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Inextensible Flow of Quaternionic Curves According to Type 2-Quaternionic Frame in the Euclidean Space

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Abstract
In this paper, we investigate inextensible flows of quaternionic curve according to type 2-quaternionic frame. We give necessary and sufficient conditions for inextensible flow of quaternionic curves. Moreover, we obtain evolution equations of the Frenet frame and curvatures according to type 2-quaternionic frame.

Keywords: *Curvature flows, Quaternionic curve, Real quaternion*

AMS Subject Classification (2020): *53C44, 53A04*

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1. Introduction

The quaternions are extensions of the complex numbers. Quaternions were defined as the quotient of two directed lines in a three dimensional space or equivalently as the quotient of two vectors by Sir William Rowan Hamilton [1]. Quaternions can be represented in various ways: as the sum of a real scalar and a real three dimensional vector, as pairs of complex numbers or as four-dimensional vectors with real components. Quaternion multiplication is generally not commutative, so quaternions are not a field.

K. Baharatti and M. Nagaraj studied quaternionic curves in three-dimensional and four-dimensional Euclidean space and obtained their Frenet formulas [2]. In analogy with the Euclidean case, A.C. Coken and A. Tuna defined Frenet formulas for the quaternionic curves in semi-Euclidean space [3]. F. Kahraman Aksoyak introduced a new version of Frenet formulas for quaternionic curves in four-dimensional Euclidean space and called it type 2-quaternionic frame [4]. After that, by using these quaternionic frames, a lot of papers about quaternionic curves have been studied [5–12].

A family of curves parametrized by time can be thought as evolving curves. The time evolution of geometric locus is investigated by using its flow. There have been various studies on flows of curves, but firstly, D.Y. Kwon and F.C. Park introduced inextensible flows of plane curves [13] and D.Y. Kwon et al. investigated inextensible flows of curves and developable surfaces in $\mathbb{R}^3$ [14]. Then in many different spaces, inextensible flows of curves are studied (see, [15–19]). Inextensible flows of curves also studied for quaternionic curves (see, [6, 10, 12]).

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Our aim is to study inextensible flows of quaternionic curve according to type 2-quaternionic frame. We give necessary and sufficient conditions for inextensible flow of quaternionic curves. Moreover, we obtain evolution equations of the Frenet frame and curvatures according to type 2-quaternionic frame.

2. Preliminaries

In this section, a brief summary of the theory of quaternions in the Euclidean space is presented.

The space of quaternions Q is isomorphic to $\mathbb{R}^4$, four-dimensional vector space over the real numbers. There are three operations in Q: addition, scalar multiplication and quaternion multiplication. Addition and scalar multiplication of quaternions are defined to be the same as in $\mathbb{R}^4$.

A real quaternion q is an expression of the form $q = ae_1 + be_2 + ce_3 + de_4$, where a, b, c and d are real numbers, and e_1, e_2, e_3 are quaternionic units which satisfy the non-commutative multiplication rules,

$$\begin{aligned} i) e_i \times e_i &= -e_4, \quad (e_4 = 1,\ 1 \le i \le 3) \\ ii) e_i \times e_j &= e_k = -e_j \times e_i, \quad (1 \le i, j \le 3), \end{aligned}$$

where (ijk) is an even permutation of (123) in the Euclidean space $\mathbb{R}^4$. Further, a real quaternion can be written as $q = S_q + V_q$, where $S_q = d$ is the scalar part and $V_q = ae_1 + be_2 + ce_3$ is the vector part of q. The product of two quaternions can be expanded as

$$p \times q = S_p S_q - < V_p, V_q > + S_p V_q + S_q V_p + V_q \wedge V_q,$$

for every $p, q \in Q$, where $<,>$ and $\wedge$ are inner product and cross product on R^3, respectively. The conjugate of the quaternion q is denoted by $\overline{q}$ and defined as

$$\overline{q} = S_q - V_q = de_4 - ae_1 - be_2 - ce_3,$$

and is called by "Hamiltonian conjugation of q". The h-inner product of two quaternions is defined by

