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Inverted Modified Lindley Dağılımı için Parametre Tahmin Yöntemlerinin Karşılaştırılması

Kübra BAĞCI GENEL[1*]

Öne Çıkanlar:
- Inverted Modified Lindley
- Tahmin

Anahtar Kelimeler:
- Cramer von Missess
- En küçük kareler
- En çok olabilirlik
- Inverted Modified Lindley
-

ÖZET:

Inverted Modified Lindley (IML) dağılımının, üstel ve Lindley dağılımlarına kıyasla daha iyi uyum sağlama yetenekleri gösterdiği önceki çalışmalarala gösterilmiştir. Bu çalışma, En Küçük Kareler (LS), Cramer von Misses (CvM) ve Maksimum Olabilirlik (ML) yöntemlerini kullanarak Inverted Modified Lindley (IML) dağılımının parametre tahminini incelemektedir. IML dağılımına ait parametrenin tahmin edilmesinde ML, LS ve CvM yöntemlerinin etkinliğini karşılaştırmak amacıyla bir Monte Carlo simülasyon çalışması yapılmıştır. Ayrıca ilgili tahmin yöntemleri kullanılarak çeşitli alanlardan gerçek veri uygulamaları sağlanmıştır. Bu yöntemlerin uyum performansı, ortalama karekök hata, belirleme katsayısı ve Kolmogorov-Smirnov testi kullanılarak değerlendirilmiştir. Uygulama sonuçlarına göre CvM metodu, IML dağılımı için dikkate alınan verileri daha bir iyi şekilde tanımlarken, simülasyon çalışması için ise, ML tahmin yöntemi öne çıkmaktadır.

A Comparison of Parameter Estimation Methods for the Inverted Modified Lindley Distribution

Highlights:
- Estimation
- Inverted Modified Lindley

Keywords:
- Cramer von Missess
- Estimation
- Inverted Modified Lindley
- Least Squares
- Maximum Likelihood

ABSTRACT:

The Inverted Modified Lindley (IML) distribution has been shown to exhibit superior fitting capabilities compared to the exponential and Lindley distributions. This study investigates the parameter estimation of the IML distribution using the Least Squares (LS), Cramer von Misses (CvM), and Maximum Likelihood (ML) methods. A Monte Carlo simulation study is conducted to compare the efficiency of the ML, LS, and CvM methods in estimating the parameters of the IML distribution. Moreover, real data applications from various fields are provided using related estimation methods. The fitting performance of these methods is evaluated using root mean squared error, coefficient of determination, and the Kolmogorov-Smirnov test. According to the application results, the CvM estimates describe the considered data for the IML distribution best, while the simulation study favors ML estimation among the considered methods.

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INTRODUCTION

One-parameter distributions have proven highly effective in statistical analysis, primarily due to their analytical simplicity. Inverted modified Lindley (IML) distribution is a one-parameter distribution introduced by Chesneau, Tomy, Gillariose, & Jamal (2020) that demonstrated improved fitting capabilities over traditional distributions such as exponential and Lindley distributions. In addition to its simpler derivation, as demonstrated in Figure 1 it exhibits versatile in shapes of hazard rate and probability density functions. Thus, the IML distribution finds application across various areas, including reliability engineering, biology, and so on (see in Chesneau et al., 2020; Kumar, Nassar, Dey, Elshahhat, & Diyali, 2022; Kumar, Yadav, & Kumar, 2023).

