{N}$. Considering the sequence in Example 3.1 in [25], we obtain that the inclusion $N_\theta^{\alpha_1}(G) \subset N_\theta^{\alpha_2}(F)$ is strict.

Corollary 2.1. *Suppose that* $F = (f_k) \subset \mathcal{M}^{ub}$ *and* $G = (g_k) \subset \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$, *and* $\alpha_1, \alpha_2 \in (0,1]$ *such that* $\alpha_1 \leq \alpha_2$. *Then the following assertions hold:*

i. *If* $\sup\limits_{u,k}\frac{f_k(u)}{g_k(u)} < \infty$ *or* $\inf\limits_{u,k}\frac{g_k(u)}{f_k(u)} > 0$ *then* $N_\theta^{\alpha_1}(G) \subset N_\theta^{\alpha_1}(F)$,

ii. *If* $\sup\limits_{u,k}\frac{f_k(u)}{g_k(u)} < \infty$ *or* $\inf\limits_{u,k}\frac{g_k(u)}{f_k(u)} > 0$ *then* $N_\theta(G) \subset N_\theta(F)$,

iii. $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}(F)$.

Note that, supremum is taken over all $u \in (0,\infty)$ *and* $k \in \mathbb{N}$ *in* (i) *and* (ii).

Corollary 2.2. *Suppose that* $F = (f_k) \subset \mathcal{M}^{ub}$ *and* $G = (g_k) \subset \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$, *and* $\alpha_1, \alpha_2 \in (0,1]$ *such that* $\alpha_1 \leq \alpha_2$. *Then the following assertions hold:*

i. *If* $\sup\limits_{u,k}\frac{f_k(u)}{u} < \infty$ *then* $N_\theta^{\alpha_1} \subset N_\theta^{\alpha_2}(F)$,

ii. *If* $\sup\limits_{u,k}\frac{f_k(u)}{u} < \infty$ *then* $N_\theta^{\alpha_1} \subset N_\theta^{\alpha_1}(F)$,

iii. *If* $\inf\limits_{u,k}\frac{f_k(u)}{u} > 0$ *then* $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}$,

iv. *If* $\inf\limits_{u,k}\frac{f_k(u)}{u} > 0$, *then* $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_1}$

v. *If* $0 < \inf\limits_{u,k}\frac{f_k(u)}{u} \leq \sup\limits_{u,k}\frac{f_k(u)}{u} < \infty$ *then* $N_\theta^{\alpha_1}(F) = N_\theta^{\alpha_1}$.

Corollary 2.3. *Suppose that* $F = (f_k) \subset \mathcal{M}^{ub}$, $\alpha_1, \alpha_2, \gamma \in (0,1]$ *such that* $\alpha_1 \leq \alpha_2 \leq \gamma$, *and* $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$. *Then the following assertions hold:*

i. *If there exists a modulus function* f *such that* $f_k \leq f$ *for every* $k \in \mathbb{N}$ *then* $N_\theta^{\alpha_1}(f) \subset N_\theta^{\alpha_2}(F)$ *holds,*

ii. *If there exists a modulus function* f *such that* $g \leq f_k$ *for every* $k \in \mathbb{N}$ *then* $N_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}(g)$ *holds,*

iii. *If there exists a modulus function f and g such that $g \leq f_k \leq f$ for every $k \in \mathbb{N}$ then $N_\theta^{\alpha_1}(f) \subset N_\theta^{\alpha_2}(F) \subset N_\theta^{\gamma}(g)$ hold.*

Proof. The proof of (i) is clear from Theorem 2.1(i) since the inequality $f_k \leq f$ for every $k \in \mathbb{N}$ implies $\sup_{u,k} \frac{f_k(u)}{f(u)} < \infty$. Similarly, the proof of (ii) is follows from Theorem 2.1(ii) since the inequality $g \leq f_k$ for every $k \in \mathbb{N}$ implies $\inf_{u,k} \frac{f_k(u)}{g(u)} > 0$. Hence, (iii) is a consequence of (i) and (ii). □

2.2 Lacunary statistically convergence using a sequence of modulus

In this section, we introduced a new concept of lacunary statistical convergence of order α by using sequences of modulus functions. By some given inclusion theorems, we establish some relations between lacunary summability and lacunary statistical convergence under certain conditions.

