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ON JORDAN ALGEBRAS THAT ARE FACTORS OF MATSUO ALGEBRAS

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Abstract. We describe all finite connected 3-transposition groups whose Matsuo algebras have nontrivial factors that are Jordan algebras. As a corollary, we show that if $\mathbb{F}$ is a field of characteristic 0, then there exist infinitely many primitive axial algebras of Jordan type $\frac{1}{2}$ over $\mathbb{F}$ that are not factors of Matsuo algebras. As an example, we prove this for an exceptional Jordan algebra over $\mathbb{F}$.

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1. Introduction

Axial algebras of Jordan type were introduced by Hall, Rehren, and Shpectorov [12] within the framework of the general theory of axial algebras [11]. The main inspiration for this theory are the Griess algebra [9], Majorana theory [16], and algebras associated with 3-transposition groups [20]. Modern results and open problems in the theory of axial algebras can be found in a recent survey [22].

Consider a commutative $\mathbb{F}$-algebra A, where $\mathbb{F}$ is a field of characteristic not equal to two. For each element a of A and $\lambda \in \mathbb{F}$, the λ-eigenspace for the adjoint operator ad_a on A is denoted by $A_\lambda(a)$. An idempotent whose adjoint operator is semisimple will be called an *axis*. If A is generated by a set of axes, then A is an *axial algebra*. An axis a is *primitive* if $A_1(a)$ is one-dimensional, i.e., spanned by a. Suppose that $\eta \in \mathbb{F}$ and $0 \neq \eta \neq 1$. The commutative $\mathbb{F}$-algebra A is a *primitive axial algebra of Jordan type* η provided it is generated by a set of primitive axes with each member a satisfying the following properties:

$$A = A_1(a) \oplus A_0(a) \oplus A_\eta(a), A_0(a)^2 \subseteq A_0(a),$$

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and for all $\delta, \epsilon \in \{\pm\}$,

$$A_\delta(a)A_\epsilon(a) \subseteq A_{\delta\epsilon}(a), \text{ where } A_+(a) = A_1(a) \oplus A_0(a) \text{ and } A_-(a) = A_\eta(a).$$

These properties generalize the Peirce decomposition for idempotents in Jordan algebras, where $\frac{1}{2}$ is replaced with η. In particular, this explains the motivation for the name of this class of axial algebras.

Another basic example of axial algebras of Jordan type are Matsuo algebras. They were introduced by Matsuo [20] and later generalized in [12]. Recall that a group G is a 3-*transposition group* if it is generated by a normal set D of involutions such that the order of the product of any pair of these involutions is not greater than three. Let η, as before, be an element of $\mathbb{F}$ distinct from 0 and 1. The Matsuo algebra $M_\eta(G, D)$ has D as its basis, where each element of D is an idempotent. Moreover, the product in $M_\eta(G, D)$ of two distinct elements $c, d \in D$ equals 0 if $|cd| = 2$ and $\frac{\eta}{2}(c+d-c^d)$ if $|cd| = 3$. It turns out that $M_\eta(G, D)$ is a primitive axial algebra of Jordan type η with generating set of primitive axes D [12]. Moreover, it is known that if $\eta \neq \frac{1}{2}$, then every primitive axial algebra of Jordan type $\eta \neq \frac{1}{2}$ is a factor algebra of a Matsuo algebra [12,13]. The case $\eta = \frac{1}{2}$ remains open.

Conjecture 1. [8, Question 1],[22, Conjecture 4.3] Every primitive axial algebra of Jordan type $\frac{1}{2}$ is either a Jordan algebra or a factor of a Matsuo algebra.

De Medts and Rehren classified Matsuo algebras that are Jordan algebras [2]. As a consequence, it can be concluded that most Matsuo algebras are not Jordan. The motivation for this paper is the following question: are there examples of axial algebras of Jordan type $\frac{1}{2}$ that are not factors of Matsuo algebras? We provide examples of such algebras among Jordan algebras. We focus on Matsuo algebras corresponding to connected 3-transposition groups (G, D), i.e., where D is a conjugacy class of 3-transpositions. If D is a union of conjugacy classes, then the Matsuo algebra on D is the direct sum of the corresponding Matsuo algebras constructed from each conjugacy class contained in D [12]. We say that two nontrivial connected 3-transposition groups (G_1, D_1) and (G_2, D_2) have the same central type if $G_1/Z(G_1)$ and $G_2/Z(G_2)$ are isomorphic as 3-transposition groups. It is easy to see that if two 3-transposition groups have the same central type, then their Matsuo algebras are isomorphic.

