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Singüler Pertürbe Özellikli Fredholm İntegro Diferansiyel Denkleminin Katman Davranışının İncelenmesi

Muhammet Enes DURMAZ

Öne Çıkanlar:

- Singüler pertürbe özellikli Fredholm integro diferansiyel denkleminde katman davranışının analizi,
- Singüler pertürbe özellikli bir problemin katman analizi,
- İntegro diferansiyel denklem

ÖZET:

Çalışma, ikinci mertebeden lineer singüler pertürbe özellikli Fredholm integro diferansiyel denklemini ele almaktadır. Bu tür problemlerin niteliksel analizi, çözümün sınır katmanlarındaki davranışının hızlı değişmesi nedeniyle oldukça zordur. Bu çalışmada sınır katmanlı Fredholm integro diferansiyel denkleminin çözümü ve çözümün birinci ve ikinci türevleri için asimptotik değerlendirmeler sunulmuştur. Elde edilen değerlendirmeler, matematiksel modelleme ve analizde uygun yaklaşık yöntemlerin geliştirilmesine ve değerlendirilmesine katkı sağlaması açısından önem taşımaktadır. Ayrıca sunulan örnek, teorik sonuçların geçerliliğine ve değerlendirmelerin doğruluğuna destek sağlamaktadır.

Anahtar Kelimeler:

- Asimptotik değerlendirme,
- Sınır katmanı,
- Singüler pertürbasyon

Survey of the Layer Behaviour of the Singularly Perturbed Fredholm Integro-Differential Equation

Highlights:

- Analyzing of the layer behavior in a singularly perturbed Fredholm integro differential equation,
- A layer analysis of a singularly perturbed problem,
- Integro differential equation

ABSTRACT:

The work handles a second order linear singularly perturbed Fredholm integro differential equation. The qualitative analysis of such problems is quite difficult due to the rapid change in behavior of the solution within the boundary layer. In this study, asymptotic estimates for the solution and its first and second derivatives of the Fredholm integro differential equation with a boundary layer have been presented. The obtained estimates have significance in their contribution to the development and evaluation of appropriate approximate methods in mathematical modeling and analysis. Furthermore, the presented example provides support for the validity of the theoretical results and the accuracy of the estimates.

Keywords:

- Asymptotic estimate,
- Boundary layer,
- Singular perturbation

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INTRODUCTION

When the largest derivative term in a differential equation (DE) is multiplied by a tiny parameter $\varepsilon \in (0,1]$, this parameter is called to be a singular perturbation parameter and the DE is called as a singularly perturbed differential equation (SPDE). Due to the presence of the perturbation parameter, layers appear at the boundaries, which are called as boundary layers. SPDEs express a wide variety of mathematical models, ranging from mathematical engineering to problems in chemical reactions, fluid dynamics, heat transfer, population dynamics, control theory, biology, aerodynamics, electrical networks and neuroscience. Further information on SPDEs can be found in the publications (Nayfeh, 1993; Kevorkian & Cole, 1981) and their references. Due to the behaviour of the boundary layer of the solution, the numerical analysis of SPDEs has always been a challenge. Such problems exhibit rapid changes in thin layers at the interior or boundary of the problem domain (Schmisser & Weiss, 1986; O'Malley, 1991; Miller et al., 2012). Standard numerical methods for solving these types of problems are widely accepted to be unsteady and fail to give accurate results when the $\varepsilon-$parameter is small. For this reason, it is crucial to develop suitable numerical techniques that converge uniformly with respect to the $\varepsilon-$parameter. A number of approaches to the numerical solution of these forms of differential equations are covered in the literature (Farrell et al., 2000; Reddy & Chakravarthy, 2004; Roos et al., 2008; Kadalbajoo & Gupta, 2010; El-Zahar, 2020).

In mathematics, physics and engineering, there are sometimes problems that cannot be expressed by a single equation. Therefore, such problems can be expressed as integral, differential or integro-differential equations, which contain more than one unknown function.

