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Değişmeli Cebirlerin Çaprazlanmış Kareleri için İzomorfizm Teoremleri

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Anahtar Kelimeler
İzomorfizm teoremi,
Alt çaprazlanmış kare,
Çaprazlanmış kare ideali.

Özet: Cebirlerin çaprazlanmış modüllerine ek bir boyut daha eklendiğinde ortaya çıkan değişmeli cebirlerin çaprazlanmış kareleri için izomorfizm teoremleri bu makalenin ana konusunu teşkil etmektedir. Bu bağlamda, çapraz kare ideal, görüntü ve bölüm çapraz kareleri gibi kavramların yanı sıra çaprazlanmış kare morfizmleri için çekirdek kavramını da kapsayan değişmeli cebirlerin çaprazlanmış karelerinin tanımı verilmiştir. Çalışma, izomorfizm teoremlerinin bu yapılara nasıl uygulandığını tartışmakta ve bu çerçeve için ayrıntılı kanıtlar sunmaktadır. Ayrıca, daha önce bu yapılarda tanımlanmamış olan bölüm çaprazlanmış kareleri gibi bazı gerekli kavramlar da sunulmakta ve bunların bazı temel özellikleri incelenmektedir. Bu çalışma, çaprazlanmış n-küpler de dahil olmak üzere bir dizi farklı yapıya olası genelleştirme fırsatları sunmaktadır.

Isomorphism Theorems for Crossed Squares of Commutative Algebras

Keywords
Isomorphism theorem,
Subcrossed square,
Crossed square ideal.

Abstract: The isomorphism theorems for crossed squares of commutative algebras, which arise when the crossed modules of algebras are given an extra dimension, are the main subject of this paper. The definition of crossed squares of commutative algebras is given in this context, encompassing ideas like the crossed square ideal, image, and quotient crossed squares, as well as the kernel for crossed square morphisms. The study discusses the way how isomorphism theorems are applied to these structures and offers detailed proofs for this framework. Moreover, some necessary concepts such as quotient crossed squares, which were not previously specified in these structures, are also presented, and some basic properties are examined. The study provides opportunities for possible generalization to a number of different structures, including crossed n-cubes.

1. Introduction

Historically, the concept of algebra and its properties have formed the foundation of mathematics. Algebras are rich algebraic structures that roughly combine two ring structures together and are among the most important structures in algebra, so much so that they are synonymous with the name of the field itself. Through years of research. it has been discovered that many structures existing in group and ring theories can also be transferred to algebras.

One of the most fundamental problems in mathematics, specifically in the field of algebra, is determining whether two structures are identical. The most straightforward way to perform this verification in algebra is by using isomorphisms. However, the answer to whether there exists an isomorphism between two algebraic structures is not always obvious.

Isomorphism theorems, which provide a reference pattern for identifying many pairs of isomorphic algebraic structures, offer an extremely useful approach to this fundamental problem in algebra and thus have numerous applications. In this regard, investigating

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whether isomorphism theorems hold for different algebraic structures constitutes a fundamental and crucial step in thoroughly understanding the related algebraic structure.

When examining the crossed modules of commutative algebras structurally, it is evident that this area continues to develop significantly. The concept of a crossed module was first defined for groups by J.H.C. Whitehead [1] in his work on homotopy groups. Since then, the concept of a crossed module has contributed significantly to various fields. M. Gerstenhaber [2] and also M. Schlessinger and S. Lichtenbaum [3] conducted different studies on the crossed modules of associative and commutative algebras. T. Lue [4] adapted the concept of semi-completeness existing for groups to the crossed module of groups. G.J. Ellis [5] further developed the concept of a crossed module dimensionally by defining crossed n-cubes for n = 1,2,3 in the group category. T. Porter [6] examined the category of crossed modules for commutative algebras in his research. K.J. Norrie [7] transferred many existing theorems and results for groups to the crossed modules of groups. Additionally, G.J. Ellis [8] provided the definition of a crossed square for commutative algebras. N.M. Shammu [9] studied the crossed modules of associative algebras algebraically and categorically, providing the definition of a crossed square for associative algebras. Later, Z. Arvasi [10], U. Ege and H.G. Akay [11] obtained significant findings on crossed modules in commutative algebras in their studies. For more information on crossed modules and crossed squares, the reader can refer to [12-22].

2. Material and Method

In this subsection, the definition and square homomorphism of crossed squares of commutative algebras are expressed.

2.1. Crossed squares of algebras

The definition of crossed squares of commutative algebras will be given below. For this, we first recall that the action of a $\mathbb{K}-$ algebra R on a $\mathbb{K}-$ algebra C is a function $R \times C \to C$ that satisfies the conditions

$$r \cdot (c_1 c_2) = (r \cdot c_1)c_2 = c_1(r \cdot c_2)$$
$$(r_1 r_2) \cdot c = r_1 \cdot (r_2 \cdot c)$$

for all $r, r_1, r_2 \in R$ and $c, c_1, c_2 \in C$. Moreover, if R is unital, it is also required that $1_R \cdot c = c$ for all $c \in C$. We also recall that a crossed module of commutative $\mathbb{K}-$ algebras is a structure of the form (C, R, ∂), where R acts on C, $\partial : C \to R$ is a $\mathbb{K}-$ algebra homomorphism and the conditions

(CM1) $\partial(r \cdot c) = r\partial(c)$

(CM2) $\partial(c_1) \cdot c_2 = c_1 c_2$

are satisfied for all $r \in R$ and $c, c_1, c_2 \in C$.

Definition 2.1.1 Let R be a unitary $\mathbb{K}-$ algebra, L, M and N be R −algebras. Given a following commutative diagram

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mathrm{v}"'] & R \end{tikzcd}$$

with a function $h: M \times N \longrightarrow L$ since L, M and N are $R-$ algebras, R acts on them. Morever, M acts on L and N through μ such that $\xi \cdot \varepsilon = \mu(\xi) \cdot \varepsilon$ and $\xi \cdot \varsigma = \mu(\xi) \cdot \varsigma$ for all $\varsigma \in N$, $\xi \in M$ and $\varepsilon \in L$ while N acts on L and M through v such that $\varsigma \cdot \varepsilon = \mathrm{v}(\varsigma) \cdot \varepsilon$ and $\varsigma \cdot \xi = \mathrm{v}(\varsigma) \cdot \xi$.