It turns out that every Matsuo algebra $M = M_\eta(G, D)$, where (G, D) is a connected 3-transposition group, has a maximal ideal $M^\perp$ containing every proper ideal of M. In fact, this ideal is the radical of a symmetric bilinear form on M (see Section 3). Clearly, if an algebra is Jordan, then every homomorphic image is

Jordan. This implies that M has Jordan factors if and only if $M/M^{\perp}$ is Jordan. In this paper we describe all algebras M satisfying the latter condition. If G is a group generated by a conjugacy class D of 3-transpositions, then we write $p^{\bullet h}$ with $p \in \{2,3\}$, for a normal p-subgroup N with $|D \cap dN| = p^h$ for all $d \in D$.

Theorem 1. *Let $\mathbb{F}$ be a field of characteristic 0 and $\eta \in \mathbb{F} \setminus \{0,1\}$. Suppose that (G,D) is a finite connected 3-transposition group and $M = M_{\eta}(G,D)$ is the Matsuo algebra constructed from (G,D) and η. If $J = M/M^{\perp}$ is a Jordan algebra, then one of the following statements holds.*

(i) *G is the cyclic group of order 2 and so $J = M$ is one-dimensional;*
(ii) *the product of every two distinct elements of D has order 3, $\eta = 2$, and J is one-dimensional;*
(iii) *$\eta = \frac{1}{2}$ and G has the same central type as one of the following 3-transposition groups:* $\mathrm{Sym}(m)$ $(m \geq 2)$, $2^{\bullet 1} : \mathrm{Sym}(m)$ $(m \geq 4)$, $3^{\bullet 1} : \mathrm{Sym}(m)$ $(m \geq 4)$, $3^2 : 2$, $O_8^+(2)$, $O_6^-(2)$, $Sp_6(2)$, ${}^+\Omega_6^-(3)$, $SU_4(2)$, $SU_5(2)$, *or* $4^{\bullet 1}SU_3(2)'$. *In particular,* $\dim J \in \{1, m^2, \frac{m(m-1)}{2}\}$, *where $m \geq 3$.*

Moreover, each of the possibilities in items $(i) - (iii)$ is realized for some M.

In Section 5, we discuss possible Matsuo algebras M satisfying the hypothesis of this theorem and the corresponding 3-transposition groups (G,D) in detail (see Proposition 5.2). Note that the case $M^{\perp} = 0$ was considered in [2]. Moreover, it was mentioned in [2, Remark 3.5] that the Weyl groups for simply-laced root systems of types E_n with $6 \leq n \leq 8$ and D_n, considered as 3-transposition groups, correspond to Matsuo algebras that have among their factors the Jordan algebra (of dimension $\frac{n(n+1)}{2}$) of all symmetric $n \times n$ matrices.

Note that for every integer $n \geq 1$, there exists a simple Jordan algebra of dimension n which is a primitive axial algebra of Jordan type $\frac{1}{2}$. As an example one can take a so-called Jordan spin factor algebra (see, for example, [13, Lemma 5.1]). This together with Theorem 1 implies the following corollary.

Corollary 1.1. *If $\mathbb{F}$ is a field of characteristic 0, then there exist infinitely many primitive axial algebras of Jordan type $\frac{1}{2}$ over $\mathbb{F}$ that are not factors of Matsuo algebras.*

In Section 6, we present another example, which is the cornerstone in the theory of Jordan algebras. Given a field $\mathbb{F}$ of characteristic 0, we show that a 27-dimensional Albert algebra over $\mathbb{F}$ is an axial algebra of Jordan type $\frac{1}{2}$ generated by four primitive axes. Theorem 1 implies that this algebra, known to be a simple Jordan algebra, is not a factor of a Matsuo algebra.

Finally, we mention two results on the status of Conjecture 1. Gorshkov and Staroletov proved that every axial algebra of Jordan type $\frac{1}{2}$ generated by at most three primitive axes is Jordan and has dimension not exceeding 9 [8]. It has recently been proved that the dimension of a 4-generated algebra does not exceed 81, which is the dimension of a 4-generated Matsuo algebra [3]. In the general case, it is not even known whether an axial algebra of Jordan type generated by a finite number of primitive axes has a finite dimension or not.

