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On Solution of Complex Equations with the Homotopy Perturbation Method

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Abstract

In this study, complex differential equations are solved using the homotopy perturbation method (HPM). Firstly, we separated the real and imaginary parts of the equation. Thus, from one unknown equation is obtained two unknown equation systems. Later, we applied HPM real and imaginary to parts of the equation. So, we have obtained real and imaginary parts of the solution.

Keywords: Homotopy perturbation method; Complex equations; Equation system.

Homotopi Pertürbasyon Metodu ile Kompleks Denklemlerin Çözümü Üzerine

Öz

Bu çalışmada kompleks denklemler homotopi pertürbasyon metodu(HPM) kullanılarak çözüldü. İlk önce denklemi reel ve imajiner kısımlarına ayırdık. Böylece bir bilinmeyenli

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denklemden iki bilnmeyenli denklem sistemi elde edildi. Daha sonra HPM'yi denklemin reel ve imajiner kısımlarına uyguladık. Böylece çözümün reel ve imajiner kısımlarını elde ettik.

Anahtar Kelimeler: Homotopi pertürbasyon metodu; Kompleks denklemler; Denklem sistemi.

1. Introduction

General solutions of some equations in real space can't be found. For example,

$$U_{xx} + U_{yy} = 0 \tag{1}$$

Laplace equation hasn't got the general solution in R^2. But

$$U_{z\bar{z}} = 0 \tag{2}$$

which is equivalent to (1) Laplace equation has got the general solution in complex space, and the solution of (2) is

$$U = f(z) + g(\bar{z}) \tag{3}$$

The most basic works in the theory of complex differential equations are "Theory of Pseudo Analytic Functions" which is written by L. Bers [1] and "Generalized Analytic Functions" which is written by I, N. Vekua [2]. There are studies related to the solution of complex differential equations. For example, complex differential equations of Lane Emden type have been solved by the differential transformation method [3]. In addition, there are studies related to approximate solutions of linear complex differential equations [4-5].

In this study, we solved the first-order constant coeffients complex differential equations with HPM. Such equations have previously been solved by integral transformations such as laplace, fourier, elzaki, and a relation for the solution has been obtained [6-8]. In addition, such equations were solved using the variational iteration method and an iteration relation was obtained [9]. In addition, linear complex differential equations with variable coefficients from the first order were solved by the adomian decomposition method and an iteration relation was obtained [10].

Many methods can be used to solve linear equations, but some of them can't be applied to the solutions of the most nonlinear equations. This situation has led us to approximate methods of finding solutions for nonlinear equations. Various numerical and analytical approach methods have been developed for such equations. Some of them are the Variational Iteration Method, Adomain Decomposition Method, Homotopy Perturbation Method, and Differential Transform Method. HPM is a mathematical method used to solve linear and nonlinear equations. The HPM, first proposed by Dr. Ji Huan He, has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions for a wide variety of problems arising in different fields. This method which is used in the solution of many linear and nonlinear ordinary and partial differential equations used in the fields of engineering, physics, applied mathematics, and quantum mechanics [11-17]. In addition, this method is also used in the field of environmental events and health [18-20]. In article [21], it has been shown that the Taylor series expansion is the same as the HPM with particular choices of the auxiliary parameters.

This paper has been organized as follows. In Section 2.1, we have mentioned the homotopy perturbation method. In Section 2.2, we have given a theorem associated with the solution of complex equations by using HPM. Moreover, we have provided some examples of the validity of the method.

2. Materials and Methods

2.1. Homotopy Perturbation Method

To illustrate the basic ideas of the method, we consider the following nonlinear differential equation

$$Au + f(r) = 0, \quad (4)$$

$$r \in \Omega$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function. The operator A can, generally speaking, be divided into two parts L and N, where L is linear, while N is nonlinear. Therefore, Eq. (4) can be rewritten as follows:

$$Lu + Nu - f(r) = 0 \quad (5)$$

The linear term is decomposed into $L+R$, where L is the highest order differential operator and R is the remainder of the linear operator. Thus the equation can be written

$$Ly + Ry + Ny = g(x) \tag{6}$$

where Ny represents the nonlinear terms. The equation type which we have investigated is linear, since there is not nonlinear term $Ny = 0$. Therefore, Eq. (6) can be written as:

$$Ly + Ry = g(x). \tag{7}$$

Let we apply HPM to the Eq. (7).

$$Ly = g(x) - p.Ry \tag{8}$$

where p is artificial parameter. The artificial parameter method assumes that the approximation of Eq. (8) can be expressed as a series of the power of p, i.e.

$$y = y_0 + py_1 + p^2 y_2 + \cdots \tag{9}$$

When $p \to 1$, Equality (9) corresponds to the approximate solution of Eq. (7).