This study will focus on the Fredholm integro-differential equation (FIDE). FIDEs can be encountered in a wide range of mathematical applications and scientific fields, including electromagnetic theory, fluid mechanics, oceanography, plasma physics, biology, artificial neural networks and financial mathematical processes (see, for example, (Abdulghani et al., 2019; Hamoud & Ghadle, 2019)). As a result, a considerable number of researchers have devoted their attention to the study of FIDEs over the years. Because exact solutions to these types of problems are difficult to obtain, researchers use appropriate analytical techniques. Some of the methods include reproducing kernel Hilbert space method (Arqub et al., 2013), Tau method (Hosseini & Shahmorad, 2003), Nyström method (Tair et al., 2021), Touchard polynomials method (Abdullah, 2021), Boole Collocation method (Dag & Bicer, 2020), Galerkin method (Chen et al., 2020), Collocation and Kantorovich methods (Tair et al. 2022), variational iteration technique (Hamoud & Ghadle, 2019), Legendre collocation matrix method (Yalcinbas et al., 2009) and parameterization method (Dzhumabaev et al., 2020).

In addition, some existence and uniqueness of the solution for both SPDEs and FIDEs are discussed in (Vougalter & Volpert, 2018; Lin et al., 2020). In (Amiraliyev et al., 2020; Amiraliyev et al., 2021) the authors have discussed an asymptotic approach for singularly perturbed Fredholm integro-differential equations (SPFIDEs). Furthermore, some numerical perspectives of these types of problems with a small term are considered in (Durmaz et al., 2022a; Durmaz et al., 2022b; Amirali et al., 2023; Panda et al., 2024). In (Cimen & Cakir, 2021) it is shown for the linear SPFIDE that the difference scheme constructed using interpolating quadrature rules converges uniformly with respect to the perturbation parameter. In research on the initial value problem for nonlinear SPFIDE, the authors proved uniform convergence with respect to the $\varepsilon-$parameter for a new difference scheme (Cakir et al., 2022). In (Cakir & Cimen, 2023), for the singularly perturbed second order Volterra integro-differential equation, it was shown that the difference scheme is uniformly convergent in the first order with respect to the perturbation parameter.

In the present paper, we provide the asymptotic estimates for the solution of (2.1)-(2.2) and its first and second derivatives. The results obtained are very important for the analysis of suitable approximation methods. To support the predicted theory, an example is given.

This study is presented in the following format: Section 1 presents the introduction, Section 2 evaluates the main findings and Section 3 illustrates the theoretical results with an example.

MATERIALS AND METHODS

In this paper, the following SPFIDE is being analyzed:

$$Lu := \varepsilon \upsilon''(\eta) + a(\eta)\upsilon'(\eta) - b(\eta)\upsilon(\eta) + \lambda\int_0^{\ell} M(\eta,\gamma)\upsilon(\gamma)\,d\gamma = f(\eta), \quad 0<\eta<\ell, \tag{1}$$

$$\upsilon(0) = \mathrm{A}, \quad \upsilon(\ell) = \mathrm{B}, \tag{2}$$

where $0<\varepsilon \ll 1$ is a perturbation and λ is a real parameter. A and B are given invariables. We presume that $M(\eta,\gamma)\big((\eta,\gamma)\in[0,\ell]\mathrm{x}[0,\ell]\big)$, $f(\eta)$, $a(\eta)\geq\alpha>0$, $b(\eta)\geq 0$, $\big(\eta\in[0,\ell]\big)$ are the sufficiently smooth functions satisfying certain regularity conditions to be specified. The solution $\upsilon(\eta)$ of (1)-(2) has in general boundary layer near $\eta=0$. We denote the maximum norm of any continuous function $q(\eta)$ on the interval by $\|q\|_\infty$.

Definition 1. (Maximum Principle) Assume that $u(0)\geq 0$ and $u(\ell)\geq 0$. Then $Lu(\eta)\geq 0$, $0<\eta<\ell$, implies that $u(\eta)\geq 0$, for all $0<\eta<\ell$.