The proof of Theorem 1 is based on the classification of 3-transposition groups and the dimensions of the eigenspaces of diagrams on the corresponding sets of 3-transpositions. The necessary definitions and results are given in Section 2. In Section 3, we provide the necessary information on Jordan and Matsuo algebras. In Section 4, we give a convenient description of the 3-transposition groups that are obtained from the symmetric group by the wreath product construction, these groups are a special case in the proof of Theorem 1. Section 5 is devoted to this proof. We emphasize that in many cases, for specific 3-transposition groups, the Jordan identity in the corresponding algebras is verified using the computer algebra system GAP [7]. Finally, in Section 6 we show that an Albert algebra over a field of characteristic different from 2 and 3 is a primitive axial algebra of Jordan type $\frac{1}{2}$.

2. Preliminaries: 3-transposition groups

Suppose that G is a group and $D \subseteq G$ is a normal set of involutions, i.e., a union of conjugacy classes of elements of order 2. If for every pair $d, e \in D$, the order of de is at most 3, then D is called a set of 3-transpositions. This notion was introduced by Fischer as a generalization of properties of transpositions in symmetric groups [4].

We say that (G, D) is a 3-transposition group if D generates G and is a set of 3-transpositions. If S is a subset of D, the diagram of S, denoted (S), is the graph whose vertices are elements of S with the pair $\{d, e\}$ forming an edge precisely when $|de| = 3$. This notion is important for 3-transpositions groups since the subgroup of G generated by S is a homomorphic image of the Coxeter group with diagram (S).

Lemma 2.1. [5, (1.2) and Lemma (2.1.1)] *Suppose that D is a set of 3-transpositions in G. Then the following statements hold.*

(i) *If H is a subgroup of G, then $D \cap H = \varnothing$ or $D \cap H$ is a set of 3-transpositions in H. If N is a normal subgroup of G, then $D \subset N$ or the nontrivial elements of DN/N form a normal set of 3-transpositions in G/N.*

(ii) *Let D_i, for $i \in I$, be the connected components of (D). Then each D_i is a conjugacy class of 3-transpositions in the group $G_i = \langle D_i \rangle$. Furthermore, the normal subgroup $\langle D \rangle$ is the central product of its subgroups G_i.*

(iii) *If $G = \langle D \rangle$, then for each $d \in D \setminus Z(G)$, the coset $dZ(G)$ meets D only in d.*

It follows that the building blocks for 3-transposition groups are groups with connected diagrams. For brevity, we say that (G, D) is a *connected 3-transposition group* if (D) is connected. Note that this is equivalent to D being a conjugacy class of G. We say that the two connected 3-transposition groups (G_1, D_1) and (G_2, D_2) have the same *central type* provided $G_1/Z(G_1)$ and $G_2/Z(G_2)$ are isomorphic as 3-transposition groups. By Lemma 2.1(iii), two connected 3-transpositions groups have the same central type if and only if their diagrams are isomorphic.

Finite connected 3-transposition groups (G, D) such that $O_2(G)O_3(G) \leq Z(G)$ were classified by Fischer in [5]. Basic examples of such groups are the following: the symmetric groups $\mathrm{Sym}(m)$ with $m = 2$ or $m \geq 5$ and D being the set of transpositions; the symplectic groups $\mathrm{Sp}_{2m}(2)$, where $m \geq 3$ and D is the set of symplectic transvections; the unitary groups $\mathrm{SU}_m(2)$, where $m \geq 4$ and D is the set of unitary transvections; the orthogonal groups $\mathrm{O}^{\epsilon}_{2m}(2)$, where D is the set of orthogonal transvections, $m \geq 3$, and either $\epsilon = +$ if the Witt index equals m or $\epsilon = -$ if the Witt index equals $m - 1$; five groups of sporadic type (in notation of [1]): Fi_{22}, Fi_{23}, Fi_{24}, $\mathrm{P}\Omega^+_8(2) : \mathrm{Sym}(3)$, $\mathrm{P}\Omega^+_8(3) : \mathrm{Sym}(3)$. There are two more infinite series of 3-transposition groups in Fischer's classification paper: ${}^+\Omega^{\pm}_m(3)$, where $m \geq 5$. Consider an orthogonal group $\mathrm{O}^{\epsilon}_{2m}(3)$ corresponding to a symmetric bilinear form $b(\cdot, \cdot)$ over a field of order 3, where ϵ is defined as above depending on the Witt index. The group ${}^+\Omega^{\epsilon}_{2m}(3)$ is then the subgroup of $\mathrm{O}^{\epsilon}_{2m}(3)$ generated by the 3-transposition conjugacy class D^+ of all reflections $d = \sigma_x$ with centers x having $b(x, x) = 1$. The corresponding odd degree group ${}^+\Omega^{\epsilon}_{2m-1}(3)$ is found within ${}^+\Omega^{\epsilon}_{2m}(3)$ as $\langle D_d \rangle$, where $D_d = C_{D^+}(d) \setminus \{d\}$ for an arbitrary 3-transposition $d \in D^+$. In what follows, we do not need explicit group constructions, but only some properties of their diagrams.