$$h(p, q) = \frac{1}{2}(p \times \overline{q} + q \times \overline{p}),$$

where h is the symmetric, non-degenerate, real-valued and bilinear form. Let p and q be two real quaternions, then $h(p, q) = 0$ if and only if p and q are $h-$orthogonal. The norm of a real quaternion q is defined by

$$\|q\|^2 = h(q, q) = a^2 + b^2 + c^2 + d^2.$$

If $q + \overline{q} = 0$, then q is called a spatial quaternion. The three-dimensional Euclidean space $\mathbb{R}^3$ is identified with the space of spatial quaternion $Q_s = \{\gamma \in Q \,|\, \gamma + \overline{\gamma} = 0\} \subset Q$ in an obvious manner.

Theorem 2.1. *Let*

$$\gamma : [0, 1] \subset \mathbb{R} \longrightarrow Q_s, \quad \gamma(s) = \sum_{i=1}^{3} \gamma_i(s) e_i, \quad (1 \le i \le 3),$$

be a smooth curve with arc-lenght parameter and $\{t, n_1, n_2\}$ be the Frenet trihedron of γ. Then Frenet equations are

$$\begin{aligned} t' &= kn_1 \\ n_1' &= -kt + rn_2 \\ n_2' &= -rn_1, \end{aligned}$$

where t is the unit tangent, n_1 is the unit principal normal, n_2 is the unit binormal vector fields, k is the principal curvature and r is the torsion of the quaternionic curve γ, [2].

Theorem 2.2. *Let*

$$\beta : [0, 1] \subset \mathbb{R} \longrightarrow Q, \quad \beta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1,$$

be a smooth curve β in Q and $\{T, N_1, N_2, N_3\}$ be the Frenet apparatus of β, then the Frenet equations are

$$\begin{aligned} T' &= KN_1 \\ N_1' &= -KT + kN_2 \\ N_2' &= -kN_1 + (r-K)N_3 \\ N_3' &= -(r-K)N_2, \end{aligned}$$

where $N_1 = t \times T$, $N_2 = n_1 \times T$, $N_3 = n_2 \times T$ and $K = \|T'(s)\|$, [2].

It is obtained the Frenet formulae in [2] and the apparatus for the curve β by making use of the Frenet formulae for a curve γ in E^3. Moreover, there are relationships between curvatures of the curves β and γ. These relations can be explained that the torsion of β is the principal curvature of the curve γ. Also, the bitorsion of β is $(r - K)$, where r is the torsion of γ and K is the principal curvature of β. These relations are only determined for quaternions, [2].

The alternative quaternionic frame for a quaternionic curve in $\mathbb{R}^4$ by using of a similar method in [2] given by Kahraman Aksoyak [4]

Theorem 2.3. *Let*

$$\zeta : [0,1] \subset R \longrightarrow Q, \quad \zeta(s) = \sum_{i=1}^{4} \gamma_i(s) e_i, \quad e_4 = 1,$$

be a smooth curve ζ in Q. The Frenet equations of $\zeta(s)$ for type 2-quaternionic frame are

$$\begin{aligned} T' &= KN_1 \\ N_1' &= -KT + -rN_2 \\ N_2' &= rN_1 + (K-k)N_3 \\ N_3' &= -(K-k)N_2, \end{aligned}$$

where $N_1 = b \times T$, $N_2 = n_1 \times T$, $N_3 = t \times T$ and $K = \|T'\|$, [4].

For further quaternions concepts see [20].

3. Flow of quaternionic curves according to type 2-quaternionic frame

Throughout this section, we investigate flow of quaternionic curve according to type 2-quaternionic frame.

Unless otherwise stated we assume that $\zeta : [0, l] \times [0, w] \rightarrow Q$ is a one parameter family of smooth quaternionic curve in Q where l is arclength of initial curve and u is the curve parametrization variable, $0 \leq u \leq l$. Let $\zeta(u, t)$ be a position vector of the semi-real quaternionic curve at time t. The arclength variation of $\zeta(u, t)$ is given by

$$s(u,t) = \int_0^u \left\| \frac{\partial \zeta}{\partial u} \right\| du = \int_0^u v du.$$

The operator $\frac{\partial}{\partial s}$ is given in term of u by $\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}$.