Lemma 1. For any $u(\eta)$ function, let $u(\eta)\in C[0,\ell]\cap C^2(0,\ell)$. Then the following estimate is true.

$$|u(\eta)| \leq |u(0)| + |u(\ell)| + \alpha^{-1}\max_{1\leq i\leq N}|Lu(\eta)|, \quad 0\leq\eta\leq\ell.$$

Proof. Let us define the $\Upsilon(\eta)$ function as follows:

$$\Upsilon(\eta) = \pm u(\eta) + |u(0)| + |u(\ell)| + \alpha^{-1}\max_{1\leq i\leq N}|Lu(\eta)|, \quad 0\leq\eta\leq\ell.$$

Then the following inequalities are satisfied

$$\Upsilon(0)\geq 0, \quad \Upsilon(\ell)\geq 0$$

and

$$L\Upsilon(\eta)\geq 0.$$

The maximum principle gives $\Upsilon(\eta)\geq 0$, for all $0<\eta<\ell$ and so inequality Lemma 1 holds.

RESULTS AND DISCUSSION

In this section, we give a priori bounds on the solution and its derivatives for the given problem (1)-(2).

Theorem 1. Assume that $a,b,f\in C^1[0,\ell]$ and $\dfrac{\partial^s M}{\partial\eta^s}\in C[0,\ell]^2$, $(s=0,1)$. Moreover

$$\lambda < \frac{\alpha \ell^{-1}}{\max\limits_{0\le\eta\le\ell} \int_0^\ell |M(\eta,\gamma)| d\gamma}. \tag{3}$$

The solution $\upsilon(\eta)$ of the problem (1)-(2) satisfies the bounds

$$\|\upsilon\|_\infty \le C, \tag{4}$$

$$\left|\upsilon^{(\tau)}(\eta)\right| \le C\left\{1+\frac{1}{\varepsilon^\tau} e^{-\frac{\alpha\eta}{\varepsilon}}\right\}, \ \eta\in[0,\ell], \ \tau=1,2. \tag{5}$$

Proof. First we show the validity of (1). By virtue of $a, f \in C^1[0,\ell]$ and $\frac{\partial^s M}{\partial \eta^s} \in C[0,\ell]^2$, according to maximum principle from (1)-(2) we have

$$\|\upsilon\|_\infty \le |A|+|B|+\alpha^{-1}\ell\|f\|_\infty + \alpha^{-1}\ell|\lambda| \max_{0\le\eta\le\ell}\int_0^\ell |M(\eta,\gamma)||\upsilon(\gamma)| d\gamma$$
$$\le |A|+|B|+\alpha^{-1}\ell\|f\|_\infty + \alpha^{-1}\ell|\lambda| \max_{0\le\eta\le\ell}\int_0^\ell |M(\eta,\gamma)| d\gamma \|\upsilon\|_\infty .$$

So, we obtain

$$\|\upsilon\|_\infty \le \frac{|A|+|B|+\alpha^{-1}\ell\|f\|_\infty}{1-\alpha^{-1}\ell|\lambda|\max\limits_{0\le\eta\le\ell}\int_0^\ell |M(\eta,\gamma)| d\gamma}$$
$$\le C.$$

Thus, taking into account (3), the correctness of the expression (4) has shown. Now, we show the proof of (5). We can rewrite the problem (1)-(2) as

$$\varepsilon\upsilon''(\eta)+a(\eta)\upsilon'(\eta)=F(\eta), \tag{6}$$

where

$$F(\eta)= f(\eta)+b(\eta)u(\eta)-\lambda\int_0^\ell M(\eta,\gamma)u(\gamma)d\gamma.$$