If for all $k \in \mathbb{K}, \varepsilon \in L$, $\xi, \xi' \in M$ $\varsigma, \varsigma' \in N$ and $r \in R$;

i) (L, M, λ), (L, N, λ'), (M, R, μ), (N, R, v) and $(L, R, \mu\lambda)$ are crossed modules of algebras.

ii) for all $\varepsilon \in L$ and $r \in R$;
$\lambda(r \cdot \varepsilon) = r \cdot \lambda(\varepsilon)$
$\lambda'(r \cdot \varepsilon) = r \cdot \lambda'(\varepsilon)$

iii) for all $k \in \mathbb{K}$, $\xi \in M$, $\varsigma \in N$;
$kh(\xi, \varsigma) = h(k\xi, \varsigma) = h(\xi, k\varsigma)$

iv) for all $\xi, \xi' \in M$, $\varsigma, \varsigma' \in N$;
$h(\xi + \xi', \varsigma) = h(\xi, \varsigma) + h(\xi', \varsigma)$
$h(\xi, \varsigma + \varsigma') = h(\xi, \varsigma) + h(\xi, \varsigma')$

v) for all $r \in R$, $\xi \in M$ and $\varsigma \in N$;
$r \cdot h(\xi, \varsigma) = h(r \cdot \xi, \varsigma) = h(\xi, r \cdot \varsigma)$

vi) for all $\xi \in M$ and $\varsigma \in N$;
$\lambda h(\xi, \varsigma) = \varsigma \cdot \xi$
$\lambda' h(\xi, \varsigma) = \xi \cdot \varsigma$

vii) for all $\varepsilon \in L$, $\xi \in M$ and $\varsigma \in N$;
$h(\lambda\varepsilon, \varsigma) = \varsigma \cdot \varepsilon$
$h(\xi, \lambda'\varepsilon) = \xi \cdot \varepsilon$

Then, following such a structures is called

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mathrm{v}"'] & R \end{tikzcd}$$

crossed square of commutative algebras.

Definition 2.1.2 Let there be a crossed square

$$\begin{tikzcd} L_1 \arrow[r, "\lambda_1"] \arrow[d, "\lambda_1'"'] & M_1 \arrow[d, "\mu_1"] \\ N_1 \arrow[r, "\nu_1"'] & R_1 \end{tikzcd}$$

$\mathcal{C}_1$ and a crossed square

$$\begin{tikzcd} L_2 \arrow[r, "\lambda_2"] \arrow[d, "\lambda_2'"'] & M_2 \arrow[d, "\mu_2"] \\ N_2 \arrow[r, "\nu_2"'] & R_2 \end{tikzcd}$$

$\mathcal{C}_2$. A crossed square homomorphism $\boldsymbol{\phi}: \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a quadruple $\boldsymbol{\phi} = (\alpha, \beta, \gamma, \delta)$ such that the following

$$\begin{tikzcd} & L_2 \arrow[rr, "\lambda_2"] \arrow[dd, "\lambda_2'"'] & & M_2 \arrow[dd, "\mu_2"] \\ L_1 \arrow[ur, "\alpha"] \arrow[rr, "\lambda_1"] \arrow[dd, "\lambda_1'"'] & & M_1 \arrow[ur, "\beta"] \arrow[dd, "\mu_1"] & \\ & N_2 \arrow[rr, "\nu_2"] & & R_2 \\ N_1 \arrow[ur, "\gamma"] \arrow[rr, "\nu_1"'] & & R_1 \arrow[ur, "\delta"'] & \end{tikzcd}$$

diagram commutes and the conditions are satisfied:
for all $\varepsilon_1 \in L_1$, $\xi_1 \in M_1, \varsigma_1 \in N_1$ and $r_1 \in R_1$
$\alpha(r_1 \cdot \varepsilon_1) = \delta(r_1) \cdot \alpha(\varepsilon_1)$,
$\beta(r_1 \cdot \xi_1) = \delta(r_1) \cdot \beta(\xi_1)$,
$\gamma(r_1 \cdot \varsigma_1) = \delta(r_1) \cdot \gamma(\varsigma_1)$
$\alpha\big(h_1(\xi_1, \varsigma_1)\big) = h_2\big(\beta(\xi_1), \gamma(\varsigma_1)\big)$.

Remark 2.1.3 Due to the commutativity of the diagram and the preservation of the actions, (α, β), (α, γ), (β, δ), (γ, δ) and (α, δ) are crossed module homomorphisms. For more information on crossed module homomorphisms, refer to [23].

The following example demonstrates that crossed modules of algebras can be considered as an extension of the concept of an ideal.

Example 2.1.4 Let R be a $\mathbb{K}-$algebra and I and J be ideals of R. Define $\mu, \nu, \lambda: I \cap J \longrightarrow I$ and $\lambda': I \cap J \longrightarrow J$ as inclision maps. For all $i \in I$ and $j \in J$, define $h: I \times J \longrightarrow I \cap J$, $h(i,j) = ij$. Then $(R, I, J, I \cap J)$

$$\begin{tikzcd} I \cap J \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & I \arrow[d, "\mu"] \\ J \arrow[r, "\nu"'] & R \end{tikzcd}$$

is a crossed square of algebras, where R acts on I, J and $I \cap J$ by multiplication. The conditions for being a crossed square are obviously satisfied in this case.

2.2. Subcrossed squares

In this subsection, the definition of sub-crossed squares of commutative algebras is presented.

Definition 2.2.1 Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

be a crossed square.
If $A \leq L, B \leq M, C \leq N, D \leq R$ such that the structure

$$\mathcal{A} = \begin{tikzcd} A \arrow[r, "\lambda|"] \arrow[d, "\lambda'|"] & B \arrow[d, "\mu|"] \\ C \arrow[r, "\nu|"'] & D \end{tikzcd}$$

λ, λ', μ, ν and h under the restriction maps is a crossed square, then $\mathcal{A}$ is called a sub-crossed square of $\mathcal{L}$. This situation is denoted by $\mathcal{A} \leq \mathcal{L}$ through the paper.

Subcossed squares naturally serve as subobjects in the category of crossed squares and crossed square homomorphisms.

3. Results

3.1. Ideal of crossed squares of algebras

In this subsection, the definition and a theorem related to the ideal of crossed squares of commutative algebras are presented.

Definition 3.1.1 Given a crossed square

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mathrm{v}"'] & R \end{tikzcd}$$

of a commutative algebras and let

$$\mathcal{E} = \begin{tikzcd} E \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & I \arrow[d, "\mu"] \\ J \arrow[r, "\mathrm{v}"'] & K \end{tikzcd}$$

be a sub-crossed square of $\mathcal{L}$. If for all $r \in R,\ \rho \in E,\ \varrho \in I,\ \sigma \in J$

i) K be a ideal of R,

ii) If for all $r \in R,\ \rho \in E,\ \varrho \in I,\ \sigma \in J$
$r \cdot \rho \in E$
$r \cdot \varrho \in I$
$r \cdot \sigma \in J$

iii) And for all $d \in K,\ \varepsilon \in L,\ \xi \in M,\ \varsigma \in N$
$d \cdot \varepsilon \in E$
$d \cdot \xi \in I$
$d \cdot \varsigma \in J$

iv) And for all $\xi \in M,\ j \in J$,
$h(\xi, j) \in E$

v) And for all $i \in I, \varsigma \in N$,
$h(i, \varsigma) \in E$

Then $\mathcal{E}$ is called an crossed square ideal of $\mathcal{L}$ or simply ideal. This situation is denoted by $\mathcal{E} \trianglelefteq \mathcal{L}$ throughout the paper.