In statistical modeling, many distributions have only one parameter, such as exponential, Rayleigh, Lindley. These models are useful for modeling data in various fields due to their desirable properties and simpler interpretations. There are also many inverse and modified versions of these distributions given in studies such as, Abouammoh & Alshingiti, 2009; Dey, Singh, Tripathi, & Asgharzadeh, 2016; Khan, 2014; Rasekhi et al., 2017 and more. The inverse transformation of random variables and assessing their usefulness in distribution modeling are widely applied. For example, Sharma, Singh, Singh, & Agiwal, (2015) introduced the inverse Lindley distribution which has one parameter, and explored its application in the survival times of head and neck cancer patients successfully, compared to the inverse Rayleigh distribution. Similarly, Abd Al-Fattah, El-Helbawy, & Al-Dayian (2017) presented the Inverted Kumaraswamy distribution by applying inverse transformation to the Kumaraswamy distribution, which has found applications in various fields by outperforming some other well-known distributions. For example, Bagci, Arslan, & Celik (2021) showed the Inverted Kumaraswamy distributions is better than the Weibull distribution modeling a wind speed data set. Likewise, the inverted Topp-Leone model has been proposed as an attractive model in reliability studies (Hassan, Elgarhy, & Ragab, 2020).

The IML distribution is obtained using $y = 1/x$ inverse transformation applied to the modified Lindley (ML) distribution by Chesneau et al., (2020). The probability density function (pdf) and the cumulative distribution function (cdf) for the IML distribution are

$$f(x) = \frac{\theta}{1+\theta}\frac{1}{x^2}e^{-\frac{2\theta}{x}}\left((1+\theta)e^{\frac{\theta}{x}} + \frac{2\theta}{x} - 1\right), x > 0, \theta > 0 \qquad (1)$$

and

$$F(x) = \left(1 + \frac{\theta}{1+\theta}\frac{1}{x}e^{-\frac{\theta}{x}}\right)e^{-\frac{\theta}{x}}, x > 0, \theta > 0 \qquad (2)$$

respectively (Chesneau et al., 2020).

The hazard rate function (hrf) and the pdf for selected values of the parameter for the IML distribution are given in Figure 1. Figure 1 demonstrates that the IML distribution displays both unimodal characteristics and the potential for right skewness. It can exhibit increasing, decreasing, constant upside-down bathtub failure rate functions.

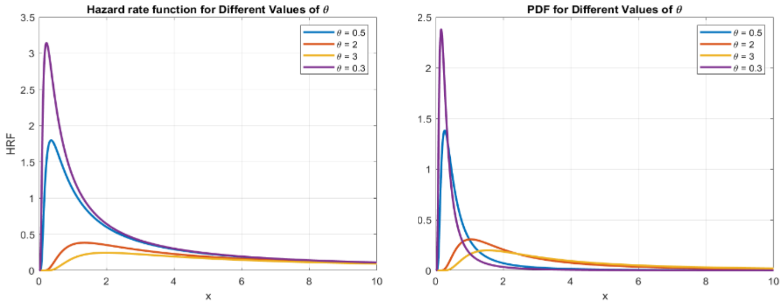

Figure 1. The hrf and pdf plots of the IML distribution for certain values of the parameter

Recently, different studies have been carried out on parameter estimation of the IML distribution. Maximum likelihood (ML) and Bayes estimation methods were adopted previously based on lower record values and censoring schemes. For example, Kumar et al., (2022) derived explicit single and product moments of order statistics from the IML distribution also utilized Best Linear Unbiased Estimators (BLUEs) in parameter estimation. In addition, Kumar et al., (2023) analyzed the IML distribution employing dual generalized order statistics. Hasaballah, Tashkandy, Bakr, Balogun, & Ramadan (2024) explored statistical inferences for product lifetimes with the IML distribution and Type-II censored data. They employed the maximum likelihood estimation (MLE), approximate confidence intervals, and Bayesian estimation with Gibbs sampling in the analysis of real data and Monte Carlo simulations to validate the accuracy and compare estimation methods.