Cuypers and Hall extended Fischer's classification in [15] by dropping the assumptions $O_2(G)O_3(G) \leq Z(G)$ and finiteness of G. As a consequence, they showed that every 3-transposition group is locally finite, i.e., every finite subset of the group generates a finite subgroup. Naturally, the groups in the general classification are extensions of the groups obtained by Fischer. For the connected 3-transposition group (G, D), we write $p^{\bullet h}$ with $p \in \{2, 3\}$, for a normal p-subgroup N with

$|D \cap dN| = p^h$ for all $d \in D$. We give a simplified formulation of the classification which is taken from [14] and sufficient for our purposes.

Theorem 2.2. (Cuypers–Hall Classification Theorem)[14, Theorem 5.3] *Let (G, D) be a finite connected 3-transposition group. Then for integral m and h, the group G has one of the central types below. Furthermore, for each G, the generating class D is uniquely determined up to an automorphism of G.*

PR1. $3^{\bullet h} : \mathrm{Sym}(2)$, *all* $h \geq 1$;
PR2(a). $2^{\bullet h} : \mathrm{Sym}(m)$, *all* $h \geq 0$, *all* $m \geq 4$;
PR2(b). $3^{\bullet h} : \mathrm{Sym}(m)$, *all* $h \geq 1$, *all* $m \geq 4$;
PR2(c). $3^{\bullet h} : 2^{\bullet 1} : \mathrm{Sym}(m)$, *all* $h \geq 1$, *all* $m \geq 4$;
PR2(d). $4^{\bullet h} : 3^{\bullet 1} : \mathrm{Sym}(m)$, *all* $h \geq 1$, *all* $m \geq 4$;
PR3. $2^{\bullet h} : \mathrm{O}^{\epsilon}_{2m}(2)$, $\epsilon = \pm$, *all* $h \geq 0$, *all* $m \geq 3$, $(m, \epsilon) \neq (3, +)$;
PR4. $2^{\bullet h} : \mathrm{Sp}_{2m}(2)$, *all* $h \geq 0$, *all* $m \geq 3$;
PR5. $3^{\bullet h +}\Omega^{\epsilon}_{m}(3)$, $\epsilon = \pm$, *all* $h \geq 0$, *all* $m \geq 5$;
PR6. $4^{\bullet h}\,\mathrm{SU}_m(2)'$, *all* $h \geq 0$, *all* $m \geq 3$;
PR7(a-e). Fi_{22}, Fi_{23}, Fi_{24}, $\mathrm{P}\Omega^{+}_{8}(2) : \mathrm{Sym}(3)$, $\mathrm{P}\Omega^{+}_{8}(3) : \mathrm{Sym}(3)$;
PR8. $4^{\bullet h} : (3 \cdot {}^{+}\Omega^{-}_{6}(3))$, *all* $h \geq 1$;
PR9. $3^{\bullet h} : (2 \times \mathrm{Sp}_6(2))$, *all* $h \geq 1$;
PR10. $3^{\bullet h} : (2 \cdot \mathrm{O}^{+}_{8}(2))$, *all* $h \geq 1$;
PR11. $3^{\bullet 2h} : (2 \times \mathrm{SU}_5(2))$, *all* $h \geq 1$;
PR12. $3^{\bullet 2h} : 4^{\bullet 1} : \mathrm{SU}_3(2)'$, *all* $h \geq 1$.

Remark 2.3. The notation **PRk** comes from [1], here P means **P**arabolic and R means **R**eflections. These abbreviations reflect how the groups arise in the classification.

In Theorem 2.2, we follow notation from [14], in particular $A : B$ means a split group extension with normal subgroup A, while $A \cdot B$ is a nonsplit group extension with normal subgroup A and quotient B. We write AB indicating that A is a normal subgroup while B is the quotient, but the extension may or may not be split.

Let V be a nonempty set and (V) a graph with V as a vertex set. The $(0,1)$-adjacency matrix of the graph will be denoted $\mathrm{AMat}((V))$, and the spectrum of the graph is the (ordered) spectrum of $\mathrm{AMat}((X))$: $\mathrm{Spec}((X)) = ((\ldots, r_i, \ldots))$.