Theorem 3.1.2 Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mathrm{v}"'] & R \end{tikzcd}$$

be a crossed square and

$$\mathcal{E} = \begin{tikzcd} E \arrow[r] \arrow[d] & I \arrow[d] \\ J \arrow[r] & K \end{tikzcd}$$

an ideal of $\mathcal{L}$. Then

a) $(E, I, \lambda) \trianglelefteq (L, M, \lambda)$
b) $(E, J, \lambda') \trianglelefteq (L, N, \lambda')$
c) $(I, K, \mu) \trianglelefteq (M, R, \mu)$
d) $(J, K, \mathrm{v}) \trianglelefteq (N, R, \mathrm{v})$
e) $(E, K, \mu\lambda) \trianglelefteq (L, R, \mu\lambda)$.

Proof:

a) Since $\mathcal{E} \leq \mathcal{L}$, (E, I, λ) is a crossed module and $E \leq L$, $I \leq M$ so $(E, I, \lambda) \leq (L, M, \lambda)$. For all $\rho \in E,\ \varrho \in I, \varepsilon \in L,\ \xi \in M$:
(i) $\xi\varrho = \mu(\xi) \cdot \varrho \in I$
(ii) $\xi \cdot \rho = \mu(\xi) \cdot \rho \in E$
(iii) $\varrho \cdot \varepsilon = \mu(\varrho) \cdot \varepsilon \in E$
Thus, $(E, I, \lambda) \trianglelefteq (L, M, \lambda)$.

b) Since $\mathcal{E} \leq \mathcal{L}$, (E, J, λ') is a crossed module and $E \leq L, J \leq N$, so $(E, J, \lambda') \leq (L, N, \lambda')$. For all $\rho \in E, \sigma \in J, \varepsilon \in L,\ \varsigma \in N$:
(i) $\varsigma\sigma = \mathrm{v}(\varsigma) \cdot \sigma \in J$
(ii) $\varsigma \cdot \rho = \mathrm{v}(\varsigma) \cdot \rho \in E$
(iii) $\sigma \cdot \varepsilon = \mathrm{v}(\sigma) \cdot \varepsilon \in E$
Thus, $(E, J, \lambda') \trianglelefteq (L, N, \lambda')$.

c) Since $\mathcal{E} \leq \mathcal{L}$, (I, K, μ) is a crossed module and $I \leq M$, $K \leq R$, so $(I, K, \mu) \leq (M, R, \mu)$. For all $\varrho \in I,\ k \in K, \xi \in M,\ r \in R$:
(i) $rk \in K$
(ii) $r \cdot \varrho \in I$
(iii) $k \cdot \xi \in I$
Thus, $(I, K, \mu) \trianglelefteq (M, R, \mu)$.

d) Since $\mathcal{E} \leq \mathcal{L}$, (J, K, v) is a crossed module and $J \leq N, K \leq R$, so $(J, K, \mathrm{v}) \leq (N, R, \mathrm{v})$. For all $\sigma \in J,\ k \in K, \varsigma \in N, r \in R$:
(i) $rk \in K$
(ii) $r \cdot \sigma \in J$
(iii) $k \cdot \varsigma \in J$
Thus $(J, K, \mathrm{v}) \trianglelefteq (N, R, \mathrm{v})$.

e) Since $\mathcal{E} \leq \mathcal{L}$, $(E, K, \mu\lambda)$ is a crossed module and $E \leq L$, $K \leq R$, so $(E, K, \mu\lambda) \leq (L, R, \mu\lambda)$. For all $\rho \in E, k \in K,\ \varepsilon \in L,\ r \in R$:
(i) $rk \in K$
(ii) $r \cdot \rho \in E$
(iii) $k \cdot \varepsilon \in E$
Thus $(E, K, \mu\lambda) \trianglelefteq (L, R, \mu\lambda)$. ■

3.2. Kernel and image of morphisms of crossed squares of algebras

In this subsection information about the kernel of image of morphisms of crossed squares of commutative algebras is provided.

Theorem 3.2.1 Let be the following two crossed squares of commutative algebras.

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

$$\mathcal{L}^* = \begin{tikzcd} L^* \arrow[r, "\lambda^*"] \arrow[d, "\lambda'^*"] & M^* \arrow[d, "\mu^*"] \\ N^* \arrow[r, "\nu^*"'] & R^* \end{tikzcd}$$

If $\boldsymbol{\phi} = (\alpha, \beta, \gamma, \delta): \mathcal{L} \longrightarrow \mathcal{L}^*$ is a crossed square homomorphism between these squares, then

$$ker\boldsymbol{\phi} := \begin{tikzcd} ker\alpha \arrow[r] \arrow[d] & ker\beta \arrow[d] \\ ker\gamma \arrow[r] & ker\delta \end{tikzcd}$$

$\ker \boldsymbol{\phi} \trianglelefteq \mathcal{L}$.

Proof: First, let's Show that $ker\boldsymbol{\phi}$ is a sub-crossed square of $\mathcal{L}$. By Proposition 3.4.27 in [23], we get

$(\ker\alpha, \ker\beta, \lambda) \leq (L, M, \lambda)$
$(\ker\alpha, \ker\gamma, \lambda') \leq (L, M, \lambda')$
$(\ker\beta, \ker\delta, \mu) \leq (M, R, \mu)$
$(\ker\gamma, \ker\delta, \nu) \leq (N, R, \nu)$
$(\ker\alpha, \ker\delta, \mu\lambda) \leq (L, R, \mu\lambda)$.

Additionally, if $\varrho \in \ker\beta, \sigma \in \ker\gamma$, then $\alpha\big(h(\varrho, \sigma)\big) = h^*\big(\beta(\varrho), \gamma(\sigma)\big) = h^*(0,0)$

and

$h^*(0,0) = h^*(0+0,0) = 2h^*(0,0)$.

Therefore, $h^*(0,0) = 0_{L^*}$ so $h(\varrho, \sigma) \in \ker\alpha$. Thus, $\ker\boldsymbol{\phi} \leq \mathcal{L}$.

