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The Relationship Between The Kauffman Bracket Polynomials And The Tutte Polynomials of (2,n)-Torus Knots

Abdulgani Şahin[1]*, Abdullah Kopuzlu[2]

[1]Ağrı İbrahim Çeçen University, Faculty of Science and Letters, Departmant of Mathematics, 04100 Ağrı TURKEY,
rukassah@gmail.com
[2]Atatürk University, Faculty of Science, Departmant of Mathematics, 25400 Erzurum TURKEY,
akopuzlu@atauni.edu.tr
*Corresponding Author

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Abstract

At knot theory there are many important invariants that are hard to calculate. They are classified as numeric, group and polynomial invariants. These invariants contribute to the problem of classification of knots. In this study, we have done a study on the polynomial invariants of the knots. First of all, for (2,n)-torus knots which is a special class of knots, we calculated their the Kauffman bracket polynomials. We have found a general formula for these calculations. Then the Tutte polynomials of graphs, which are marked with a {+} or {-} sign each on edge, of (2,n)-torus knots have been computed. Some results have been obtained at the end of these calculations. While these researches have been studied, figures and regular diagrams of knots have been applied so much. During the first calculation, we have used skein diagrams and relations of the Kauffman polynomial. In the second calculation, the Tutte polynomials of (2,n)-torus knots have been computed, at the end of the operation some general formulas have been introduced. For (2,n)-torus knots the marked graphs have been gotten by using regular diagrams of them. Thus the Tutte polynomials of the ones have been computed as a diagrammatic by recursive formulas that can be defined by deletion-contraction operations. Finally, it has been obtained that there is a correlate among the Tutte polynomials and the Kauffman bracket polynomials of (2,n)-torus knots.

Keywords — Kauffman polynomial, Knots, Knot graph, Torus knots, Tutte polynomial.

1. Introduction

Knot theory has gained more importance, especially since the beginning of the 20th century. Because knot theory has found many applications in fields such as biology, chemistry and physics. Topologists have developed various knot invariants and worked extensively to solve the basic similarity problem of knots. Their goals have been to find more sensitive knot invariants that reveal that the two knots are different.

In 1987, Vaughan Jones found a new polynomial invariant. This invention has given new impetus to the theory. It was immediate described in algebraic discipline. Its combinatorics definition from graph theory to knot theory provided more utility than itself. Kauffman described his polynomial and bracket polynomial using Jones' works.

In 1954 a new polynomial for graphs, which is a two-variable polynomial and has got an important role in mathematics, statiscal physics, biology and theoretical computer science is devised by William Tutte. It can be defined for each undirected graph and gives information about how the graph can be connected. In 1988, Kauffman described marked graphs' Tutte polynomials.

2. Materials and Methods

2.1 Knot

In space a knot is defined as a simple closed curve. More mathematically, a knot is the embedding of the circle $\boldsymbol{S}^{\boldsymbol{1}}$ in $\mathbb{R}^{\boldsymbol{3}}$ (or $\boldsymbol{S}^{\boldsymbol{3}}$) [1]. A knot is closed curve in space which does intersect itself nowhere.

A knot could be limned without an intersection point whereon the trivial torus. Then this knot is called torus knot. A way to get the trivial torus, it is to take a cylinder whose floor is the unit circle C_1 and top is the unit circle C_2 and then in $\mathbb{R}^3$ C_1 and C_2 stick together to form a trivial knot C

Celal Bayar University Journal of Science
Volume 13, Issue 4, p 929-932

in the central axis of the cylinder (see Figure 1). This torus can also be called the standard solid torus. If a knot is formed so as to surround that standard solid torus q times on meridian and p times on longitude the knot is called torus knot $K_{p,q}$ of the type (p,q). Here, q and p must be prime between them [1].

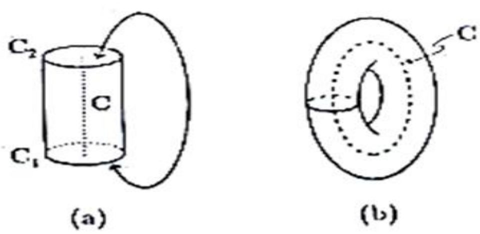


Figure 1 The trivial torus

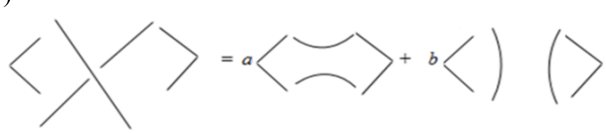
2.2 Graphs

V is the vertex set and $E \subseteq V \times V$ is the edge set, a graph is represented by (V,E) binary. Only undirected graphs will be taken into account here. A loop is an edge formed on the same vertex and a bridge is an edge which is just a path (there is no another path) between two vertices.