Taking into account (4) evidently we get relation

$$|F(\eta)| \le \left| f(\eta)+b(\eta)\upsilon(\eta)-\lambda\int_0^\ell M(\eta,\gamma)\upsilon(\gamma)d\gamma\right|$$
$$\le |f(\eta)|+|b(\eta)\upsilon(\eta)|+|\lambda|\int_0^\ell |M(\eta,\gamma)||\upsilon(\gamma)|d\gamma$$
$$\le C.$$

From the equation (6), we can write

$$\upsilon'(\eta)=\upsilon'(0)e^{-\frac{1}{\varepsilon}\int_0^\eta a(\gamma)d\gamma} + \frac{1}{\varepsilon}\int_0^\eta F(\zeta) e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma} d\zeta. \tag{7}$$

We need an estimate in (7) for $\upsilon'(0)$. For this reason, integrating this equality over $(0,\ell)$, we have

$$\upsilon(\ell)-\upsilon(0)=\upsilon'(0)\int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\eta a(\gamma)d\gamma}d\eta+\frac{1}{\varepsilon}\int_0^\ell\int_0^\eta F(\zeta)e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma}d\zeta d\eta,$$

$$\mathrm{B}-\mathrm{A}=\upsilon'(0)\int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\eta a(\gamma)d\gamma}d\eta+\frac{1}{\varepsilon}\int_0^\ell\int_0^\eta F(\zeta)e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma}d\zeta d\eta.$$

Rearranging this equality we obtain

$$\upsilon'(0)=\frac{\mathrm{B}-\mathrm{A}-\frac{1}{\varepsilon}\int_0^\ell\int_0^\eta F(\zeta)e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma}d\zeta d\eta}{\int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\eta a(\gamma)d\gamma}d\eta}. \tag{8}$$

For integral in the denominator, we acquire

$$\begin{aligned}\int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\eta a(\gamma)d\gamma}d\eta &\geq \int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\eta \tilde{a}(\gamma)d\gamma}d\eta\\ &=\int_0^\ell e^{-\frac{\tilde{a}\eta}{\varepsilon}}d\eta\\ &=\frac{\varepsilon}{\tilde{a}}\left(1-e^{-\frac{\tilde{a}\ell}{\varepsilon}}\right)\\ &\equiv c_1\varepsilon,\end{aligned} \tag{9}$$

where is $\tilde{a}=\max_{\eta\in(0,1]}a(\eta)$. Applying the mean value theorem to integral term in the numerator in (8), we get

$$\begin{aligned}\left|\frac{1}{\varepsilon}\int_0^\ell\int_0^\eta F(\zeta)e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma}d\zeta d\eta\right| &\leq \frac{1}{\varepsilon}\int_0^\ell\int_0^\eta |F(\zeta)|e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma}d\zeta d\eta\\ &\leq \frac{\|F\|_\infty}{\varepsilon}\int_0^\ell\int_0^\eta e^{-\frac{1}{\varepsilon}\int_\zeta^\eta \alpha d\gamma}d\zeta d\eta\\ &\leq \frac{\|F\|_\infty}{\varepsilon}\int_0^\ell\int_0^\eta e^{-\frac{\alpha(\eta-\zeta)}{\varepsilon}}d\zeta d\eta\\ &=\frac{\|F\|_\infty}{\varepsilon}\frac{\varepsilon}{\alpha}\int_0^\ell\left(1-e^{-\frac{\alpha\eta}{\varepsilon}}\right)d\eta\\ &\leq \frac{\|F\|_\infty\ell}{\alpha}\\ &\equiv c_2.\end{aligned} \tag{10}$$

By considering the estimates (9) and (10) in (8), we can write

$$\left|\upsilon'(0)\right| \le \frac{|\mathrm{A}|+|\mathrm{B}|+\frac{1}{\varepsilon}\int_0^\ell\int_0^\eta |F(\zeta)| e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma} d\zeta d\eta}{\int_0^\ell e^{-\frac{1}{\varepsilon}\int_0^\eta a(\gamma)d\gamma} d\eta}$$
$$\le \frac{|\mathrm{A}|+|\mathrm{B}|+c_2}{c_1}$$
$$\equiv \frac{C_1}{\varepsilon}.$$