Definition 3.1. Let ζ be smooth quaternionic curve. Any flow of ζ can be given by

$$\frac{\partial \zeta}{\partial t} = g_1 T + g_2 N_1 + g_3 N_2 + g_4 N_3, \tag{3.1}$$

where g_1, g_2, g_3 and g_4 are scalar speed functions of ζ.

In Q, the inextensible condition of the length of the curve can be expressed by [13]

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0. \tag{3.2}$$

Definition 3.2. A quaternionic curve evolution $\zeta(u,t)$ and its flow $\frac{\partial \zeta}{\partial t}$ in Q are said to be inextensible if

$$\frac{\partial}{\partial t}\left\|\frac{\partial \zeta}{\partial u}\right\| = 0.$$

Lemma 3.1. *The evolution equation for the speed v according to type 2-quaternionic frame is given by*

$$\frac{\partial v}{\partial t} = \frac{\partial g_1}{\partial u} - v\kappa g_2. \tag{3.3}$$

Proof. As $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ are commutative and $v^2 = h\left(\frac{\partial \zeta}{\partial u}, \frac{\partial \zeta}{\partial u}\right)$, we have

$$2v\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}h\left(\frac{\partial \zeta}{\partial u}, \frac{\partial \zeta}{\partial u}\right) = 2h\left(\frac{\partial \zeta}{\partial u}, \frac{\partial}{\partial u}\left(\frac{\partial \zeta}{\partial t}\right)\right).$$

By using the equations of type 2-quaternionic frame, we obtain

$$\frac{\partial v}{\partial t} = \frac{\partial g_1}{\partial u} - v\kappa g_2.$$

□

Theorem 3.1. *The flow of quaternionic curve is inextensible according to type 2-quaternionic frame if and only if*

$$\frac{\partial g_1}{\partial s} = \kappa g_2. \tag{3.4}$$

Proof. Let the flow of quaternionic curve be inextensible. From equation (3.2) and (3.3), we have

$$\frac{\partial}{\partial t}s(u,t) = \int_0^u \frac{\partial v}{\partial t}du = \int_0^u \left(\frac{\partial g_1}{\partial u} - v\kappa g_2\right)du = 0.$$

This clearly forces

$$\frac{\partial g_1}{\partial s} = \kappa g_2.$$

□

Lemma 3.2. *Let the flow of $\zeta(u,t)$ be inextensible. Derivatives of the elements of type 2-quaternionic frame with respect to evolution parameter can be given as follows;*

$$\begin{aligned}
\frac{\partial T}{\partial t} &= \left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right)N_1 + \left(-g_2 r + \frac{\partial g_3}{\partial s} - g_4(\kappa - k)\right)N_2 \\
&\quad + \left(g_3(\kappa - k) + \frac{\partial g_4}{\partial s}\right)N_3, \\
\frac{\partial N_1}{\partial t} &= -\left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right)T + \psi_1 N_2 + \psi_2 N_3, \\
\frac{\partial N_2}{\partial t} &= \left(g_2 r - \frac{\partial g_3}{\partial s} + g_4(\kappa - k)\right)T - \psi_1 N_1 + \psi_3 N_3, \\
\frac{\partial N_3}{\partial t} &= -\left(g_3(\kappa - k) + \frac{\partial g_4}{\partial s}\right)T - \psi_2 N_1 - \psi_3 N_2,
\end{aligned}$$

where $\psi_1 = h\left(\frac{\partial N_1}{\partial t}, N_2\right), \psi_2 = h\left(\frac{\partial N_1}{\partial t}, N_3\right), \psi_3 = h\left(\frac{\partial N_2}{\partial t}, N_3\right)$.