Suppose that (G, D) is a connected 3-transposition group. Hall and Shpectorov determined in [14] the spectrum of the diagram (D) in all cases of Theorem 2.2.

Before formulating their result, it is necessary to introduce some notation and conventions.

Clearly, the all-one vector 1 is an eigenvector of AMat((V)) with eigenvalue k if and only if (V) is regular of degree k. If (V) is connected, then the Perron–Frobenius Theorem implies that k is the largest eigenvalue and the corresponding eigenspace has dimension one. Following [14], we list k first in the spectrum and separate it from the rest of eigenvalues by a semicolon. We use the convention that $[t]^c$ indicates an eigenvalue t of multiplicity c and $[t]^*$ means that the eigenvalue t has multiplicity such that the total multiplicity of all eigenvalues is equal to the size of V.

Theorem 2.4. [14] *Let (G, D) be a finite 3-transposition group from the conclusion of Theorem 2.2. Then the size of (D) and its spectrum are as in the second and third columns of Table 1, respectively.*

Label	Size	Spectrum
PR1	3^h	$((3^h-1;[-1]^{3^h-1}))$
PR2(a)	$2^{h-1}m(m-1)$	$((2^{h+1}(m-2);[2^h(m-4)]^{m-1},[0]^*,$ $[-2^{h+1}]^{m(m-3)/2}))$
PR2(b)	$3^hm(m-1)/2$	$((3^h(2m-3)-1;[3^h(m-3)-1]^{m-1},[-1]^*,$ $[-3^h-1]^{m(m-3)/2}))$
PR2(c)	$3^hm(m-1)$	$((3^h(4m-7)-1;[3^h(2m-7)-1]^{m-1},$ $[3^h-1]^{m(m-1)/2},[-1]^*,[-3^{h+1}-1]^{m(m-3)/2}))$
PR2(d)	$3(2^{2h-1})m(m-1)$	$((4^h(6m-10);[4^h(3m-10)]^{m-1},[0]^*,$ $[-4^h]^{m(m-1)},[-4^{h+1}]^{m(m-3)/2}))$
PR3 $\epsilon=+$	$2^h(2^{2m-1}-2^{m-1})$	$((2^h(2^{2m-2}-2^{m-1});[2^{h+m-1}]^{(2^m-1)(2^{m-1}-1)/3},$ $[0]^*,[-2^{h+m-2}]^{(2^{2m}-4)/3}))$
$\epsilon=-$	$2^h(2^{2m-1}+2^{m-1})$	$((2^h(2^{2m-2}+2^{m-1});[2^{h+m-2}]^{(2^{2m}-4)/3},$ $[0]^*,[-2^{h+m-1}]^{(2^m+1)(2^{m-1}+1)/3}))$
PR4	$2^h(2^{2m}-1)$	$((2^{2m-1+h};[2^{m-1+h}]^{2^{2m-1}-2^{m-1}-1},$ $[0]^*,[-2^{h+m-1}]^{2^{2m-1}+2^{m-1}-1}))$
PR5		
odd $m\geq 5$, $\epsilon=+$	$3^h(3^{m-1}-3^{(m-1)/2})/2$	$((3^h(3^{m-2}-2\cdot 3^{(m-3)/2})-1;[3^{(m-3)/2+h}-1]^f,$ $[-1]^*,[-3^{(m-3)/2+h}-1]^g))$ for $f=(3^{m-1}-1)/4$ and $g=(3^{m-1}-1-2(3^{(m-1)/2}+1))/4$
odd $m\geq 5$, $\epsilon=-$	$3^h(3^{m-1}+3^{(m-1)/2})/2$	$((3^h(3^{m-2}+2\cdot 3^{(m-3)/2})-1;[3^{(m-3)/2+h}-1]^f,$ $[-1]^*,[-3^{(m-3)/2+h}-1]^g))$ for $f=(3^{m-1}-1+2(3^{(m-1)/2}-1))/4$ and $g=(3^{m-1}-1)/4$
even $m\geq 6$, $\epsilon=+$	$3^h(3^{m-1}-3^{(m-2)/2})/2$	$((3^{m-2+h}-1;[3^{(m-4)/2+h}-1]^f,$ $[-1]^*,[-3^{(m-2)/2+h}-1]^g))$ for $f=(3^m-9)/8$ and $g=(3^{m/2}-1)(3^{(m-2)/2}-1)/8$
even $m\geq 6$, $\epsilon=-$	$3^h(3^{m-1}+3^{(m-2)/2})/2$	$((3^{m-2+h}-1;[3^{(m-2)/2+h}-1]^f,$ $[-1]^*,[-3^{(m-4)/2+h}-1]^g))$ for $f=(3^{m/2}+1)(3^{(m-2)/2}+1)/8$