As noted previously, the IML distribution exhibits appealing properties (see also Chesneau et al., 2020) Nonetheless, there are limited studies examining various methods for estimating the parameter of the IML distribution. It is acknowledged that minimum distance estimators tend to be less affected by unusual observations (Donoho & Liu, 1988). The minimum distance estimators can serve as alternatives to the MLE in some cases in the literature (see Arslan, Acitas, & Senoglu, 2022; Bagci, Erdogan, Arslan, & Celik, 2022). To the best of the author's knowledge, the LS and CvM estimations for the IML distribution have not been implemented previously. Motivated by these reasons, in this study, a classical method namely, the Least Squares (LS) and Cramer von Misses (CvM) estimation methods are utilized, and the MLE method is included in the analysis as well. A Monte Carlo simulation study is considered for varying parameter values and sample sizes in addition to real data applications.

MATERIALS AND METHODS

In this section, the data and MLE, CvM, and LS methods are described.

Data

In this study, three different data from various fields are included in the analysis. The first dataset is sourced from The Open University (1993), detailing the prices of 31 children's wooden toys available at a Suffolk craft shop in April 1991 obtained from Chesneau et al. (2020). The second dataset comprises the time intervals between failures for a repairable item, obtained from Murthy, Xie, & Jiang, (2004). The third application involves the utilization of data on vinyl chloride concentrations (mg/L) from monitoring wells designated for clean upgrading, which is obtained by Bhaumik & Gibbons (2006). Observations for these data are provided as follows.

First Dataset : 4.2, 1.12, 1.39, 2, 3.99, 2.15, 1.74, 5.81, 1.7, 0.5, 0.99, 11.5, 5.12, 0.9, 1.99, 6.24, 2.6, 3, 12.2, 7.36, 4.75, 11.59, 8.69, 9.8, 1.85, 1.99, 1.35, 10, 0.65, 1.45.

Second Dataset: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46, 2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30, 1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

Third Dataset: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

Estimation Methods

Let $X_1, X_2, \ldots, X_n$ be a random sample following the IML distribution and $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are the order statistics for this sample.

ML estimation

The ML estimations are obtained by maximizing the following loglikelihood function $\ell(\theta)$ for the parameter θ. Since the ML estimation is provided in Chesneau et al. (2020) previously, more details on MLE for the IML distribution can be found in Chesneau et al. (2020). To estimate the parameter, iterative techniques are used.

$$\ell(\theta) = \log[L(\theta)] = \mathrm{nlog}(\theta) - \mathrm{nlog}(1+\theta) - 2\theta \sum_{i=1}^{n} \frac{1}{x_i} + \sum_{i=1}^{n} \log\left[(1+\theta)\mathrm{e}^{\theta/x_i} + \frac{2\theta}{x_i} - 1\right] - 2\sum_{i=1}^{n} \log(x_i)\,. \tag{3}$$

CvM estimation

The parameter estimation can be derived by minimizing the subsequent objective function. Here, $(\cdot)$ is the cdf of the IML distribution provided in Equation (2)

$$CvM = \sum_{i=1}^{n} [F(x_{(i)}; \theta) - \frac{2i-1}{2n}]^2 + \frac{1}{12n} \tag{4}$$

The nonlinear equation in Equation (5) yields a CvM estimate for the parameter θ. Here, iterative techniques are used to obtain an estimation of the parameter θ.

$$\frac{\partial CvM}{\partial \theta} = 2\left(\frac{2i-1}{2n} - e^{-\frac{\theta}{y}}\left(\frac{\theta e^{-\frac{\theta}{y}}}{y(\theta+1)} + 1\right)\right)\left(e^{-\frac{\theta}{y}}\left(\frac{\theta e^{-\frac{\theta}{y}}}{y(\theta+1)^2} - \frac{e^{-\frac{\theta}{y}}}{y(\theta+1)} + \frac{a e^{-\frac{\theta}{y}}}{y^2(\theta+1)}\right) + \frac{e^{-\frac{\theta}{y}}\left(\frac{\theta e^{-\frac{\theta}{y}}}{y(\theta+1)}+1\right)}{y}\right) = 0 \tag{5}$$