2.3 The Kauffman Bracket Polynomial

A polynomial denoted by $\langle L \rangle$ is assigned to each unoriented knot L, which this polynomial has three independent polynomial variables a, b, c. So that the following conditions are satisfied:

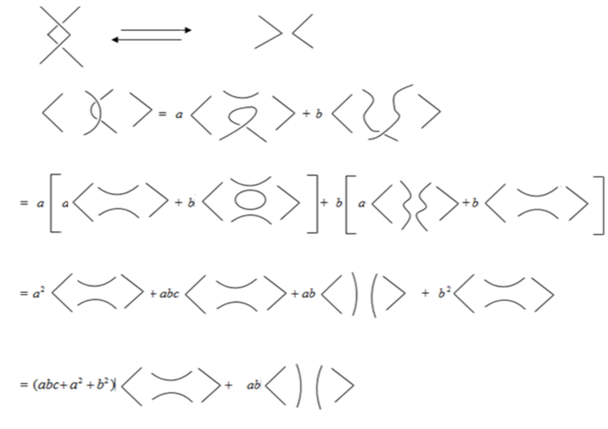
1)



or

2) $\langle L \cup \bigcirc \rangle = c\langle L \rangle$
3) $\langle \bigcirc \rangle = 1$

Now the relation between the mutables a, b, c is given. That polynomial is constant according to the Reidemeister move Ω_2. If the Reidemeister move Ω_2 is dealt with we have:



If $ab = 1$ and $a^2 + b^2 + abc = 0$ the polynomial will be constant according to the Reidemeister move Ω_2. Therefore we have:

$$b = a^{-1}$$
$$a^2 + b^2 + abc = 0$$
$$\Rightarrow c = -a^2 - b^2$$
$$\Rightarrow c = -a^2 - a^{-2}.$$

2.4 The Tutte Polynomial

To characterize the Tutte polynomial of a graph, edge deletion and edge contraction operations must be known. Edge deletion is shown with $G - e$. Edge contraction is shown with G/e. To perform the edge contraction operation edge deleting is firstly applied.

For a graph (G,V) the Tutte polynomial is as follows [2]:

$$T(G;x,y) = \begin{cases} 1, & E(G) = \emptyset \\ xT(G/e), & e \in E \text{ and } e \text{ is a bridge} \\ yT(G-e), & e \in E \text{ and } e \text{ is a loop} \\ T(G-e;x,y) + T(G/e;x,y), & e \in E \text{ and } e \text{ is neither a loop nor a bridge} \end{cases}$$

2.5 For Signed Graph Tutte Polynomial

There is a corresponding regular knot diagram for each signed planar graph. Conversely the regular diagrams of the knots correspond individually to the signed planar graphs [3].

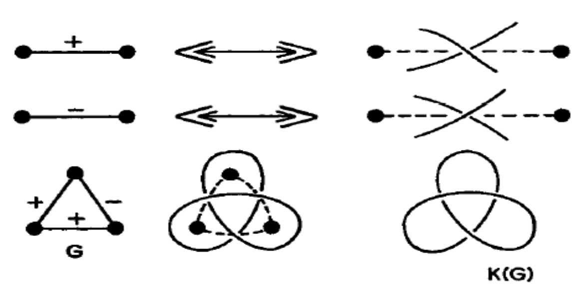


Figure 2 (The rule of how to get signed graphs)

Celal Bayar University Journal of Science
Volume 13, Issue 4, p 929-932

G is a signed graph. Mark(e) indicates the sign of the edge e of G. It may be (+) or (-). This edge e can be a loop or a bridge. We will indicate the number of positive bridges by $i_+ = i_+(G)$, the number of negative bridges by $i_- = i_-(G)$, the number of positive loops by $l_+ = l_+(G)$ and the number of negative loops by $l_- = l_-(G)$ in G. For signed graphs it can be defined the polynomial of $Q[G] = Q[G](a, b, d)$, where G' and G'' are the reduced graphs obtained using deletion-contraction and $X = a + bd$, $Y = ad + b$ [3]:

(1) In G, providing that neither an edge e is a loop nor it is a bridge in that case

$Q[G] = aQ[G'] + bQ[G'']$ if mark$(e) < 0$,
$Q[G] = bQ[G'] + aQ[G'']$ if mark$(e) > 0$.

(2) Providing that each edge of G is either a bridge or a loop and G is connected in that case

$Q[G] = X^{i_+ + l_-} Y^{i_- + l_+}$.

(3) Providing that G is equal to cloven fusion of graphs G_1 and G_2 in that case $Q[G] = dQ[G_1]Q[G_2]$.

That polynomial contains the Tutte polynomials for constant sign of random graphs [3].

3. Results and Discussion

3.1 (2,n)-Torus Knots' Kauffman Polynomials

We showed torus knots $K_{2,n}$ of the type $(2, n)$ in the following way (see Figure 3).