Proof. Let $\frac{\partial \zeta}{\partial t}$ be inextensible. Then, considering that $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ are commutative, we get

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial \zeta}{\partial s}\right) = \frac{\partial}{\partial s}\left(\frac{\partial \zeta}{\partial t}\right) = \frac{\partial}{\partial s}\left(g_1 T + g_2 N_1 + g_3 N_2 + g_4 N_3\right),$$

substituting (3.4) in the last equation, we have

$$\frac{\partial T}{\partial t} = \left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right) N_1 + \left(-g_2 r + \frac{\partial g_3}{\partial s} - g_4\left(\kappa - k\right)\right) N_2 + \left(g_3\left(\kappa - k\right) + \frac{\partial g_4}{\partial s}\right) N_3.$$

Now, if we consider orthogonality of $\{T, N_1, N_2, N_3\}$, then we get

$$\begin{aligned}
0 &= \frac{\partial}{\partial t} h\left(T, N_1\right) = h\left(\frac{\partial T}{\partial t}, N_1\right) + h\left(T, \frac{\partial N_1}{\partial t}\right) \\
&= \left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right) + h\left(T, \frac{\partial N_1}{\partial t}\right), \\
0 &= \frac{\partial}{\partial t} h\left(T, N_2\right) = h\left(\frac{\partial T}{\partial t}, N_2\right) + h\left(T, \frac{\partial N_2}{\partial t}\right) \\
&= \left(-g_2 r + \frac{\partial g_3}{\partial s} - g_4\left(\kappa - k\right)\right) + h\left(T, \frac{\partial N_2}{\partial t}\right), \\
0 &= \frac{\partial}{\partial t} h\left(T, N_3\right) = h\left(\frac{\partial T}{\partial t}, N_3\right) + h\left(T, \frac{\partial N_3}{\partial t}\right) \\
&= \left(g_3\left(\kappa - k\right) + \frac{\partial g_4}{\partial s}\right) + h\left(T, \frac{\partial N_3}{\partial t}\right), \\
0 &= \frac{\partial}{\partial t} h\left(N_1, N_2\right) = h\left(\frac{\partial N_1}{\partial t}, N_2\right) + h\left(N_1, \frac{\partial N_2}{\partial t}\right) \\
&= \psi_1 + h\left(N_1, \frac{\partial N_2}{\partial t}\right), \\
0 &= \frac{\partial}{\partial t} h\left(N_1, N_3\right) = h\left(\frac{\partial N_1}{\partial t}, N_3\right) + h\left(N_1, \frac{\partial N_3}{\partial t}\right) \\
&= \psi_2 + h\left(N_1, \frac{\partial N_3}{\partial t}\right), \\
0 &= \frac{\partial}{\partial t} h\left(N_2, N_3\right) = h\left(\frac{\partial N_2}{\partial t}, N_3\right) + h\left(N_2, \frac{\partial N_3}{\partial t}\right) \\
&= \psi_3 + h\left(N_2, \frac{\partial N_3}{\partial t}\right),
\end{aligned}$$

which brings about that

$$\begin{aligned}
\frac{\partial N_1}{\partial t} &= -\left(g_1\kappa + \frac{\partial g_2}{\partial s} + g_3 r\right) T + \psi_1 N_2 + \psi_2 N_3, \\
\frac{\partial N_2}{\partial t} &= \left(g_2 r - \frac{\partial g_3}{\partial s} + g_4\left(\kappa - k\right)\right) T - \psi_1 N_1 + \psi_3 N_3, \\
\frac{\partial N_3}{\partial t} &= -\left(g_3\left(\kappa - k\right) + \frac{\partial g_4}{\partial s}\right) T - \psi_2 N_1 - \psi_3 N_2,
\end{aligned}$$

where $\psi_1 = h\left(\frac{\partial N_1}{\partial t}, N_2\right), \psi_2 = h\left(\frac{\partial N_1}{\partial t}, N_3\right), \psi_3 = h\left(\frac{\partial N_2}{\partial t}, N_3\right).$ □

Theorem 3.2. *Let the flow of $\zeta\left(u, t\right)$ be inextensible. Then the evolution equation of κ is*

$$\frac{\partial \kappa}{\partial t} = \frac{\partial g_1}{\partial s}\kappa + g_1\frac{\partial \kappa}{\partial s} + \frac{\partial^2 g_2}{\partial s^2} + 2\frac{\partial g_3}{\partial s} r + g_3\frac{\partial r}{\partial s} - g_2 r^2 - g_4 r\left(\kappa - k\right).$$