LS estimation

The LS estimation of the parameter θ is obtained by minimizing the following function with respect to the parameter.

$$LS = \sum_{i=1}^{n} \left(\frac{i}{n+1} - (1 + \theta \frac{1}{1+\theta} \frac{1}{y} e^{-\frac{\theta}{y}}) e^{-\frac{\theta}{y}}\right)^2 \tag{6}$$

The nonlinear equation below yields an LS estimate for the parameter θ. Here iterative techniques are used to obtain estimation of the parameter θ.

$$\frac{\partial LS}{\partial \theta} = -2\left(e^{-\frac{\theta}{x}}\left(\frac{\theta e^{-\frac{\theta}{x}}}{x(\theta+1)^2} - \frac{e^{-\frac{\theta}{x}}}{x(\theta+1)} + \frac{\theta e^{-\frac{\theta}{x}}}{x^2(\theta+1)}\right) + \frac{e^{-\frac{\theta}{x}}\left(\frac{\theta e^{-\frac{\theta}{x}}}{x(\theta+1)}+1\right)}{x}\right)\left(e^{-\frac{\theta}{x}}\left(\frac{\theta e^{-\frac{\theta}{x}}}{x(\theta+1)} + 1\right) - \frac{i}{n+1}\right) = 0 \tag{7}$$

Simulation Study

In this subsection, a Monte-Carlo simulation study is conducted to compare the estimation methods' efficiencies and examine if the estimations are applicable in different conditions. The simulations are run 1000 times considering sample sizes n=25, 50, 100, and 500. The selected parameter values are $\theta = 0.3$, 0.5, 2, and 3. These values are chosen by looking up the related literature to ensure the simulation accurately reflects the real-world scenario. Estimates are calculated by using the "fminsearch" function in the Matlab R2021a optimization toolbox. The performances are compared using mean, variance, and Mean Squared Error (MSE) criteria for the ML, LS, and CvM estimation methods. The MSE is formulated as follows.

$$MSE(\hat{\theta}) = E(\theta - \hat{\theta})^2 \quad (8)$$

Table 1. The Simulation Results

Method	Mean	Variance	MSE		Mean	Variance	MSE
n=25, θ =0.5				n=25 θ =2			
MLE	0.509471	0.006735	0.006824	MLE	2.04229	0.130672	0.13246
LS	0.504842	0.007537	0.00756	LS	2.038945	0.166597	0.168114
CvM	0.506528	0.007555	0.007597	CvM	2.047908	0.167586	0.169881
n=50, θ =0.5				n=50, θ =2			
MLE	0.502339	0.003358	0.003364	MLE	2.038514	0.063793	0.065276
LS	0.502048	0.003953	0.003957	LS	2.034189	0.07326	0.074429
CvM	0.502904	0.003959	0.003967	CvM	2.038987	0.073562	0.075082
n=100, θ =0.5				n=100, θ =2			
MLE	0.501921	0.001504	0.001507	MLE	2.009743	0.032157	0.032252
LS	0.500729	0.001712	0.001713	LS	2.01138	0.042017	0.042147
CvM	0.501164	0.001714	0.001715	CvM	2.013732	0.042087	0.042276
n=500, θ =0.5				n=500, θ =2			
MLE	0.500585	0.000335	0.000336	MLE	2.002023	0.00587	0.005874
LS	0.500463	0.000377	0.000378	LS	2.001421	0.007079	0.007081
CvM	0.500551	0.000378	0.000378	CvM	2.001898	0.007082	0.007085
n=25, θ =3				n=25, θ =0.3			
MLE	3.100257	0.339193	0.349245	MLE	0.307	0.002437	0.002486
LS	3.071424	0.412546	0.417647	LS	0.304749	0.002812	0.002834
CvM	3.08666	0.415672	0.423182	CvM	0.305747	0.00282	0.002853
n=50, θ =3				n=50 θ =0.3			
MLE	3.028521	0.160343	0.161156	MLE	0.304569	0.001168	0.001188
LS	3.019026	0.184984	0.185346	LS	0.304358	0.001351	0.00137
CvM	3.02664	0.185835	0.186544	CvM	0.30484	0.001352	0.001376
n=100, θ =3				n=100, θ =0.3			
MLE	3.014135	0.075792	0.075992	MLE	0.3015	0.000616	0.000618
LS	3.007577	0.089665	0.089723	LS	0.300825	0.000687	0.000688
CvM	3.011494	0.089879	0.090011	CvM	0.301069	0.000688	0.000689
n=500, θ =3				n=500, θ =0.3			
MLE	3.001633	0.014596	0.014598	MLE	0.300655	0.000103	0.000103
LS	3.000802	0.017865	0.017865	LS	0.30068	0.000118	0.000119
CvM	3.001597	0.017875	0.017877	CvM	0.300729	0.000118	0.000119