2.2. Solution of Complex Equations with HPM

In this section, we will study the solution of complex equations from the first order with HPM. Firstly, let us give complex derivatives equality from kind real derivatives.

Definition 1: Let $w = w(z,\bar{z})$ be a complex function. Here $z = x + iy$, $w(z,\bar{z}) = u(x,y) + iv(x,y)$. First order derivatives according to z and $\bar{z}$ of $w(z,\bar{z})$ are defined as follows:

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right), \frac{\partial w}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) \tag{10}$$

Theorem 1: Let A, B, C are real constants, $F(z,\bar{z})$ is a polynomial of $z, \bar{z}$ and $w = u + iv$ is a complex function. Then the solution of

$$A\frac{\partial w}{\partial z} + B\frac{\partial w}{\partial \bar{z}} + Cw = F(z,\bar{z}) \tag{11}$$

$$w(x,0) = f(x) \tag{12}$$

is given as:

$$w(z,\bar{z}) = u(x,y) + iv(x,y) = \sum_{n=0}^{\infty}\big(u_n(x,y) + iv_n(x,y)\big)$$

$$u_{n+1} = \int\left(\frac{(A+B)\dfrac{\partial v_n}{\partial x} + 2Cv_n}{A-B}\right)dy$$

$$v_{n+1} = \int\left(\frac{(A+B)\dfrac{\partial u_n}{\partial x} + 2Cu_n}{B-A}\right)dy$$

$$u_0 = f(x), v_0 = 0$$

Proof: We can obtain the following equality in (11) using (10) equality.

$$A\frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) + B\frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right) + Cw = F(z,\bar{z}). \tag{13}$$

If we write $w = u + iv$ in (13), then following equality is obtained.

$$A\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} - i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right] + B\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right] + 2C(u+iv)$$
$$= 2F_1(x,y) + 2iF_2(x,y)$$

If we seperate (13) as real and imaginer parts, then following equation system is obtained

$$(A+B)\frac{\partial u}{\partial x} + (A-B)\frac{\partial v}{\partial y} + 2Cu = 2F_1(x,y)$$

$$(A+B)\frac{\partial v}{\partial x} + (B-A)\frac{\partial u}{\partial y} + 2Cv = 2F_2(x,y)$$

$$\frac{\partial v}{\partial y} = \frac{2F_1(x,y) - (A+B)\dfrac{\partial u}{\partial x} - 2Cu}{A-B}$$

$$\frac{\partial u}{\partial y} = \frac{2F_2(x,y) - (A+B)\dfrac{\partial v}{\partial x} - 2Cv}{B-A}$$

Now let's apply the homotopy perturbation technique.

$$\frac{\partial v}{\partial y}=\frac{2F_1(x,y)}{A-B}-p\frac{(A+B)\frac{\partial u}{\partial x}+2Cu}{A-B}$$

$$\frac{\partial u}{\partial y}=\frac{2F_2(x,y)}{B-A}-p\frac{(A+B)\frac{\partial v}{\partial x}+2Cv}{B-A}$$

where $u=u_0+pu_1+p^2u_2+\cdots$ and $v=v_0+pv_1+p^2v_2+\cdots$

In these equalities, we equal to coeffients of the same powers of p then we get the following equalities.

$$u_0=\int\frac{2F_2(x,y)}{B-A}dy+f(x)$$

$$v_0=\int\frac{2F_1(x,y)}{A-B}dy$$

$$\frac{\partial v_{n+1}}{\partial y}=\frac{(A+B)\frac{\partial u_n}{\partial x}+2Cu_n}{B-A}$$

$$\frac{\partial u_{n+1}}{\partial y}=\frac{(A+B)\frac{\partial v_n}{\partial x}+2Cv_n}{A-B}$$

$$\frac{\partial u_1}{\partial y}=p\left[\frac{(A+B)\frac{\partial v_0}{\partial x}+2Cv_0}{A-B}\right]$$

$$\frac{\partial v_1}{\partial y}=p\left[\frac{(A+B)\frac{\partial u_0}{\partial x}+2Cu_0}{B-A}\right]$$

$$\frac{\partial u_2}{\partial y}=p\left[\frac{(A+B)\frac{\partial v_1}{\partial x}+2Cv_1}{A-B}\right]$$

$$\frac{\partial v_2}{\partial y}=p\left[\frac{(A+B)\frac{\partial u_1}{\partial x}+2Cu_1}{B-A}\right]$$