		and $g = (3^m - 9)/8$
PR6		
even $m \geq 4$	$4^h(2^{2m-1} + 2^{m-1} - 1)/3$	$((2^{2h+2m-3}; [2^{2h+m-3}]^f, [0]^*, [-2^{2h+m-2}]^g))$ for $f = 8(2^{2m-3} - 2^{m-2} - 1)/9$ and $g = 4(2^{2m-3} + 7(2^{m-3}) - 1)/9$
odd $m \geq 3$	$4^h(2^{2m-1} - 2^{m-1} - 1)/3$	$((2^{2h+2m-3}; [2^{2h+m-2}]^f, [0]^*, [-2^{2h+m-3}]^g))$ for $f = 4(2^{2m-3} - 7(2^{m-3}) - 1)/9$ and $g = 8(2^{2m-3} + 2^{m-2} - 1)/9$
PR7(a)	3510	$((2816; [8]^{3080}, [-64]^{429}))$
PR7(b)	31671	$((28160; [8]^{30888}, [-352]^{782}))$
PR7(c)	306936	$((275264; [80]^{249458}, [-352]^{57477}))$
PR7(d)	360	$((296; [8]^{105}, [-4]^{252}, [-64]^2))$
PR7(e)	3240	$((2888; [8]^{2457}, [-28]^{780}, [-352]^2))$
PR8	$126 \cdot 4^h$	$((5 \cdot 4^{h+2}; [2^{2h+3}]^{35}, [0]^*, [-4^{h+1}]^{90}))$
PR9	$63 \cdot 3^h$	$((11 \cdot 3^{h+1} - 1; [5 \cdot 3^h - 1]^{27}, [-1]^*, [-3^{h+1} - 1]^{35}))$
PR10	$120 \cdot 3^h$	$((19 \cdot 3^{h+1} - 1; [3^{h+2} - 1]^{35}, [-1]^*, [-3^{h+1} - 1]^{84}))$
PR11	$165 \cdot 3^{2h}$	$((43 \cdot 3^{2h+1} - 1; [3^{2h+2} - 1]^{44}, [-1]^*, [-3^{2h+1} - 1]^{120}))$
PR12	$36 \cdot 3^{2h}$	$((11 \cdot 3^{2h+1} - 1; [3^{2h} - 1]^{27}, [-1]^*, [-3^{2h+1} - 1]^8))$

Table 1: Spectra of diagrams

Before finishing this section, we introduce an alternative view of the elements of the set of 3-transpositions. The *Fischer space* of a 3-transposition group (G, D) is a point-line geometry $\Gamma(G, D)$ whose point set is D and where distinct points c and d are collinear if and only if $|cd| = 3$. Observe that any two collinear points c and d lie in a unique common line, which consists of c, d, and the third point $e = c^d = d^c$. It follows from the definition that the connected components of the Fischer space coincide with the conjugacy classes of G contained in D. In particular, the Fischer space is connected if and only if the diagram (D) is connected.

3. Preliminaries: Jordan and Matsuo algebras

Throughout this section we assume that $\mathbb{F}$ is a field of characteristic not 2. Recall that a commutative $\mathbb{F}$-algebra J is called Jordan if any two of its elements x and y satisfy the identity $(x^2y)x - x^2(yx) = 0$. If x, y, z are three elements in an $\mathbb{F}$-algebra, then their associator is $(x, y, z) := (xy)z - x(yz)$. The associator is convenient when writing identities, for example the Jordan identity $(x^2y)x - x^2(yx) = 0$ can be rewritten as $(x^2, y, x) = 0$. To show that an algebra is Jordan we will use the linearized Jordan identity.

Lemma 3.1. [21, Proposition 1.8.5(1)] *Let $\mathbb{F}$ be a field of characteristic not* 2 *and* 3. *Then a commutative $\mathbb{F}$-algebra J is a Jordan algebra if and only if $(xz, y, w) + (zw, y, x) + (wx, y, z) = 0$ for all elements x, y, z, w in J.*

Suppose that A is an $\mathbb{F}$-algebra and $a \in A$. For an element $\lambda \in \mathbb{F}$ denote by $A_\lambda(a)$, the λ-eigenspace of the (left) adjoint operator of a: $A_\lambda(a) = \{b \in A \mid ab = \lambda b\}$.