Upon reviewing the simulation results, it can be seen that the MSEs of all estimates decrease as the sample size increases. These observation suggest that the considered estimation methods may be well-suited for data fitting purposes. Moreover, the MSE of the MLE seems to approach zero more quickly than those of the LS and CvM methods. It can be also inferred, for the lower values of the parameter, the MSE's are much smaller.

The simulation results evaluated for each parameter as follows. According to Table 1 when $\theta = 0.5$, the ML estimation provided lower MSE values for all sample sizes and the LS and CvM methods performed similarly.

Similarly, when $\theta = 3$, the ML estimation method performs better than the CvM and LS methods, and the LS estimates also provided smaller MSEs than the CvM.

When $\theta = 2$ for all sample sizes considered the MLE method is slightly better than the other two rivals. Similarly when $\theta = 0.3$ the MLE method is slightly better considering MSE values for all sample sizes considered.

Overall, it can be observed that the LS method generally exhibits lower values of MSE than the CvM method, and the results of the simulation study favor the MLE method. However, since the LS and CvM methods performed very well, their modeling performances are worth examining with real data applications.

RESULTS AND DISCUSSION

In this section, the data used in the application are presented. Then various data are modeled using IML distribution considering the ML, CvM, and LS methods. The results are compared using several criteria including root mean squared error (RMSE), coefficient of determination (R^2), and the Kolmogorov-Smirnov goodness of fit test statistic (KS). Lower values of RMSE and KS test statistic and higher values of R^2, and KS p-values demonstrate better fitting performance.

Applications

In this subsection, the IML distribution is fitted to the given data, and comparisons for the estimation methods are provided in Tables 2, 3, and 4 for the first, second, and third data, respectively. In addition, fitted densities are visualized in Figures 2, 3, and 4 for the estimation methods. To implement the analysis, Matlab R2021 software and its functions are used.

Table 2. Estimated parameter and fitting criteria results for the first data set

Method	KS(p-value)	R^2	RMSE	$\hat{\theta}$
ML	0.122502(0.7134)	0.982059	0.037387	2.153682
LS	0.115218(0.7787)	0.979678	0.039575	2.048729
CvM	0.121582(0.7218)	0.982138	0.037277	2.140409

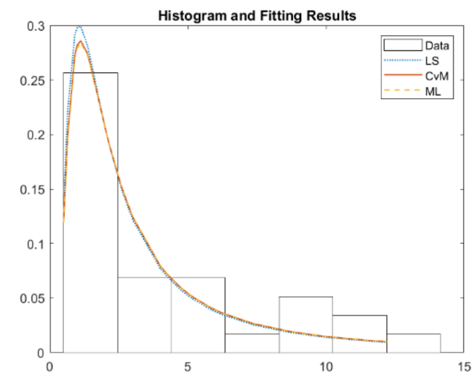

Figure 2. Estimation methods fitting plot for the first data set

When Table 2 examined although the ML and CvM methods were performed very closely it can be seen that CvM estimates stand out in more criteria for dataset 1. In addition, according to Figure 2 the CvM and ML method are fitted very similarly as well.