By taking the absolute value in (7), we get

$$\left|\upsilon'(\eta)\right| \le \left|\upsilon'(0)\right| e^{-\frac{1}{\varepsilon}\int_0^\eta a(\gamma)d\gamma} + \frac{1}{\varepsilon}\int_0^\eta |F(\zeta)| e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma} d\zeta$$
$$\le \frac{C_1}{\varepsilon} e^{-\frac{1}{\varepsilon}\int_0^\eta \alpha d\gamma} + \frac{1}{\varepsilon}\|F\|_\infty \int_0^\eta e^{-\frac{1}{\varepsilon}\int_\zeta^\eta \alpha d\gamma} d\zeta$$
$$= \frac{C_1}{\varepsilon} e^{-\frac{\alpha\eta}{\varepsilon}} + \frac{\|F\|_\infty}{\alpha}$$
$$\le C\left(1+\frac{1}{\varepsilon} e^{-\frac{\alpha\eta}{\varepsilon}}\right)$$

which reach at the proof of (5) for $\tau = 1$.
Now, to obtain (5) estimation for $\tau = 2$, differentiating (1) we get

$$\varepsilon\upsilon'''(\eta)+a(\eta)\upsilon''(\eta)=\Psi(\eta), \tag{11}$$

where

$$\Psi(\eta) = f'(\eta)+b'(\eta)\upsilon(\eta)+b(\eta)\upsilon'(\eta)-a'(\eta)\upsilon'(\eta)-\lambda\int_0^\ell \frac{\partial}{\partial\eta} M(\eta,\gamma)\upsilon(\gamma)d\gamma.$$

It is obvious that

$$\left|\Psi(\eta)\right| \le C\left(1+\frac{1}{\varepsilon} e^{-\frac{\alpha\eta}{\varepsilon}}\right).$$

From the relation (11), we get

$$\upsilon''(\eta) = \upsilon''(0) e^{-\frac{1}{\varepsilon}\int_0^\eta a(\gamma)d\gamma} + \frac{1}{\varepsilon}\int_0^\eta \Psi(\zeta) e^{-\frac{1}{\varepsilon}\int_\zeta^\eta a(\gamma)d\gamma} d\zeta. \tag{12}$$

We need an estimate in (12) for $\upsilon''(0)$. From (1) we have

$$\left|\upsilon''(0)\right| = \frac{1}{\varepsilon}\left| f(0)+b(0)\upsilon(0)-a(0)\upsilon'(0)-\lambda\int_0^\ell M(0,\gamma)\upsilon(\gamma)d\gamma\right|$$
$$\le \frac{C_1}{\varepsilon^2}.$$

Taking the absolute value in (12), we get

$$\left|\upsilon''(\eta)\right| \le \left|\upsilon''(0)\right| e^{-\frac{1}{\varepsilon}\int_0^{\eta} a(\gamma)d\gamma} + \frac{1}{\varepsilon}\int_0^{\eta}\left|\Psi(\zeta)\right| e^{-\frac{1}{\varepsilon}\int_{\zeta}^{\eta} a(\gamma)d\gamma} d\zeta$$

$$\le \frac{C_1}{\varepsilon^2} e^{-\frac{1}{\varepsilon}\int_0^{\eta}\alpha d\gamma} + \frac{C_1}{\varepsilon}\int_0^{\eta}\left(1+\frac{1}{\varepsilon}e^{-\frac{\alpha\zeta}{\varepsilon}}\right) e^{-\frac{\alpha(\eta-\zeta)}{\varepsilon}} d\zeta.$$