We firstly define a density with the help of sequence of modulus functions and order $\alpha \in (0,1]$ as follows:

Definition 2.2. The density of $A \subseteq \mathbb{N}^+$ with respect to a sequence of unbounded modulus functions $F = (f_k) \subset \mathcal{M}^{ub}$ and order $\alpha \in (0,1]$ is defined by the following limit

$$\delta_{F_\alpha}(A) = \lim_{r\to\infty} \frac{f_r(|\{k \leq r : k \in A\}|)}{f_r(r^\alpha)}$$

whenever the limit exists. The abbreviation for this density is referred to as F_α-density.

Remark 2.3. Obviously,

i. If $\alpha = 1$ and $f_k(x) = x$ for all $k \in \mathbb{N}$ then $F_\alpha-$density is reduced to the natural density (see *[3]*),

ii. If $\alpha \in (0,1]$ and $f_k(x) = x$ for all $k \in \mathbb{N}$ then $F_\alpha-$density is reduced to the $\alpha-$density (see *[7]*),

iii. If $\alpha = 1$ and $f_k(x) = f(x)$ for all $k \in \mathbb{N}$ and for $f \in \mathcal{M}$ then $F_\alpha-$density is reduced to the $f-$density (see *[22]*),

iv. If $\alpha \in (0,1]$ and $f_k(x) = f(x)$ for all $k \in \mathbb{N}$ and for $f \in \mathcal{M}$ then $F_\alpha-$density is reduced to the f_α-density (see [23]).

Similar to other density types, we may provide an alternative form of the lacunary statistical convergence in relation to the $F_\alpha-$density in the following manner.

Definition 2.3. Suppose that $F = (f_k) \subset \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$. The sequence $(z_k) \subset \mathbb{C}$ is F^α-lacunary statistically convergent (shortly $S_\theta^\alpha(F)$-convergent) to some $l \in \mathbb{C}$ provided that for every $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{f_r(h_r^\alpha)} f_r(|\{k \in I_r : |z_k - l| \geq \varepsilon\}|) = 0.$$

holds and this is denoted by $z_k \to l(S_\theta^\alpha(F))$ or $S_\theta^\alpha(F) - \lim_k z_k = l$. The class of all $S_\theta^\alpha(F)$-convergent sequences is denoted by $S_\theta^\alpha(F)$, i.e.

$$S_\theta^\alpha(F) = \{(z_k) : \lim_{r\to\infty} \frac{1}{f_r(h_r^\alpha)} f_r(|\{k \in I_r : |z_k - l| \geq \varepsilon\}|) = 0 \text{ for some } l \in \mathbb{C}\}.$$

Now, we can establish some inclusion theorems between $F-$lacunary summability of order α and $F-$lacunary statistically convergence of order α.

Theorem 2.2. *Suppose that $F = (f_k) \subset \mathcal{M}^{ub}$, $G = (g_k) \subset \mathcal{M}^{ub}$, $\alpha_1, \alpha_2 \in (0,1]$ such that $\alpha_1 \leq \alpha_2$ and $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$. If $\inf_{u,k} \frac{f_k(u)}{g_k(u)} > 0$ and $\lim_{u\to\infty} \frac{g_k(u)}{u} > 0$ for all k, then $N_\theta^{\alpha_1}(F)$-summability implies $S_\theta^{\alpha_2}(G)-$statistically convergence, i.e. $N_\theta^{\alpha_1}(F) \subset S_\theta^{\alpha_2}(G)$.*

Proof. Choose a sequence (z_k) which is $N_\theta^{\alpha_1}(F)$-summable to $l \in \mathbb{C}$. From the assumption, $q = \inf_{u,k} \frac{f_k(u)}{g_k(u)} > 0$ implies that $qg_k(u) \leq f_k(u)$ holds for every $k \in \mathbb{N}$ and for every $u \in \mathbb{R}^+ \cup \{0\}$. Due to (z_k) is $N_\theta^{\alpha_1}(F)-$summable to $l \in \mathbb{C}$, we have □