Table 3. Estimated parameter and fitting criteria results for the second data set

Method	KS(p-value)	R^2	RMSE	$\hat{\theta}$
ML	0.139395(0.557525)	0.937258	0.062812	0.922262
LS	0.133713(0.609556)	0.943439	0.060104	0.934013
CvM	0.133032(0.615863)	0.959702	0.050337	1.01983

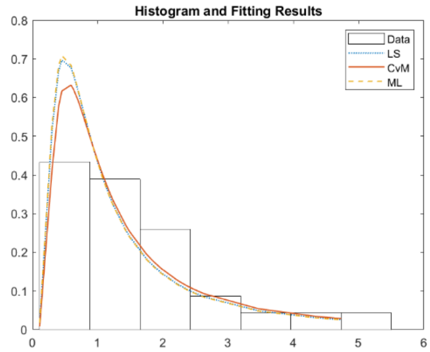

Figure 3. Estimation methods fitting plot for the second data set

According to Table 3, the CvM method performed better for all of the criteria in modeling the second dataset. Also from Figure 3, it can be seen that ML and LS methods overfitted the data at the peak of the distribution.

Table 4. Estimated parameter and fitting criteria results for the third data set

Method	KS(p-value)	R^2	RMSE	$\hat{\theta}$
ML	0.193049(0.138677)	0.899053	0.099107	0.704007
LS	0.196431(0.126391)	0.914371	0.093426	1.18378
CvM	0.112336(0.742425)	0.969335	0.05534	0.923025

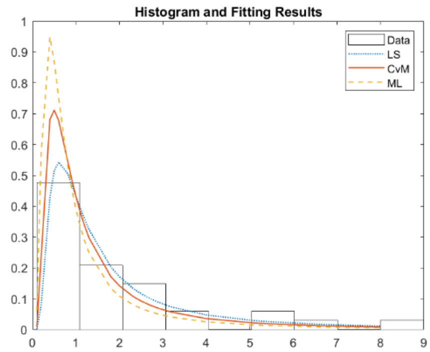

Figure 4. Estimation methods fitting plot for the third data set

According to Table 4, the CvM method performed better for all of the criteria in modeling the third dataset. Moreover, from Figure 4, it can be seen that the ML method overfitted the data at the peak of the distribution, and although the CvM estimation overfitted at the peak of the distribution, it described the data for the rest of the distribution better.

Accurate parameter estimation is crucial for effectively describing data. Different estimation methods can stand out depending on the data and its specific characteristics. For this study, it can be said that the LS and CvM estimations provided more accurate estimations for the parameter of the IML distribution than the ML method for considered data. For the first data set, using the LS estimation resulted in a higher p-value compared to previous studies (Kumar et al., 2023), which used order statistics in the estimation the parameter of the IML distribution. For the second data set, the CvM estimation for the parameter fitted better than reported in Kumar et al., (2023) as well. Overall, while ML remains a powerful tool for parameter estimation, CvM offers significant benefits in practical scenarios involving skewed data. No additional studies have been included in the discussion, as the other studies in the literature (overviewed in the Introduction section) are based on estimation for record values or censoring schemes.

CONCLUSION

In this study, the LS, CvM, and ML methods are utilized for estimating the parameter of the IML distribution and the methods' modeling performances compared through simulation study and real data applications. According to the Monte Carlo simulation study, it was observed that the MLE method generally outperformed the LS and CvM estimations with the slightest difference. In the case of real data applications, the CvM method demonstrated superior performance compared to its rivals. In conclusion, this study contributes to the literature by presenting the LS and CvM estimation methods for the IML distribution, alongside the conventional MLE method and applying these methods data from various fields successfully.

Conflict of Interest

The article's author declares that there is no conflict of interest.

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