Proof. Since $\frac{\partial}{\partial s}\left(\frac{\partial T}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial s}\right)$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial T}{\partial t}\right) &= \left(-g_1\kappa^2-\frac{\partial g_2}{\partial s}\kappa-g_3\kappa r\right)T \\
&+\left(\frac{\partial g_1}{\partial s}\kappa+g_1\frac{\partial \kappa}{\partial s}+\frac{\partial^2 g_2}{\partial s^2}+2\frac{\partial g_3}{\partial s}r+g_3\frac{\partial r}{\partial s}-g_2r^2-g_4r\left(\kappa-k\right)\right)N_1 \\
&+\left(-g_1\kappa r-2\frac{\partial g_2}{\partial s}r-g_3r^2+g_2\frac{\partial r}{\partial s}+\frac{\partial^2 g_3}{\partial s^2}\right. \\
&\left.-2\frac{\partial g_4}{\partial s}\left(\kappa-k\right)-g_4\frac{\partial\left(\kappa-k\right)}{\partial s}-g_3\left(\kappa-k\right)^2\right)N_2 \\
&+\left(-g_2r\left(\kappa-k\right)+2\frac{\partial g_3}{\partial s}\left(\kappa-k\right)-g_4\left(\kappa-k\right)^2\right. \\
&\left.+g_3\frac{\partial\left(\kappa-k\right)}{\partial s}+\frac{\partial^2 g_4}{\partial s^2}\right)N_3
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial s}\right) &= \frac{\partial}{\partial t}\left(\kappa N_1\right)=\frac{\partial\kappa}{\partial t}N_1+\kappa\frac{\partial N_1}{\partial t} \\
&=\left(-g_1\kappa^2-\frac{\partial g_2}{\partial s}\kappa-g_3\kappa r\right)T+\frac{\partial \kappa}{\partial t}N_1+\psi_1\kappa N_2 \\
&+\psi_2\kappa N_3.
\end{aligned}$$

From equality of the component of N_1 in above equations, we obtain

$$\frac{\partial\kappa}{\partial t}=\frac{\partial g_1}{\partial s}\kappa+g_1\frac{\partial \kappa}{\partial s}+\frac{\partial^2 g_2}{\partial s^2}+2\frac{\partial g_3}{\partial s}r+g_3\frac{\partial r}{\partial s}-g_2r^2-g_4r\left(\kappa-k\right).$$

□

Corollary 3.1. *In theorem (3.2), from rest of the equality, we get*

$$\begin{aligned}
\kappa\psi_1 &= -g_1\kappa r-2\frac{\partial g_2}{\partial s}r-g_3r^2+g_2\frac{\partial r}{\partial s}+\frac{\partial^2 g_3}{\partial s^2}-2\frac{\partial g_4}{\partial s}\left(\kappa-k\right)-g_4\frac{\partial\left(\kappa-k\right)}{\partial s}-g_3\left(\kappa-k\right)^2, \\
\kappa\psi_2 &= -g_2r\left(\kappa-k\right)+2\frac{\partial g_3}{\partial s}\left(\kappa-k\right)-g_4\left(\kappa-k\right)^2+g_3\frac{\partial\left(\kappa-k\right)}{\partial s}+\frac{\partial^2 g_4}{\partial s^2}.
\end{aligned}$$

Theorem 3.3. *Let the flow of $\zeta\left(u,t\right)$ be inextensible. Then the evolution equation of r is*

$$\frac{\partial r}{\partial t}=g_2\kappa r-\frac{\partial g_3}{\partial s}\kappa+g_4\kappa\left(\kappa-k\right)-\frac{\partial\psi_1}{\partial s}+\psi_2\left(\kappa-k\right).$$

Proof. Noticing that $\frac{\partial}{\partial s}\left(\frac{\partial N_1}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{\partial N_1}{\partial s}\right)$, it is seen that