Lemma 3.2. (Peirce decomposition)[21, Section 6.1] *Suppose that e is an idempotent in a Jordan algebra J. Then the following statements hold.*

(i) $J = J_1(e) \oplus J_0(e) \oplus J_{1/2}(e)$;

(ii) $J_1(e) + J_0(e)$ *is a subalgebra of J and, moreover,* $J_1(e)^2 \subseteq J_1(e)$, $J_0(e)^2 \subseteq J_0(e)$, *and* $J_1(e)J_0(e) = (0)$;

(iii) $J_{1/2}(e)^2 \subseteq J_0(e) + J_1(e)$ *and* $J_{1/2}(e)(J_0(e) + J_1(e)) \subseteq J_{1/2}(e)$.

Suppose that $\eta \in \mathbb{F}$ and $\eta \neq 0, 1$. Fix a 3-transposition group (G, D). The *Matsuo algebra* $M_\eta(G, D)$ over $\mathbb{F}$, corresponding to (G, D) and η, has the point set D as its basis. Multiplication is defined on D as follows:

$$c \cdot d = \begin{cases} c, & \text{if } c = d; \\ 0, & \text{if } |cd| = 2; \\ \frac{\eta}{2}(c + d - e), & \text{if } |cd| = 3 \text{ and } e = c^d = d^c. \end{cases}$$

We use the dot for the algebra product to distinguish it from the multiplication in the group G. It turns out that the assertions of Lemma 3.2 hold for every Matsuo algebra $M_\eta(G, D)$. This means that Matsuo algebras are examples of axial algebras of Jordan type η (see [12, Theorem 6.4] for details).

The Matsuo algebra $M = M_\eta(G, D)$ admits a bilinear symmetric form $(\cdot, \cdot)$ that associates with the algebra product, i.e., $(u \cdot v, w) = (u, v \cdot w)$ for arbitrary algebra elements u, v, and w (so-called Frobenius form). This form is given on the basis D by the following:

$$(c, d) = \begin{cases} 1, & \text{if } c = d; \\ 0, & \text{if } |cd| = 2; \\ \frac{\eta}{2}, & \text{if } |cd| = 3. \end{cases}$$

The radical $M^\perp$ of the form is the set of elements orthogonal to M:

$$M^\perp = \{u \in M \mid (u, v) = 0 \text{ for all } v \in M\}.$$

Since the form associates with the algebra product, $M^\perp$ is an ideal in M.

One can define a graph on the set D, called the *projection graph*, where distinct involutions d and e are adjacent whenever $(d, e) \neq 0$. By the definition of the form on M, if G is connected, then this graph is connected. It follows from [18, Corollary 4.15] that in this case $M^\perp$ includes all proper ideals of M.

Proposition 3.3. *Suppose that $M = M_\eta(G, D)$ is a Matsuo algebra. If the diagram (D) is connected, then each ideal of M lies in the radical $M^\perp$.*

It turns out that the values of η for which the radical is nonzero are easier to find in terms of the adjacency matrix $\mathrm{AMat}((D))$ than in terms of the form. The following statement will be used to calculate the dimension of the radical.

Lemma 3.4. [1] *Let $\mathbb{F}$ be a field of characteristic not 2 and $\eta \in \mathbb{F} \setminus \{0,1\}$. Suppose that (G,D) is a 3-transposition group and $M = M_\eta(G,D)$ is the Matsuo algebra for (G,D). Fix some order of elements of D and denote by $\mathcal{M}$ the Gram matrix of the Frobenius form of M with respect to D and by $\mathcal{A}$ the adjacency matrix $\mathrm{AMat}((D))$. Then ζ is an eigenvalue of $\mathcal{A}$ with multiplicity k if and only if $1+\frac{\eta}{2}\zeta$ is an eigenvalue of $\mathcal{M}$ with multiplicity k.*

Proof. By definitions of $\mathcal{M}$ and $\mathcal{A}$, we see that $\mathcal{M} = I + \frac{\eta}{2}\mathcal{A}$. Now the statement follows from the fact that the Jordan normal forms of these matrices are related by a similar equation. □