Now, let's show that $\ker\boldsymbol{\phi}$ is an ideal.

i) By Proposition 2.19 (i) in [23], $\ker\delta \trianglelefteq R$.

ii) For all $r \in R$, $\rho \in ker\,\alpha$, $\varrho \in ker\,\beta$, $\sigma \in ker\,\gamma$:

$\alpha(r \cdot \rho) = \delta(r) \cdot \alpha(\rho) = \delta(r) \cdot 0 = 0$, so $r \cdot \rho \in \ker\alpha$.
$\beta(r \cdot \varrho) = \delta(r) \cdot \beta(\varrho) = \delta(r) \cdot 0 = 0$, so $r \cdot \varrho \in \ker\beta$.
$\gamma(r \cdot \sigma) = \delta(r) \cdot \gamma(\sigma) = \delta(r) \cdot 0 = 0$, so $r \cdot \sigma \in \ker\gamma$.

iii) For all $d \in \ker\delta$, $\varepsilon \in L, \xi \in M$ ve $\varsigma \in N$:

$\alpha(d \cdot \varepsilon) = \delta(d) \cdot \alpha(\varepsilon) = 0 \cdot \alpha(\varepsilon) = 0$,
$\beta(d \cdot \xi) = \delta(d) \cdot \beta(\xi) = 0 \cdot \beta(\xi) = 0$
and
$\gamma(d \cdot \varsigma) = \delta(d) \cdot \gamma(\varsigma) = 0 \cdot \gamma(\varsigma) = 0$

Therefore, $d \cdot \rho \in \ker\alpha$, $d \cdot \varrho \in \ker\beta$ and $d \cdot \sigma \in \ker\gamma$.

iv) For all $\xi \in M, \sigma \in \ker\gamma$:

$$\alpha\big(h(\xi, \sigma)\big) = h^*\big(\beta(\xi), \gamma(\sigma)\big) = h^*(\beta(\xi), 0)$$

and

$$h^*(\beta(\xi), 0) = h^*(\beta(\xi), 0+0) = h^*(\beta(\xi), 0) + h^*(\beta(\xi), 0),$$

then $h^*(\beta(\xi), 0) = 0_{L^*}$. So $h(\xi, \sigma) \in \ker\alpha$.

v) For all $\varrho \in \ker\beta$, $\varsigma \in N$: $\alpha\big(h(\varrho, \varsigma)\big) = h\big(\beta(\varrho), \gamma(\varsigma)\big) = h\big(0, \gamma(\varsigma)\big) = 0$

Thus $h(\varrho, \varsigma) \in \ker\alpha$. ■

Theorem 3.2.2 Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

$$\mathcal{L}^* = \begin{tikzcd} L^* \arrow[r, "\lambda^*"] \arrow[d, "\lambda'^*"] & M^* \arrow[d, "\mu^*"] \\ N^* \arrow[r, "\nu^*"'] & R^* \end{tikzcd}$$

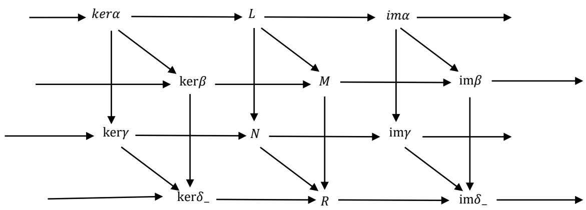
be two crossed squares and $\boldsymbol{\phi} = (\alpha, \beta, \gamma, \delta): \mathcal{L} \longrightarrow \mathcal{L}^*$ be a crossed square homomorphism. Then,

$$\mathrm{im}\boldsymbol{\phi} := \begin{tikzcd} \mathrm{im}\alpha \arrow[r] \arrow[d] & \mathrm{im}\beta \arrow[d] \\ \mathrm{im}\gamma \arrow[r] & \mathrm{im}\delta \end{tikzcd}$$

$\mathrm{im}\boldsymbol{\phi} \leq \mathcal{L}^*$.

Proof: By Proposition 3.4.28 in [23], we have:
$(\mathrm{im}\alpha, \mathrm{im}\beta, \lambda^*) \leq (L^*, M^*, \lambda^*)$
$(\mathrm{im}\alpha, \mathrm{im}\gamma, \lambda'^*) \leq (L^*, M^*, \lambda'^*)$
$(\mathrm{im}\beta, \mathrm{im}\delta, \mu^*) \leq (M^*, R^*, \mu^*)$
$(\mathrm{im}\gamma, \mathrm{im}\delta, \mathrm{v}^*) \leq (N^*, R^*, \mathrm{v}^*)$
$(\mathrm{im}\alpha, \mathrm{im}\delta, \mu^*\lambda^*) \leq (L, R, \mu^*\lambda^*)$.
Additionally, if $\varrho^* \in \mathrm{im}\beta$ and $\sigma^* \in \mathrm{im}\gamma$, there exist $\varrho \in M$ and $\sigma \in N$ such that $\varrho^* = \beta(\varrho)$ and $\sigma^* = \gamma(\sigma)$.
$h^*(\varrho^*, \sigma^*) = h^*\big(\beta(\varrho), \gamma(\sigma)\big) = \alpha\big(h(\varrho, \sigma)\big)$.
Thus $h^*(\varrho^*, \sigma^*) \in \mathrm{im}\alpha$.
As a consequence, im $\boldsymbol{\phi}$ is shown to be a subcrossed square of $\mathcal{L}^*$. ■

As a direct result of this theorem and the previous theorem, the exact sequence of crossed squares can be obtained.



Theorem 3.2.3 Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mathrm{v}"'] & R \end{tikzcd}$$

be a crossed square.

i) Given a family of sub-crossed squares

$$\mathcal{A}_i = \begin{tikzcd} A_i \arrow[r] \arrow[d] & B_i \arrow[d] \\ C_i \arrow[r] & D_i \end{tikzcd}$$

of $\mathcal{L}$, in this situation, then

$$\bigcap \mathcal{A}_i := \begin{tikzcd} \bigcap A_i \arrow[r] \arrow[d] & \bigcap B_i \arrow[d] \\ \bigcap C_i \arrow[r] & \bigcap D_i \end{tikzcd}$$

$\bigcap \mathcal{A}_i$ is a sub-crossed square of $\mathcal{L}$.

ii) Given a family of crossed squares ideals

$$\mathcal{E}_i = \begin{tikzcd} E_i \arrow[r] \arrow[d] & I_i \arrow[d] \\ J_i \arrow[r] & K_i \end{tikzcd}$$

of $\mathcal{L}$, then,

$$\bigcap \mathcal{E}_i := \begin{tikzcd} \bigcap E_i \arrow[r] \arrow[d] & \bigcap I_i \arrow[d] \\ \bigcap J_i \arrow[r] & \bigcap K_i \end{tikzcd}$$

$\bigcap \mathcal{E}_i$ is an ideal of $\mathcal{L}$.