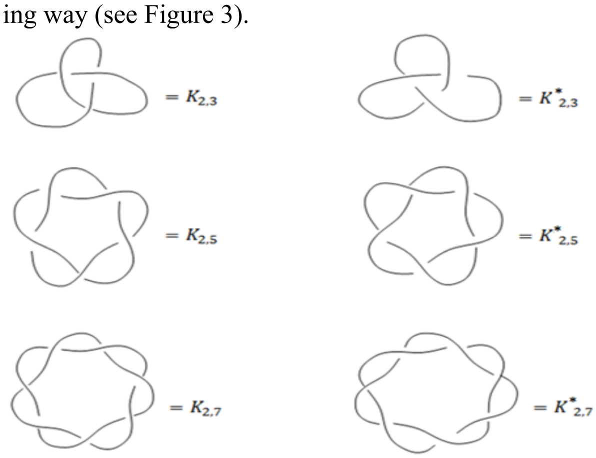


Figure 3 Regular diagrams of $K_{2,n}$

Then the Kauffman bracket polynomials of $K_{2,n}$ and $K^*_{2,n}$ calculate as defined in Section 2.3. It is obtained the following results:

Theorem 1: For the Kauffman bracket polynomials of $K_{2,n}$ the following general formula is found [4]:

$$P_{K_{2,n}}(a) = aP_{K_{2,n-1}}(a) + (-1)^{n-1}a^{-3n+2} \qquad \textbf{(3.1)}$$

where $\boldsymbol{P_{K_{2,1}}(a) = -a^3}$.

Theorem 2: For the Kauffman bracket polynomials of $K^*_{2,n}$ the following general formula is found [4]:

$$P_{K^*_{2,n}}(a) = a^{-1}P_{K^*_{2,n-1}}(a) + (-1)^{n-1}a^{3n-2} \qquad \textbf{(3.2)}$$

where $P_{K^*_{2,1}}(a) = -a^{-3}$.

3.2 The Tutte Polynomials of (2,n)-Torus Knots' Signed Graphs

Isomorphic graphs of (2,n)-torus knots $K_{2,n}$ have been gotten using regular diagrams. The signed graphs of (2,n)-torus knots are gotten with (+) or (-) signs as shown in Figure 2. Then the Tutte polynomials of this signed graphs calculate as defined in subsection 2.5. It is obtained the following results:

Theorem 3: For signed graphs of (2,n)-torus knots $K_{2,n}$, which every edges have {-} sign, the following general formula is found about the Tutte polynomials of them [5]:

$$Q[G] = a(\textstyle\sum_{k=1}^{n-1} b^{k-1}Y^{n-k}) + b^{n-1}X \qquad \textbf{(3.3)}$$

Theorem 4: For signed graphs of (2,n)-torus knots $K^*_{2,n}$, which every edges have {+} sign, the following general formula is found about the Tutte polynomials of them [5]:

$$Q[G^*] = b(\textstyle\sum_{k=1}^{n-1} a^{k-1}X^{n-k}) + a^{n-1}Y \qquad \textbf{(3.4)}$$

4. Conclusion

The relationship indicated in the title is determined and it is expressed in the following conclusions for torus knots of the type $(2, n)$:

Conclusion 1: If we put $\boldsymbol{X = -a^{-3}}$, $\boldsymbol{Y = a^3}$ and $\boldsymbol{b = -a^{-1}}$ in the relation (3.3) we have the same result as the Kauffman bracket polynomial of the corresponding knot in the relation (3.1). For example; for $\boldsymbol{K_{2,3}}$:

$$Q[G] = a\left(\sum_{k=1}^{n-1} b^{k-1}Y^{n-k}\right) + b^{n-1}X$$
$$Q[G] = aY^2 + abY + b^2X$$
$$Q[G] = a(a^3)^2 + a(-a^{-1})a^3 + (-a^{-1})^2(-a^{-3})$$
$$Q[G] = a^7 - a^3 - a^{-5} = P_{K_{2,3}}(a).$$

Conclusion 2: If we put $\boldsymbol{X = -a^{-3}}$, $\boldsymbol{Y = -a^3}$ and $\boldsymbol{b = a^{-1}}$ in the relation (3.4) we have the same result as the Kauffman bracket polynomial of the corresponding knot the relation (3.2). For example; for $\boldsymbol{K^*_{2,5}}$:

$$\boldsymbol{Q}[\boldsymbol{G}^{*}] = \boldsymbol{b}\left(\sum_{\boldsymbol{k}=\boldsymbol{1}}^{\boldsymbol{n}-\boldsymbol{1}} \boldsymbol{a}^{\boldsymbol{k}-\boldsymbol{1}}\boldsymbol{X}^{\boldsymbol{n}-\boldsymbol{k}}\right) + \boldsymbol{a}^{\boldsymbol{n}-\boldsymbol{1}}\boldsymbol{Y}$$

$$Q[G^{*}] = a^{4}Y + a^{3}bX + a^{2}bX^{2} + abX^{3} + bX^{4}$$

$$Q[G^{*}] = a^{4}(-a^{3}) + a^{3}a^{-1}(-a^{-3}) + a^{2}a^{-1}(-a^{-3})^{2} + aa^{-1}(-a^{-3})^{3} + a^{-1}(-a^{-3})^{4}$$

$$Q[G^{*}] = -a^{7} - a^{-1} + a^{-5} - a^{-9} + a^{-13} = P_{K_{2,5}^{*}}(a).$$

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