Therefore, we obtain

$$\left|\upsilon''(\eta)\right| \le \frac{C_1}{\varepsilon^2} e^{-\frac{\alpha\eta}{\varepsilon}} + C_1\alpha^{-1}\left(1-e^{-\frac{\alpha\eta}{\varepsilon}}\right) + \frac{C_1}{\varepsilon^2}\int_0^{\eta} e^{-\frac{\alpha\eta}{\varepsilon}} d\zeta$$

$$\le C_1\left(1+\frac{1}{\varepsilon^2}e^{-\frac{\alpha\eta}{\varepsilon}}\right) + \frac{C_1\eta}{\varepsilon^2}e^{-\frac{\alpha\eta}{\varepsilon}},$$

which proves (5) for $\tau = 2.$

Example 1. We contemplate the special problem with

$$\varepsilon\upsilon''(\eta) + \upsilon'(\eta) - (1+2\eta)\upsilon(\eta) + \frac{1}{2}\int_0^1 \eta\upsilon(\gamma)d\gamma = \delta_1\eta, \quad 0<\eta<1,$$

$$\mathrm{A}=0, \quad \mathrm{B}=1+\delta_1,$$

where

$$\delta_1 = \varepsilon^2\left(\frac{1}{2}-\varepsilon\right) + \frac{1}{6}\left(\varepsilon+\frac{1}{4}\right) + \frac{\varepsilon^2-\frac{1}{4}}{e^{\frac{1}{\varepsilon}}-1}.$$

The exact solution is given by

$$\upsilon(\eta) = \eta^2 + \eta - 2\eta\varepsilon + \frac{(2\varepsilon-1)\left(1-e^{-\frac{\eta}{\varepsilon}}\right)}{1-e^{-\frac{1}{\varepsilon}}} + \delta_1\eta$$

which is in agreement with the theoretical bounds described above. Below is the graph of the exact solution for $\varepsilon = 0.1,\ 0.05,\ 0.02,\ 0.002$ values.

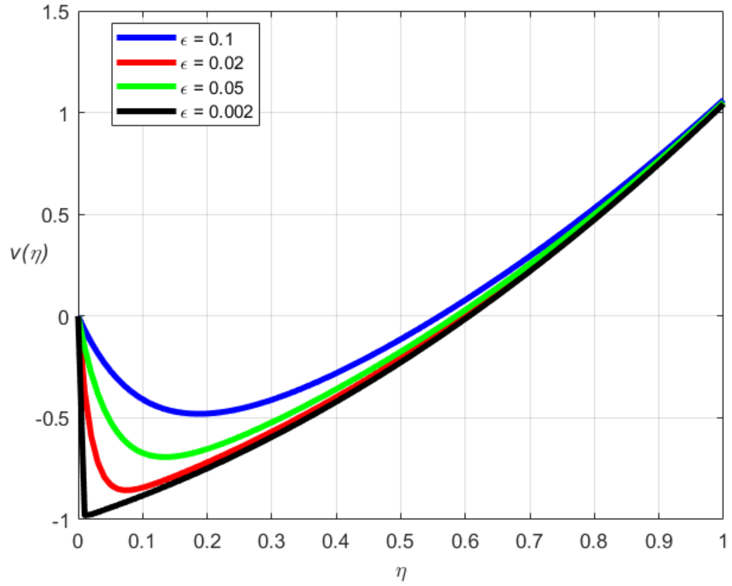

Figure 1. Solution of example for different values ε

CONCLUSION

In this work, a boundary value problem for FIDEs with a boundary layer has been taken considered. It has been determined how the solution behaves in the boundary layer and how its first and second derivatives behave. These estimates are important because they contribute to the development and evaluation of appropriate approximation methods in mathematical modelling and analysis. By understanding the details of the behaviour of the solution and its derivatives, researchers can design more accurate and efficient numerical methods to solve similar problems in applied mathematics. An example is given that is consistent with the theoretical analysis and demonstrates the practical applicability of the theoretical results. Future work could extend these results to more complex systems and explore the effect of different types of boundary conditions on the solution and its derivatives.

Conflict of Interest

The author of the article declares that he has no conflict of interest.

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