Proof: As a result of Proposition 3.4.24 in [23] given for crossed modules, it is seen that each edge of the structures $\bigcap \mathcal{A}_i$ and $\bigcap \mathcal{E}_i$ are crossed modules. Therefore, it is sufficient to check the following for the proof of this result.

i) For all $\varrho \in \bigcap B_i$ and $\sigma \in \bigcap C_i$ since $h(\varrho, \sigma) \in A_i$ for each i, it follows that $h(\varrho, \sigma) \in \bigcap A_i$.

ii) For all $\xi \in M$ and $\sigma \in \bigcap J_i$ since $h(\xi, \sigma) \in E_i$ for each i, it follows that $h(\xi, \sigma) \in \bigcap E_i$

iii) For all $\varrho \in \bigcap I_i$, $\varsigma \in N$ since $h(\varrho, \varsigma) \in E_i$ for each i, then $h(\varrho, \varsigma) \in \bigcap E_i$.■

Theorem 3.2.4 Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mathrm{v}"'] & R \end{tikzcd}$$

be a crossed square of commutative algebras.

i) Given a sub-crossed square $\mathcal{A} \leq \mathcal{L}$ and an ideal $\mathcal{E} \trianglelefteq \mathcal{L}$, respectively.

$$\mathcal{A} = \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

$$\mathcal{E} = \begin{tikzcd} E \arrow[r] \arrow[d] & I \arrow[d] \\ J \arrow[r] & K \end{tikzcd}$$

The crossed square $\mathcal{A} + \mathcal{E}$

$$\mathcal{A} + \mathcal{E} := \begin{tikzcd} A+E \arrow[r] \arrow[d] & B+I \arrow[d] \\ C+J \arrow[r] & D+K \end{tikzcd}$$

defined as their sum is a sub-crossed square of $\mathcal{L}$.

ii) If the crossed squares for each i,

$$\mathcal{E}_i = \begin{tikzcd} E_i \arrow[r] \arrow[d] & I_i \arrow[d] \\ J_i \arrow[r] & K_i \end{tikzcd}$$

are ideal of $\mathcal{L}$, then the crossed square

$$\sum \mathcal{E}_i := \begin{tikzcd} \sum E_i \arrow[r] \arrow[d] & \sum I_i \arrow[d] \\ \sum J_i \arrow[r] & \sum K_i \end{tikzcd}$$

defined as their sum is also an ideal of $\mathcal{L}$.

Proof: From Theorem 3.4.26 in [23], it is seen that each edge of $\mathcal{A} + \mathcal{E}$ is a sub-crossed module of the corresponding edge of $\mathcal{L}$, and each edge of $\sum \mathcal{E}_i$ is a crossed ideal of $\mathcal{L}$. Therefore, it is sufficient to show the following.

i) For all $\varrho \in B, \varrho' \in I, \sigma \in C, \sigma' \in J$:
$$h(\varrho + \varrho', \sigma + \sigma') = h(\varrho, \sigma) + h(\varrho, \sigma') + h(\varrho', \sigma) + h(\varrho' + \sigma')$$
and since $h(\varrho, \sigma) \in A$, $h(\varrho, \sigma')$, $h(\varrho', \sigma)$, $h(\varrho', \sigma') \in A + E$, thus $h(\varrho + \varrho', \sigma + \sigma') \in A + E$.

ii) For all $\xi \in M$ and $\sum \sigma_i \in \sum J_i$: $h(\xi, \sum \sigma_i) = \sum h(\xi, \sigma_i) \in \sum E_i$. Similarly, for all $\varsigma \in N$ and $\sum \varrho_i \in \sum I_i$:
$h(\sum \varrho_i, \varsigma) = \sum h(\varrho_i, \varsigma) \in \sum E_i$ ■

Theorem 3.2.5 Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mathrm{v}"'] & R \end{tikzcd}$$

be a crossed square of commutative algebras and

$$\mathcal{E} = \begin{tikzcd} E \arrow[r] \arrow[d] & I \arrow[d] \\ J \arrow[r] & K \end{tikzcd}$$

an ideal of $\mathcal{L}$;

For $\varepsilon + E \in L/_E$, $\forall\, \xi + I \in M/_I$ ve $\forall\, \varsigma + J \in N/_J$:

$\bar{\lambda}: L/_E \longrightarrow M/_I$, $\bar{\lambda}(\varepsilon + E) = \lambda(\varepsilon) + I$

$\bar{\lambda'}: L/_E \longrightarrow N/_J$, $\bar{\lambda'}(\varepsilon + E) = \lambda'(\varepsilon) + J$

$\bar{\mu}: M/_I \longrightarrow R/_K$, $\bar{\mu}(\xi + I) = \mu(\xi) + K$

$\bar{\mathrm{v}}: N/_J \longrightarrow R/_K$, $\bar{\mathrm{v}}(\varsigma + J) = \mathrm{v}(\varsigma) + K$

$\bar{h}: {}^{M}/_{I} \times {}^{N}/_{J} \longrightarrow {}^{L}/_{E}$, $\bar{h}(\xi + I, \varsigma + J) = h(\xi, \varsigma) + E$

With these functions,

$$
{}^{\mathcal{L}}/_{\mathcal{E}} := \begin{tikzcd}
{}^{L}/_{E} \arrow[r, "\bar{\lambda}"] \arrow[d, "\bar{\lambda}'"] & {}^{M}/_{I} \arrow[d, "\bar{\mu}"] \\
{}^{N}/_{J} \arrow[r, "\bar{\nu}"'] & {}^{R}/_{K}
\end{tikzcd}
$$

the structure is a crossed square.

Proof:

i) From Proposition 3.5.1 in [23], it is seen that each edge of ${}^{\mathcal{L}}/_{\mathcal{E}}$ is a crossed module. It is sufficient to show that $\bar{h}$ is well-defined and satisfies the conditions related to crossed squares.

For all $\xi, \xi' \in M$ and $\varsigma, \varsigma' \in N$, let $\xi + I = \xi' + I$ and $\varsigma + J = \varsigma' + J$. Thus $\xi - \xi' \in I$, $\varsigma - \varsigma' \in J$. Therefore, since $\xi - \xi' \in I$, $h(\xi - \xi', \varsigma) \in E$. Similarly, $\varsigma - \varsigma' \in J$, $h(\xi', \varsigma - \varsigma') \in E$. Thus,

$$
\begin{aligned}
& h(\xi - \xi', \varsigma) + h(\xi', \varsigma - \varsigma') \\
&= h(\xi, \varsigma) - h(\xi', \varsigma) + h(\xi', \varsigma) - h(\xi', \varsigma) \\
&= h(\xi, \varsigma) - h(\xi', \varsigma') \in E
\end{aligned}
$$

Therefore, $h(\xi, \varsigma) + E = h(\xi', \varsigma') + E$, or equivalently

$\bar{h}(\xi + I, \varsigma + J) = \bar{h}(\xi' + I, \varsigma' + J)$.