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial N_1}{\partial t}\right) &= \left(-\frac{\partial g_1}{\partial s}\kappa-g_1\frac{\partial\kappa}{\partial s}+\frac{\partial^2 g_2}{\partial s^2}+\frac{\partial g_3}{\partial s}r+g_3\frac{\partial r}{\partial s}\right)T \\
&+\left(-g_1\kappa^2+\frac{\partial g_2}{\partial s}\kappa+g_3\kappa r+\psi_1 k\right)N \\
&+\left(\frac{\partial\psi_1}{\partial s}-\psi_2\left(\kappa-k\right)\right)N_2 \\
&+\left(\psi_1\left(\kappa-k\right)+\frac{\partial\psi_2}{\partial s}\right)N_3
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial N_1}{\partial s}\right) &= \frac{\partial}{\partial t}\left(-\kappa T - rN_2\right)\\
&= \left(-\frac{\partial \kappa}{\partial t} - g_2 r^2 + \frac{\partial g_3}{\partial s} r - g_4 r\left(\kappa - k\right)\right) T\\
&+ \left(-g_1\kappa^2 - \frac{\partial g_2}{\partial s}\kappa - g_3\kappa r + \psi_1 r\right) N_1\\
&+ \left(g_2 r\kappa - \frac{\partial g_3}{\partial s}\kappa + g_4\kappa\left(\kappa - k\right) - \frac{\partial r}{\partial t}\right) N_2\\
&+ \left(-g_3\kappa\left(\kappa - k\right) - \frac{\partial g_4}{\partial s}\kappa - \psi_3 r\right) N_3.
\end{aligned}$$

From above equations, we get

$$\frac{\partial r}{\partial t} = g_2\kappa r - \frac{\partial g_3}{\partial s}\kappa + g_4\kappa\left(\kappa - k\right) - \frac{\partial \psi_1}{\partial s} + \psi_2\left(\kappa - k\right).$$

□

Corollary 3.2. *In theorem (3.3), from rest of the equality, we obtain*

$$\psi_1\left(\kappa - k\right) = -\frac{\partial \psi_2}{\partial s} - g_3\kappa\left(\kappa - k\right) - \frac{\partial g_4}{\partial s}\kappa - \psi_3 r.$$

Theorem 3.4. *Let the flow of $\zeta(u,t)$ be inextensible. Then the evolution equation of $(\kappa - k)$ is*

$$\frac{\partial\left(\kappa - k\right)}{\partial t} = -\psi_2 r + \frac{\partial \psi_3}{\partial s}.$$

Proof. Noticing that $\frac{\partial}{\partial s}\left(\frac{\partial N_2}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial N_2}{\partial s}\right)$, it is seen that

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial N_2}{\partial t}\right) &= \left(\frac{\partial g_2}{\partial s} r + g_2\frac{\partial r}{\partial s} - \frac{\partial^2 g_3}{\partial s^2} + \frac{\partial g_4}{\partial s}\left(\kappa - k\right) + g_4\frac{\partial\left(\kappa - k\right)}{\partial s} + \psi_1\kappa\right) T\\
&+ \left(g_2\kappa r - \frac{\partial g_3}{\partial s}\kappa + g_4\kappa\left(\kappa - k\right) - \frac{\partial \psi_1}{\partial s}\right) N_1\\
&+ \left(\psi_1 r - \psi_3\left(\kappa - k\right)\right) N_2\\
&+ \left(\frac{\partial \psi_3}{\partial s}\right) N_3
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial N_2}{\partial s}\right) &= \frac{\partial}{\partial t}\left(rN_1 + \left(\kappa - k\right)N_3\right)\\
&= \left(-g_1 r\kappa - \frac{\partial g_2}{\partial s} r - g_3 r^2 - g_3\left(\kappa - k\right)^2 - \frac{\partial g_4}{\partial s}\left(\kappa - k\right)\right) T\\
&+ \left(\frac{\partial r}{\partial t} - \psi_2\left(\kappa - k\right)\right) N_1\\
&+ \left(\psi_1 r - \psi_3\left(\kappa - k\right)\right) N_2\\
&+ \left(\psi_2 k + \frac{\partial\left(\kappa - k\right)}{\partial t}\right) N_3.
\end{aligned}$$

From above equations, we obtain

$$\frac{\partial\left(\kappa - k\right)}{\partial t} = -\psi_2 r + \frac{\partial \psi_3}{\partial s}.$$

□

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