$$\begin{aligned}\frac{1}{h_r^{\alpha_1}}\sum_{k\in I_r} f_k(|z_k-l|) &\geq q\frac{1}{h_r^{\alpha_1}}\sum_{k\in I_r} g_k(|z_k-l|)\\ &\geq q\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r} g_k(|z_k-l|)\\ &= q\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r,|z_k-l|\geq\varepsilon} g_k(|z_k-l|)+q\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r,|z_k-l|<\varepsilon} g_k(|z_k-l|)\\ &\geq q\frac{1}{h_r^{\alpha_2}}\sum_{k\in I_r,|z_k-l|\geq\varepsilon} g_k(|z_k-l|)\\ &\geq q\frac{1}{h_r^{\alpha_2}}\left|\{k\epsilon I_r : |z_k-l|\geq\varepsilon\}\right|\ g_r(\varepsilon).\end{aligned}$$

where $g_r(\varepsilon)=\inf\limits_{k\in I_r} g_k(\varepsilon)$. Since $|\{k\in I_r : |z_k-l|\geq\varepsilon\}|\in\mathbb{Z}^+$, the following inequality holds:

$$\begin{aligned}\frac{1}{h_r^{\alpha_1}}\sum_{k\in I_r} f_k(|z_k-l|) &\geq \frac{1}{h_r^{\alpha_2}}\inf_{k\in I_r} g_k(|\{k\in I_r : |z_k-l|\geq\varepsilon\}|)\frac{\inf\limits_{k\in I_r} g_k(\varepsilon)}{\inf\limits_{k\in I_r} g_k(1)}q\\ &= \frac{g_r(|\{k\in I_r : |z_k-l|\geq\varepsilon\}|)}{g_r\left(h_r^{\alpha_2}\right)}\frac{g_r\left(h_r^{\alpha_2}\right)}{h_r^{\alpha_2}}\frac{g_r(\varepsilon)}{g_r(1)}q.\end{aligned}$$

As the limit $r\to\infty$, we conclude that $(z_k)\in N_\theta^{\alpha_1}(F)$ implies $(z_k)\in S_\theta^{\alpha_2}(G)$.

Remark 2.4. However, $S_\theta^{\alpha_2}(G)$ −statistically convergent a sequence do not need to be $N_\theta^{\alpha_1}(F)$ −summable. This observation is evident by referring to Example 3.2 in [25] where we consider $f_k(u)=g_k(u)=u$ for all $k\in\mathbb{N}$.

Corollary 2.4. *Suppose that $F\in\mathcal{M}^{ub}$, $\theta=(k_r)\in\mathcal{LS}(\mathbb{Z})$, and $\alpha_1,\alpha_2\in(0,1]$ such that $\alpha_1\leq\alpha_2$. If $\lim\limits_{u\to\infty}\frac{f_k(u)}{u}>0$ holds for all $k\in\mathbb{N}$ then $N_\theta^{\alpha_1}(F)$-summability implies $S_\theta^{\alpha_2}(F)$-statistical convergence, i.e. $N_\theta^{\alpha_1}(F)\subseteq S_\theta^{\alpha_2}(F)$.*

Proof. Proof is clear by taking $F=G$ in the last Theorem 2.2. □

Corollary 2.5. *Suppose that $F,G\in\mathcal{M}^{ub}$, $\alpha\in(0,1]$ and $\theta=(k_r)\in\mathcal{LS}(\mathbb{Z})$. If $\inf\limits_{u,k}\frac{f_k(u)}{g_k(u)}>0$ and $\lim\limits_{u\to\infty}\frac{g_k(u)}{u}>0$ for all $k\in\mathbb{N}$, then $N_\theta^{\alpha}(F)$-summability implies $S_\theta^{\alpha}(G)$-statistical convergence, i.e. $N_\theta^{\alpha}(F)\subseteq S_\theta^{\alpha}(G)$.*

Proof. It is consequence of Theorem 2.2 by taking $\alpha_2=\alpha$. □

Corollary 2.6. *Suppose that $F\in\mathcal{M}^{ub}$, $\alpha\in(0,1]$ and $\theta=(k_r)\in\mathcal{LS}(\mathbb{Z})$. If $\inf\limits_{u,k}\frac{f_k(u)}{u}>0$ then $N_\theta^{\alpha}(F)$-summability implies S_θ^{α}-statitically convergence, i.e. $N_\theta^{\alpha}(F)\subseteq S_\theta^{\alpha}$ and particularly, $N_\theta^{\alpha}(F)\subseteq S_\theta$ whenever $\alpha=1$.*