Hence, $\bar{h}$ is well-defined.

ii) For all $r + K \in {}^{R}/_{K}$, $\varepsilon + E \in {}^{L}/_{E}$:

$$
\begin{aligned}
\bar{\lambda}((r + K) \cdot (\varepsilon + E)) &= \bar{\lambda}(r \cdot \varepsilon + E) \\
&= \lambda(r \cdot \varepsilon) + I = r \cdot \lambda(\varepsilon) + I \\
&= (r + K) \cdot (\lambda(\varepsilon) + I) \\
&= (r + K) \cdot \bar{\lambda}(\varepsilon + E).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\bar{\lambda}'((r + K) \cdot (\varepsilon + E)) &= \bar{\lambda}'(r \cdot \varepsilon + E) \\
&= \lambda'(r \cdot \varepsilon) + J = r \cdot \lambda'(\varepsilon) + J \\
&= (r + K) \cdot (\lambda'(\varepsilon) + J) \\
&= (r + K) \cdot \bar{\lambda}'(\varepsilon + E).
\end{aligned}
$$

iii) For all $\xi + I \in {}^{M}/_{I}, \varsigma + J \in {}^{N}/_{J}, k \in K$;

$$
\begin{aligned}
k\bar{h}(m + I, \varsigma + J) &= k(h(\xi, \varsigma) + E) \\
&= kh(\xi, \varsigma) + E = h(k\xi, \varsigma) + E \\
&= h(\xi, k\varsigma) + E \\
&= \bar{h}(k\xi + I, \varsigma + J) \\
&= \bar{h}(\xi + I, k\varsigma + J) \\
&= \bar{h}(k(\xi + I), \varsigma + J) \\
&= \bar{h}(\xi + I, k(\varsigma + J)).
\end{aligned}
$$

iv) For all $\xi + I$, $\xi' + I \in {}^{M}/_{I}$ and $\varsigma + J \in {}^{N}/_{J}$;

$$
\begin{aligned}
& \bar{h}(\xi + I + \xi' + I, \varsigma + J) \\
&= \bar{h}(\xi + \xi' + I, \varsigma + J) \\
&= h(\xi + \xi', \varsigma) + E \\
&= h(\xi, \varsigma) + h(\xi', \varsigma) + E \\
&= h(\xi, \varsigma) + E + h(\xi', \varsigma) + E \\
&= \bar{h}(\xi + I, \varsigma + J) + \bar{h}(\xi' + I, \varsigma + J).
\end{aligned}
$$

For all $\xi + I \in {}^{M}/_{I}$, $\varsigma + J$ and $\varsigma' + J \in {}^{N}/_{J}$;

$$
\begin{aligned}
& \bar{h}(\xi + I, \varsigma + J + \varsigma' + J) \\
&= \bar{h}(\xi + I, \varsigma + \varsigma' + J) \\
&= h(\xi, \varsigma + \varsigma') + E \\
&= h(\xi, \varsigma) + h(\xi, \varsigma') + E \\
&= h(\xi, \varsigma) + E + h(\xi, \varsigma') + E \\
&= \bar{h}(\xi + I, \varsigma + J) + \bar{h}(\xi + I, \varsigma' + J).
\end{aligned}
$$

v) For all $r + K \in {}^{R}/_{K}$, $\xi + I \in {}^{M}/_{I}$ and $\varsigma + J \in {}^{N}/_{J}$;

$$
\begin{aligned}
& (r + K) \cdot \bar{h}(\xi + I, \varsigma + J) \\
&= (r + K) \cdot (h(\xi, \varsigma) + E) \\
&= r \cdot h(\xi, \varsigma) + E \\
&= h(r \cdot \xi, \varsigma) + E \\
&= \bar{h}(r \cdot \xi + I, \varsigma + J) \\
&= \bar{h}((r + K) \cdot (\xi + I), \varsigma + J).
\end{aligned}
$$

Similarly,

For all $r + K \in {}^{R}/_{K}$, $\forall\, \xi + I \in {}^{M}/_{I}$ and $\varsigma + J \in {}^{N}/_{J}$;

$$
\begin{aligned}
& (r + K) \cdot \bar{h}(\xi + I, \varsigma + J) \\
&= (r + K) \cdot (h(\xi, \varsigma) + E) \\
&= r \cdot h(\xi, \varsigma) + E \\
&= h(\xi, r \cdot \varsigma) + E \\
&= \bar{h}(\xi + I, r \cdot \varsigma + J) \\
&= \bar{h}(\xi + I, (r + K) \cdot (\varsigma + J)).
\end{aligned}
$$

vi) For all $\xi + I \in {}^{M}/_{I}, \varsigma + J \in {}^{N}/_{J}$;

$$
\begin{aligned}
\bar{\lambda}(\bar{h}(\xi + I, \varsigma + J) &= \bar{\lambda}(h(\xi, \varsigma) + E) \\
&= \lambda(h(\xi, \varsigma)) + I \\
&= \varsigma \cdot \xi + I \\
&= (\varsigma + J) \cdot (\xi + I).
\end{aligned}
$$

Similarly,

$\xi + I \in {}^{M}/_{I}, \varsigma + J \in {}^{N}/_{J}$;

$$
\begin{aligned}
\bar{\lambda}'(\bar{h}(\xi + I, \varsigma + J) &= \bar{\lambda}'(h(\xi, \varsigma) + E) \\
&= \lambda'(h(\xi, \varsigma)) + J \\
&= \xi \cdot \varsigma + J = (\xi + I) \cdot (\varsigma + J).
\end{aligned}
$$

vii) For all $\varepsilon + E \in {}^{L}/_{E}$ and $\varsigma + J \in {}^{N}/_{J}$;

$$
\begin{aligned}
\bar{h}(\bar{\lambda}(\varepsilon + E), n + J) &= \bar{h}(\lambda(\varepsilon) + I, \varsigma + J) \\
&= h(\lambda(\varepsilon), \varsigma) + E \\
&= \varsigma \cdot \varepsilon + E \\
&= (\varsigma + J) \cdot (\varepsilon + E).
\end{aligned}
$$

Similarly, for all $\xi + I \in {}^{M}/_{I}$ and $\varepsilon + E \in {}^{L}/_{E}$;

$$
\begin{aligned}
& h\left(\xi + I, \bar{\lambda}'(\varepsilon + E)\right) \\
&= \bar{h}(\xi + I, \lambda'(\varepsilon) + J) \\
&= h(\xi, \lambda'(\varepsilon)) + E \\
&= \xi \cdot \varepsilon + E = (\xi + I) \cdot (\varepsilon + E). \blacksquare
\end{aligned}
$$

3.3. Quotient crossed squares of algebras

In this subsection, the definition of quotient crossed squares of commutative algebras is provided.

Definition 3.3.1 Let $\mathcal{E} \trianglelefteq \mathcal{L}$. The crossed square $\mathcal{L}/_{\mathcal{E}}$ given in the theorem above is called the quotient of $\mathcal{L}$ by $\mathcal{E}$.

Corollary 3.3.2 For a crossed square $\mathcal{L}$, a necessary and sufficient condition for $\mathcal{E}$ to be an ideal of $\mathcal{L}$ is that $\mathcal{E}$ is equal to the kernel of a crossed square homomorphism $\boldsymbol{\phi}: \mathcal{L} \longrightarrow \mathcal{L}'$.

3.4. Isomorphism theorems for crossed squares of algebras

In this subsection, the first, second, and third isomorphism theorems for crossed squares of unitary commutative algebras are stated and proven.