$$\vdots$$

$$\frac{\partial u}{\partial y}=\frac{2F_2(x,y)}{B-A}+p\left[\frac{(A+B)\frac{\partial v_0}{\partial x}+2Cv_0}{A-B}\right]+p\left[\frac{(A+B)\frac{\partial v_1}{\partial x}+2Cv_1}{A-B}\right]+\cdots$$

$$=\frac{2F_2(x,y)}{B-A}+p\sum_{n=0}^{\infty}\frac{(A+B)\frac{\partial v_n}{\partial x}+2Cv_n}{A-B}$$

$$\frac{\partial v}{\partial y}=\frac{2F_1(x,y)}{A-B}+p\left[\frac{(A+B)\frac{\partial u_0}{\partial x}+2Cu_0}{B-A}\right]+p\left[\frac{(A+B)\frac{\partial u_1}{\partial x}+2Cu_1}{B-A}\right]+\cdots$$

$$=\frac{2F_1(x,y)}{A-B}+p\sum_{n=0}^{\infty}\frac{(A+B)\frac{\partial u_n}{\partial x}+2Cu_n}{B-A}$$

$$u(x,y)=\int\left[\frac{2F_2(x,y)}{B-A}+p\sum_{n=0}^{\infty}\frac{(A+B)\frac{\partial v_n}{\partial x}+2Cv_n}{A-B}\right]dy+f(x)$$

$$v(x,y)=\int\left[\frac{2F_1(x,y)}{A-B}+p\sum_{n=0}^{\infty}\frac{(A+B)\frac{\partial u_n}{\partial x}+2Cu_n}{B-A}\right]dy$$

Example 1: [10] Solve the following problem

$$4\frac{\partial w}{\partial z}+\frac{\partial w}{\partial \bar{z}}=0$$

with the condition $w(x,0)=-\frac{1}{3x}$.

Solution: Clearly the coefficients of equation which are as follows

$$A = 4, B = 1, C = 0, F = 0$$

$$u_0=-\frac{1}{3x}, v_0=0$$

$$u_1=0, \frac{\partial v_1}{\partial y}=-\frac{5}{3}p\frac{1}{3x^2}, v_1=-\frac{5yp}{9x^2}$$

$$v_2=0, \frac{\partial u_2}{\partial y}=\frac{5}{3}p\left(\frac{10yp}{9x^3}\right), u_2=\frac{25y^2p^2}{27x^3}$$

$$u_3 = 0, v_3 = \frac{125p^3y^3}{81x^4}$$

$$\vdots$$

$$v_{2n} = 0, u_{2n} = \frac{(-1)^{n-1}(5yp)^{2n}}{(3x)^{2n+1}}, v_{2n+1} = \frac{(-1)^{n-1}(5yp)^{2n+1}}{(3x)^{2n+1}}$$

$$w(z,\bar{z}) = \lim_{p\to 1}\sum_{n=0}^{\infty}(u_n + iv_n) = \lim_{p\to 1}(u_0 + iv_0 + u_1 + iv_1 + u_2 + iv_2 + \cdots)$$

$$= -\frac{1}{3x} - \frac{5iy}{9x^2} + \frac{25y^2}{27x^3} + \frac{125iy^3}{81x^4} + \cdots = \frac{-\frac{1}{3x}}{1-\frac{5iy}{3x}} = \frac{1}{5iy-3x} = \frac{1}{z-4\bar{z}}$$

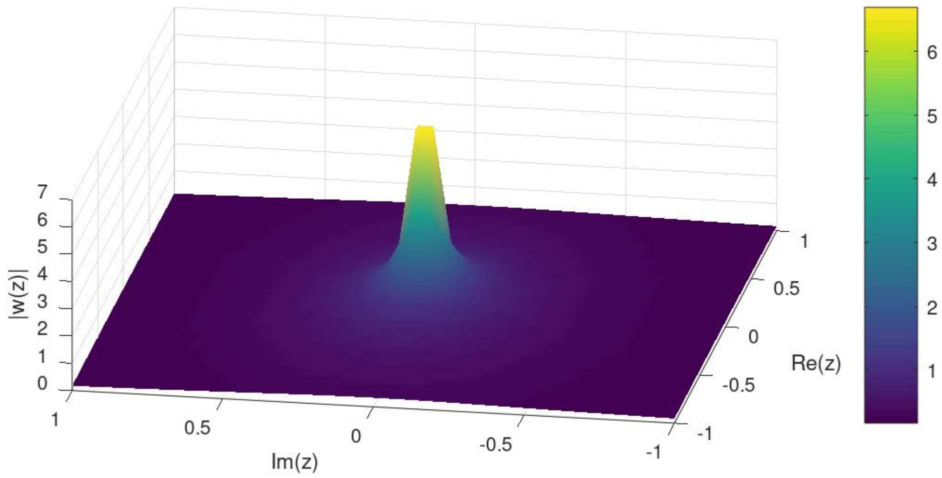

Figure 1: Graph of the absolute value of $w(z,\bar{z}) = \frac{1}{z-4\bar{z}}$ when the real and imaginary parts of z are in the range [-1,1].