Proof. Proof is clear by taking $g_k(u)=u$ for all $k\in\mathbb{N}$ and $\alpha=\alpha_2$ in Corollary 2.4. □

Theorem 2.3. *Suppose that $F=(f_k), G=(g_k)\in\mathcal{M}^{ub}$, $0<\alpha_1\leq\alpha_2\leq 1$, and $\theta=(k_r),\psi=(s_r)\in\mathcal{LS}(\mathbb{Z})$ such that $I_r\subset J_r$ for each $r\in\mathbb{N}$. If $\sup\limits_{u,k}\frac{g_k(u)}{u}<\infty$ and $\lim\limits_{r\to\infty}\frac{s_r-s_{r-1}}{(k_r-k_{r-1})^{\alpha_2}}=1$, then each $S_\theta^{\alpha_1}(F)$-convergent bounded sequence is $N_\psi^{\alpha_2}(G)$ −summable, i.e. $\ell_\infty\cap S_\theta^{\alpha_1}(F)\subset N_\psi^{\alpha_2}(G)$.*

Proof. Suppose that $I_r=(k_{r-1},k_r]$, $J_r=(s_{r-1},s_r]$, $h_r=k_r-k_{r-1}$, $v_r=s_r-s_{r-1}$ and $0<\alpha_1\leq\alpha_2\leq 1$. Choose $(z_k)\in\ell_\infty\cap S_\theta^{\alpha_1}(F)$ such that $z_k\to l(S_\theta^{\alpha_1}(F))$. Firstly, we will show that $S_\theta^{\alpha_1}(F)\subset S_\theta^{\alpha_1}$. Since $(z_k)\in S_\theta^{\alpha_1}(F)$, for every $\varepsilon>0$, we have

$$\lim_{r\to\infty}\frac{1}{f_r(h_r^{\alpha_1})}f_r(|\{k\in I_r : |z_k-l|\geq\varepsilon\}|)=0.$$

Hence, given $p\in\mathbb{N}$ we can find a natural number r_0 such that, □

$$f_r(|\{k\epsilon I_r : |z_k - l| \geq \varepsilon\}|) \leq \frac{1}{p} f_r(h_r^{\alpha_1}) \leq \frac{1}{p} p f_r(\frac{h_r^{\alpha_1}}{p}) = f_r(\frac{h_r^{\alpha_1}}{p})$$

for $r > r_0$. Since f_r are increasing modulus functions, we have,

$$\frac{1}{h_r^{\alpha_1}} |\{k\epsilon I_r : |z_k - l| \geq \varepsilon\}| \leq \frac{1}{p}.$$

It means that the inclusion $S_\theta^{\alpha_1}(F) \subset S_\theta^{\alpha_1}$ holds and hence $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset \ell_\infty \cap S_\theta^{\alpha_1}$. From the assumptions $\lim_{r\to\infty} \frac{v_r}{h_r^{\alpha_2}} = 1$ and $I_r \subset J_r$ for each $r \in \mathbb{N}$, we have $\ell_\infty \cap S_\theta^{\alpha_1} \subset N_\psi^{\alpha_2}$ (see *Theorem 2.14, [14]*) and $N_\theta^{\alpha_2} \subset N_\psi^{\alpha_2}(G)$ holds by Corollary 2.2(ii) since the assumption $\sup_{u,k} \frac{g_k(u)}{u} < \infty$. It follows that $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset N_\psi^{\alpha_2}(G)$ holds.

Remark 2.5. For $F, G \in \mathcal{M}^{ub}$, the inclusion $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_2}(G)$ may be strict. This fact can be seen from Example 3.3 in *[25]* if we take $f_k(u) = g_k(u) = u$ for all $k \in \mathbb{N}$.