Theorem 3.4.1 First Isomorphism Theorem for Crossed Squares) Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "v"'] & R \end{tikzcd}$$

and,

$$\mathcal{L}^* = \begin{tikzcd} L^* \arrow[r, "\lambda^*"] \arrow[d, "\lambda'^*"] & M^* \arrow[d, "\mu^*"] \\ N^* \arrow[r, "v^*"'] & R^* \end{tikzcd}$$

are two crossed squares of commutative algebras and $\boldsymbol{\phi} = (\alpha, \beta, \gamma, \delta): \mathcal{L} \longrightarrow \mathcal{L}^*$ be a crossed square homomorphism. Then $\mathcal{L}/_{ker\boldsymbol{\phi}} \cong im\boldsymbol{\phi}$.

Proof: From Theorem 3.6.1 in [23] for all $l + ker\alpha \in L/_{ker\alpha}$, $\xi + ker\beta \in M/_{ker\beta}$, $\varsigma + ker\gamma \in N/_{ker\gamma}$ and $r + ker\delta \in R/_{ker\delta}$:

$\bar{\alpha}: L/_{ker\alpha} \longrightarrow im\alpha$, $\bar{\alpha}(\varepsilon + ker\alpha) = \alpha(\varepsilon)$

$\bar{\beta}: M/_{ker\beta} \longrightarrow im\beta$, $\bar{\beta}(\xi + ker\beta) = \beta(\xi)$

$\bar{\gamma}: N/_{ker\gamma} \longrightarrow im\gamma$, $\bar{\gamma}(\varsigma + ker\gamma) = \gamma(\varsigma)$

$\bar{\delta}: R/_{ker\delta} \longrightarrow im\delta$, $\bar{\delta}(r + ker\delta) = \delta(r)$

Since the pair $(\bar{\alpha}, \bar{\beta}), (\bar{\alpha}, \bar{\gamma}), (\bar{\beta}, \bar{\delta}), (\bar{\gamma}, \bar{\delta})$ and $(\bar{\alpha}, \bar{\delta})$ are isomorphism of crossed modules, it is sufficient to examine the condition for the functions $\bar{h}: M/_{ker\beta} \times N/_{ker\gamma} \longrightarrow L/_{ker\alpha}$ and $h^*: im\beta \times im\gamma \longrightarrow im\alpha$.

$$\begin{aligned} \bar{\alpha}\left(\bar{h}(\xi + ker\beta, \varsigma + ker\gamma)\right) &= \bar{\alpha}(h(\xi, \varsigma) + ker\alpha) \\ &= \alpha\big(h(\xi, \varsigma)\big) = h^*\big(\beta(\xi), \gamma(\varsigma)\big) \\ &= h^*\left(\bar{\beta}(\xi + ker\beta), \bar{\gamma}(\varsigma + ker\gamma)\right). \blacksquare \end{aligned}$$

Proposition 3.4.2 Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"] & M \arrow[d, "\mu"] \\ N \arrow[r, "v"'] & R \end{tikzcd}$$

be a crossed square of algebras. In this situation, if

$$\mathcal{E} = \begin{tikzcd} E \arrow[r] \arrow[d] & I \arrow[d] \\ J \arrow[r] & K \end{tikzcd}$$

$$\mathcal{A} = \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

$\mathcal{A} \leq \mathcal{E} \trianglelefteq \mathcal{L}$ and $\mathcal{A} \trianglelefteq \mathcal{L}$, then $\mathcal{A} \trianglelefteq \mathcal{E}$ and $\mathcal{E}/_{\mathcal{A}} \trianglelefteq \mathcal{L}/_{\mathcal{A}}$.

Proof: Since $\mathcal{A} \leq \mathcal{E}$, it is sufficient to show that $\mathcal{A} \trianglelefteq \mathcal{E}$.

i) If $D \leq K \trianglelefteq R$ and $D \trianglelefteq R$, then $D \trianglelefteq K$, by Proposition 2.21 in [23].

ii) For all $k \in K$, $\rho \in A$, $\varrho \in B$ and $\sigma \in C$ since $k \in R$:
$k \cdot \rho \in A, k \cdot \varrho \in B, k \cdot \sigma \in C$.

iii) For all $d \in D, \rho \in E, \varrho \in I, \sigma \in J$ since $\rho \in L$, $\varrho \in M, \sigma \in N$ and $\mathcal{A} \trianglelefteq \mathcal{L}$:
$d \cdot \rho \in A, d \cdot \varrho \in B, d \cdot \sigma \in C$.

iv) For all $i \in I$ and $\sigma \in C$ since $i \in M$ $h(i, \sigma) \in A$.

v) For all $\varrho \in B$ and $j \in J$ since $j \in N$ $h(\varrho, j) \in A$.

Thus $\mathcal{A} \trianglelefteq \mathcal{H}$.

Now, let's show that $\mathcal{E}/_{\mathcal{A}} \trianglelefteq \mathcal{L}/_{\mathcal{A}}$.

Since $E/_A \subseteq L/_A$, $I/_B \subseteq M/_B$, $J/_C \subseteq N/_C$, $K/_D \subseteq R/_D$ and $\mathcal{E}/_{\mathcal{A}}$ is a crossed square, and since the boundary homomorphism of $\mathcal{E}/_{\mathcal{A}}$ are restrictions of the boundary homomorphisms of $\mathcal{L}/_{\mathcal{A}}$, $\mathcal{E}/_{\mathcal{A}} \leq \mathcal{L}/_{\mathcal{A}}$.

i) By Proposition 2.21 in [23], ${}^{K}/_{D} \trianglelefteq {}^{R}/_{D}$.

ii) For all $r+D \in {}^{R}/_{D}$, $\rho + A \in {}^{E}/_{A}$, $\varrho + B \in {}^{I}/_{B}$ and $\sigma + C \in {}^{J}/_{C}$:
$(r+D)\cdot(\rho+A) = r\cdot\rho + A \in {}^{E}/_{A}$,
$(r+D)\cdot(\varrho+B) = r\cdot\varrho + B \in {}^{I}/_{B}$,
$(r+D)\cdot(\sigma+C) = r\cdot\sigma + C \in {}^{J}/_{C}$.

iii) For all $k+D \in {}^{K}/_{D}$, $\varepsilon + A \in {}^{L}/_{A}$, $\xi + B \in {}^{M}/_{B}$, $\varsigma + C \in {}^{N}/_{C}$:
$(k+D)\cdot(\varepsilon+A) = k\cdot\varepsilon + A \in {}^{E}/_{A}$,
$(k+D)\cdot(\xi+B) = k\cdot\xi + B \in {}^{I}/_{B}$,
$(k+D)\cdot(\varsigma+C) = k\cdot\varsigma + C \in {}^{J}/_{C}$.

iv) For all $\xi + B \in {}^{M}/_{B}$, $j + C \in {}^{J}/_{C}$:
$\bar{h}(\xi+B, j+C) = h(\xi, j) + A \in {}^{E}/_{A}$.

v) For all $i + B \in {}^{I}/_{B}$, $\varsigma + C \in {}^{N}/_{C}$:
$\bar{h}(i+B, \varsigma+C) = h(i,\varsigma) + A \in {}^{E}/_{A}$. ■