Example 2: [6] Solve the following problem

$$2\frac{\partial w}{\partial z} - \frac{\partial w}{\partial \bar{z}} = 4z + 1$$

with the condition

$$w(x,0) = x^2 + 5x.$$

Solution:

The coefficients of equation which are $A = 2, B = -1, C = 0$ and $F(z,\bar{z}) = 4z + 1 = 4x + 1 + 4iy$

$$u_0 = x^2 + 5x + \int \frac{8y}{-3} dy = x^2 + 5x - \frac{4y^2}{3}, v_0 = \int \frac{8x+2}{3} dy = \frac{8xy + 2y}{3}$$

$$u_1 = p\frac{4y^2}{9}, v_1 = -\frac{1}{3}p(2xy + 5y)$$

$$u_2 = -\frac{1}{9}p^2y^2, v_2 = 0$$

$$u_n = v_n = 0 \ (n \geq 3)$$

$$w(z,\bar{z}) = \lim_{p\to 1}\sum_{n=0}^{\infty}(u_n + iv_n) = \lim_{p\to 1}(u_0 + iv_0 + u_1 + iv_1 + u_2 + iv_2 + \cdots)$$

$$= x^2 + 5x - \frac{4y^2}{3} + i\frac{8xy + 2y}{3} + \frac{4y^2}{9} - \frac{1}{3}i(2xy + 5y) - \frac{1}{9}y^2$$

$$= x^2 - y^2 + 2ixy + 3(x - iy) + 2(x + iy) = z^2 + 3\bar{z} + 2z.$$

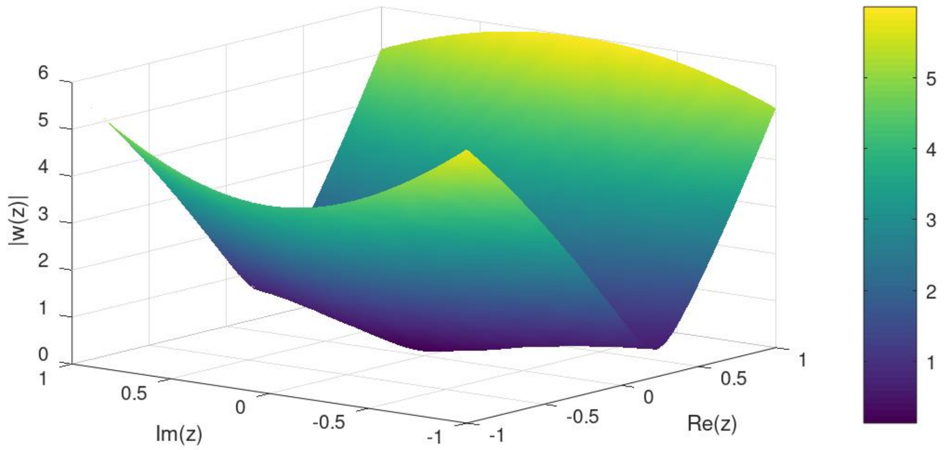

Figure 2: Graph of the absolute value of $w(z,\bar{z}) = z^2 + 3\bar{z} + 2z$ when the real and imaginary parts of z are in the range [-1,1].

3. Results and Discussion

In this study, we have studied solutions of first order with constant coefficients complex partial differential equations by using HPM. We know that such equations can be solved by the lagrange system used for semi-linear equations from the first order. Our aim is that an alternative approach solution is obtained for such equations by using HPM. We applied the iteration relation on two samples. In the first example, an infinite geometric series was obtained, and the solution was reached. In the second example, it was seen that all terms after the fourth iteration were zero and thus a solution in the form of polynomials. The results obtained are consistent with the literature.

4. Conclusion

In this study, it is thought that the iteration relationship obtained for the solution of first-order complex equations with constant coefficients can also be obtained for higher-order equations with constant and variable coefficients. We know that HPM is applied for nonlinear equations. We think that this study can also be applied to non-linear complex equations.

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