Corollary 2.7. *Suppose that* $F = (f_k) \in \mathcal{M}^{ub}$, $\theta = (k_r), \psi = (\omega_r) \in \mathcal{LS}(\mathbb{Z})$ *such that* $I_r \subset J_r$ *for every* $r \in \mathbb{N}$, *and* $0 < \alpha_1 \leq \alpha_2 \leq 1$ *. If* $\sup_{u,k} \frac{f_k(u)}{u} < \infty$ *for all* $k \in \mathbb{N}$ *and* $\lim_{r\to\infty} \frac{v_r}{h_r^{\alpha_2}} = 1$, *then the following assertions hold:*

i. $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset N_\psi^{\alpha_2}(F)$,

ii. $\ell_\infty \cap S_\theta^{\alpha_1}(F) \subset N_\psi^{\alpha_1}(F)$,

iii. $\ell_\infty \cap S_\theta^{\alpha_1} \subset N_\psi^{\alpha_1}(F)$.

In case modulus functions are bounded, a result similar to the above can be obtained.

Theorem 2.4. *Suppose that* $F = (f_k) \in \mathcal{M}^{ub}$, $\theta = (k_r) \in \mathcal{LS}(\mathbb{Z})$ *and* $0 < \alpha_1 \leq \alpha_2 \leq 1$*. Then,* $S_\theta^{\alpha_1}(F) \subset N_\theta^{\alpha_1}(F)$ *holds provided that* $\sup_{u>0} \sup_{n\in\mathbb{N}} f_n(u) < \infty$.

Proof. Assume that $\sup_{u>0} \sup_{n\in\mathbb{N}} f_n(u) < \infty$ holds and define $T = \sup_{u>0} T(u)$ where $T(u) = \sup_{n\in\mathbb{N}} f_n(u)$. Choose an arbitrary element $z = (z_k) \in S_\theta^{\alpha_1}(F)$ which is $S_\theta^{\alpha_1}(F)-$convergent to $l \in \mathbb{C}$. As shown in the proof of Theorem 2.3, $S_\theta^{\alpha_1}(F) \subset S_\theta^{\alpha_1}$ is satisfied. Hence $\lim_{r\to\infty} \frac{1}{h_r^{\alpha_1}} |\{k\epsilon I_r : |z_k - l| \geq \varepsilon\}| = 0$ holds. On the other hand, we obtain the following inequality:

$$\begin{aligned} \frac{1}{h_r^{\alpha_1}} \sum_{k\in I_r} f_k(|z_k - l|) &= \frac{1}{h_r^{\alpha_1}} \sum_{k\in I_r, |z_k - l| \geq \varepsilon} f_k(|z_k - l|) + \frac{1}{h_r^{\alpha_1}} \sum_{k\in I_r, |z_k - l| < \varepsilon} f_k(|z_k - l|) \\ &\leq \frac{1}{h_r^{\alpha_1}} T |\{k\epsilon I_r : |z_k - l| \geq \varepsilon\}| + \frac{1}{h_r^{\alpha_1}} h_r T. \end{aligned}$$

Taking the limit $r \to \infty$, it follows that $\lim_{r\to\infty} \frac{1}{h_r^{\alpha_1}} \sum_{k\in I_r} f_k(|z_k - l|) = 0$, i.e. $z = (z_k) \in N_\theta^{\alpha_1}(F)$. □

3. Concluding remarks and future directions

In this study, the categories of strongly lacunary summable sequences and lacunary statistically convergent sequences of numbers were introduced by employing a sequence of modulus functions. Furthermore, inclusion theorems have been established to compare these sets, which depend on parameters such as α, lacunary sequences, and sequences of modulus.

Statistical convergence is frequently employed in applied mathematics. Typically, a sequence is considered to converge statistically to a point when the majority of its elements approximate that point closely. However, achieving this majority often necessitates disregarding many terms in practice. In numerous applications, this approach to statistical convergence can be overly abrupt, resulting in the exclusion of elements from the sequence. Employing modulus functions offers a precise method for maintaining terms without discarding them.

As similar to other types of density functions, in this study, a density function defined by a sequence of unbounded modules and a real number has been used. By using a sequence of module functions instead of a single constant module function, the number of neglected terms will be much lower. Therefore, it can be considered as a method to somewhat improve statistical convergence and summability methods.

This research paper could be a resource for obtaining further advanced results. For example, by selecting different sequences of modulus functions used in various applied fields, application-specific sequence spaces can be obtained.

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References

[1] Zygmund, A.: Trigonometric Series, Cambridge University Press, Cambridge, (1979).

[2] Steinhaus, H.: *Sur la convergence ordinaire et la convergence asymptotique*. Colloquium Mathematicum. **2**, 73-74 (1951).