Theorem 3.4.3 (Second Isomorphism Theorem for Crossed Squares) Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

be a crossed square of commutative algebras and

$$\mathcal{E} = \begin{tikzcd} E \arrow[r] \arrow[d] & I \arrow[d] \\ J \arrow[r] & K \end{tikzcd}$$

$$\mathcal{A} = \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

$${}^{\mathcal{L}}/_{\mathcal{A}}\Big/{}^{\mathcal{E}}/_{\mathcal{A}} \cong {}^{\mathcal{L}}/_{\mathcal{E}}.$$

Proof: From Theorem 3.6.2 in [23], for all $\varepsilon + A + {}^{E}/_{A} \in {}^{L}/_{A}\big/{}^{E}/_{A}$, $\xi + B + {}^{I}/_{B} \in {}^{M}/_{B}\big/{}^{I}/_{B}$, $\varsigma + C + {}^{I}/_{C} \in {}^{N}/_{C}\big/{}^{J}/_{C}$ and $r + D + {}^{K}/_{D} \in {}^{R}/_{D}\big/{}^{K}/_{D}$:

α: ${}^{L}/_{A}\big/{}^{E}/_{A} \to {}^{L}/_{E}$, $\alpha\left(\varepsilon + A + {}^{E}/_{A}\right) = \varepsilon + E$

β: ${}^{M}/_{B}\big/{}^{I}/_{B} \to {}^{M}/_{I}$, $\beta\left(\xi + B + {}^{I}/_{B}\right) = \xi + I$

γ: ${}^{N}/_{C}\big/{}^{J}/_{C} \to {}^{N}/_{J}$, $\gamma\left(\varsigma + C + {}^{I}/_{C}\right) = \varsigma + J$

δ: ${}^{R}/_{D}\big/{}^{K}/_{D} \to {}^{R}/_{K}$, $\delta\left(r + D + {}^{K}/_{D}\right) = r + K$

since the pair (α,β), (α,γ), (β,δ), (γ,δ) and (α,δ) are isomorphisms of crossed modules, it is sufficient to examine the condition for the functions $\bar{\bar{h}}$: ${}^{M}/_{B}\big/{}^{I}/_{B} \times {}^{N}/_{C}\big/{}^{J}/_{C} \to {}^{L}/_{A}\big/{}^{E}/_{A}$ and $\bar{h}$: ${}^{M}/_{I} \times {}^{N}/_{J} \to {}^{L}/_{E}$.

$$\begin{aligned} \alpha\left(\bar{\bar{h}}\left(\xi + B + {}^{I}/_{B}, \varsigma + C + {}^{J}/_{C}\right)\right) &= \alpha\left(\bar{h}(\xi+B, \varsigma+C) + {}^{E}/_{A}\right) \\ &= \alpha\left(h(\xi,\varsigma) + A + {}^{E}/_{A}\right) \\ &= h(\xi,\varsigma) + E \\ &= \bar{h}(\xi + I, \varsigma + B) \\ &= \bar{h}\left(\beta\left(\xi + B + {}^{I}/_{B}\right), \gamma\left(\varsigma + C + {}^{J}/_{C}\right)\right). \ ■ \end{aligned}$$

Theorem 3.4.4 (Third Isomorphism Theorem for Crossed Squares) Let

$$\mathcal{L} = \begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & R \end{tikzcd}$$

be a crossed square of commutative algebras. And,

$$\mathcal{E} = \begin{tikzcd} E \arrow[r] \arrow[d] & I \arrow[d] \\ J \arrow[r] & K \end{tikzcd}$$

$$\mathcal{A} = \begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

if $\mathcal{A} \leq \mathcal{L}$ and $\mathcal{E} \trianglelefteq \mathcal{L}$, then ${\mathcal{A}}/{\mathcal{A} \cap \mathcal{E}} \cong {\mathcal{A} + \mathcal{E}}/{\mathcal{E}}$.

Proof: By Theorem 3.6.3 in [23],

α: ${A}/{A \cap E} \longrightarrow {A + E}/{E}$, $\alpha(\rho + A \cap E) = \rho + E$

β: ${B}/{B \cap I} \longrightarrow {B + I}/{I}$, $\beta(\varrho + B \cap I) = \varrho + I$

γ: ${C}/{C \cap J} \longrightarrow {C + J}/{J}$, $\gamma(\sigma + C \cap J) = \sigma + J$

δ: ${D}/{D \cap K} \longrightarrow {D + K}/{K}$, $\delta(d + D \cap K) = r + K$

for all $\rho + A \cap E \in {A}/{A \cap E}$, $\varrho + B \cap I \in {B}/{B \cap I}$, $\sigma + C \cap J \in {C}/{C \cap J}$ and $d + D \cap K \in {D}/{D \cap K}$.

Since the pair (α, β), (α, γ), (β, δ), (γ, δ) and (α, δ) are isomorphisms of crossed modules, it is sufficient to show the condition for the functions $\bar{h}_1 : {B}/{B \cap I} \times {C}/{C \cap J} \longrightarrow {A}/{A \cap E}$ and $\bar{h}_2 : {B + I}/{I} \times {C + J}/{J} \longrightarrow {A + E}/{E}$.

$$\begin{aligned} \alpha\left(\bar{h}_1(\varrho + B \cap I, \sigma + C \cap J)\right) &= \alpha(h(\varrho, \sigma) + A \cap E) \\ &= h(\varrho, \sigma) + E = \bar{h}_2(\varrho + I, \sigma + J) \\ &= \bar{h}_2(\beta(\varrho + B \cap I), \gamma(\sigma + C \cap J)). \end{aligned}$$

■

4. Discussion and Conclusion

In this study, the isomorphism theorems, which hold significant importance in the theory of crossed modules of algebras in algebra theory, are explicitly proven since they have only been stated in the literature. Subsequently, by using the isomorphism theorems of crossed modules of algebras, the isomorphism theorems in the crossed squares of commutative algebras are stated and proven. Furthermore, to achieve this, several auxiliary concepts, such as the quotient crossed square, which had not been previously defined in these structures, are introduced, and their fundamental properties are examined.

This study can be adapted or generalized to various other structures in numerous ways. For instance, one could investigate whether the isomorphism theorems apply to structures similar to crossed squares, such as crossed cubes and crossed n-cubes, or examine certain algebraic results, whether found in this thesis or not, within the category of crossed squares.

It is believed that this work will serve as a fundamental resource in mathematics and engineering, especially in the fields of algebra and algebraic topology, and will provide a valuable contribution to advanced studies on commutative algebras. Additionally, it will shed significant light on the discovery of numerous new topics and guide contemporary structures.

Declaration of Ethical Code

In this study, we undertake that all the rules required to be followed within the scope of the "Higher Education Institutions Scientific Research and Publication Ethics Directive" are complied with, and that none of the actions stated under the heading "Actions Against Scientific Research and Publication Ethics" are not carried out.

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