[3] Fast, H.: *Sur la convergence statistique*. Colloquium Mathematicum. **2**, 241-244 (1951).

[4] Schoenberg, I.J.: *The integrability of certain functions and related summability methods*. The American Mathematical Monthly. **66**, 361-375 (1959).

[5] Salat, T.: *On statistically convergent sequences of real numbers*. Mathematica Slovaca. **30**, 139-150 (1980).

[6] Fridy, J.: *On statistical convergence*. Analysis. **5**, 301-313 (1985).

[7] Çolak, R.: Statistical convergence of order α. In:Modern Methods in Analysis and Its Applications. Anamaya Publishers, New Delhi (2010).

[8] Çolak, R., Bektaş, Ç.A.: *$\lambda-$ Statistical convergence of order α*. Acta Mathematica Scientia. **31**(3), 953-959 (2011).

[9] Fridy, J., Orhan, C.: *Lacunary statistical convergence*. Pacific Journal of Mathematics. **160**(1), 43-51 (1993).

[10] Fridy, J., Orhan, C.: *Lacunary statistical summability*. Journal of Mathematical Analysis and Applications. **173**(2), 497-504 (1993).

[11] Connor, J.: *On strong matrix summability with respect to a modulus and statistical convergence*. Canadian Mathematical Bulletin. **32**(2), 194-198 (1989).

[12] Çolak, R.: *Lacunary strong convergence of difference sequences with respect to a modulus function*. Filomat. **17**, 9-14 (2003).

[13] Şengül, H., Et, M.: *On lacunary statistical convergence of order α*. Acta Mathematica Scientia. **34**(2), 473-482 (2014).

[14] Et, M., Şengül, H.: *Some Cesàro-Type Summability of order and Lacunary Statistical Convergence of order α*. Filomat. **28**, 1593-1602 (2014).

[15] Pehlivan, S., Fisher, B., *Lacunary strong convergence with respect to a sequence of modulus functions*. Commentationes Mathematicae Universitatis Carolinae. **36**(1), 69-76 (1995).

[16] Nakano, H.: *Concave modulars*. Journal of the Mathematical Society of Japan. **5**, 29-49 (1953).

[17] Ruckle, W.H.: *FK-Spaces in which the sequence of coordinate vectors is bounded*. Canadian Journal of Mathematics. **25**, 973-978 (1973).

[18] Bhardwaj, V.K., Singh,N.: *On some sequence spaces defined by a modulus*. Indian Journal of Pure and Applied Mathematics. **30**, 809-817 (1999).

[19] Ghosh, D., Srivastava, P.D.: *On some vector valued sequence spaces defined using a modulus function*. Indian Journal of Pure and Applied Mathematics. **30**, 819-826 (1999).

[20] Maddox, I.J.: *Sequence spaces defined by a modulus*. Mathematical Proceedings of the Cambridge Philosophical Society. **100**, 161-166 (1986).

[21] Pehlivan, S., Fisher, B.: *Some sequence spaces defined by a modulus function*. Mathematica Slovaca. **45**(3), 275-280 (1995).

[22] Aizpuru, A., Listán-Garcia, M.C., Rambla-Barreno, F.: *Density by moduli and statistical convergence*. Quaestiones Mathematicae. **37**, 525-530 (2014).

[23] Bhardwaj, V.K., Dhawan,S.: *f-Statistical convergence of order α and strong Cesàro summability of order with respect to a modulus*. Journal of Inequalities and Applications. **332** (2015).

[24] León-Saavedra, F., Listán-García, M.d.C., Pérez Fernández, F.J., de la Rosa, M.P.R.: *On statistical convergence and strong Cesàro convergence by moduli*. Journal of Inequalities and Applications. **298** (2019).

[25] İbrahim, İ.S., Çolak, R.: *On strong Lacunary summability of order with respect to modulus functions*. Math. Comp. Sci. Series. **48**(1), 127-136 (2021).

[26] Freedman, A.R., Sember, J.J., Raphael, M.: *Some Cesàro-Type summability spaces*. Proceedings of the London Mathematical Society. **37**(3), 508-520 (1978).

[27] Şengül, H., Et, M.: *f-Lacunary statistical convergence and strong f-Lacunary summability of order α*. Filomat. **32**(13), 4513-4521 (2018).

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