Corollary 3.5. *Let $\mathcal{M}$ and $\mathcal{A}$ be as in Lemma 3.4. If $\eta = \frac{1}{2}$, then the multiplicity of 0 in the spectrum of $\mathcal{M}$ is equal to that of -4 in the spectrum of $\mathcal{A}$.*

Proof. From the bijection between eigenvalues of $\mathcal{M}$ and $\mathcal{A}$ in Lemma 3.4, we find that 0 corresponds to ζ such that $1 + \frac{1}{4}\zeta = 0$, that is $\zeta = -4$. □

De Medts and Rehren classified Matsuo algebras that are Jordan algebras in [2]. Yabe corrected a gap in the case when the characteristic of the field equals 3 [25]. For simplicity and since we are mainly interested in characteristic zero, we state the result when the field characteristic is not three.

Theorem 3.6. [2, Main Theorem] *Let $\mathbb{F}$ be a field, $\mathrm{char}(\mathbb{F}) \neq 2,3$, and let J be a Jordan algebra over $\mathbb{F}$ which is also a Matsuo algebra. Then J is a direct product of Matsuo algebras $J_i = M_{1/2}(G_i, D_i)$ corresponding to 3-transposition groups (G_i, D_i), where for each i,*

- (i) *either $G_i = \mathrm{Sym}(n)$, and J_i is the Jordan algebra of $n \times n$ symmetric matrices over F with zero row sums;*
- (ii) *$G_i \simeq 3^2 : 2$, and J_i is the Jordan algebra of hermitian 3×3 matrices over the quadratic étale extension $E = \mathbb{F}[x]/(x^2+3)$.*

[1] This lemma was mentioned by S. Shpectorov in his talk at Axial seminar, 12/10/21, https://sites.google.com/view/axial-algebras/home

4. Preliminaries: wreath product

In this section, we discuss 3-transposition groups that correspond to types **PR2(a-e)** in Theorem 2.2. All these groups can be constructed from the wreath product of a group whose elements have orders not exceeding 3 and a symmetric group.

Denote the base group of the wreath product $G = T \operatorname{wr} \mathrm{Sym}(n)$ by B, i.e., $B = T^n$. The natural injection ι_i of T as the i-th direct factor T_i of B is given by $\iota_i(t) = t_i$, where $1 \leq i \leq n$. The projection π_i of B onto T induced by the i-th factor is given by $\pi_i(b) = b(i)$. We identify $\mathrm{Sym}(n)$ with the complement to B in G which acts naturally on the indices from $\{1, \dots, n\}$. Let $Wr(T, n)$ be the subgroup $\langle d^G \rangle$ of G, where d is a transposition of the complement to B. Note that the factor group $G/Wr(T, n)$ is isomorphic to the abelian group T/T', in particular $Wr(T, n)$ can be a proper subgroup of G. The following statement describes when d^G is a class of 3-transpositions.

Proposition 4.1. [26, Theorem 6], [10, Prop. 8.1] *Suppose that T is a finite group and $G = T \operatorname{wr} \mathrm{Sym}(n)$. Fix a transposition d of* $\mathrm{Sym}(n)$. *Then d^G is a class of 3-transpositions in G if and only if each element of T has order* 1, 2, *or* 3.

Note that the groups T with restrictions as in the proposition were classified in [23]. The next two lemmas are well known and describe how we deal with points and lines of the Fischer space of $Wr(T, n)$.

Lemma 4.2. [19, Lemma 3.2] *Consider the wreath product $G = T \operatorname{wr} \mathrm{Sym}(n)$ and a transposition $d \in \mathrm{Sym}(n)$. Then d^G consists of elements $t_i t_j^{-1}(i, j)$, where $t \in T$ and $1 \leq i < j \leq n$.*

Notation. We write $t.(i, j)$ for the 3-transposition $t_i t_j^{-1}(i, j)$ from Lemma 4.2. Since $t.(i, j) = t^{-1}.(j, i)$, we will usually assume that $i < j$.

Lemma 4.3. [19, Lemma 3.3] *Suppose that each element of T has order* 1, 2, *or* 3. *Then each line of the Fischer space of $Wr(T, n)$ coincides with one of the following sets.*

(i) $\{t.(i, j), s.(j, k), ts.(i, k)\}$, *where $s, t \in T$ and $1 \leq i < j < k \leq n$;*
(ii) $\{t.(i, j), s.(i, j), st^{-1}s.(i, j)\}$, *where $s, t \in T$, $|st^{-1}| = 3$, and $1 \leq i < j \leq n$.*

Now we focus on 3-transposition groups $Wr(p, n)$, where p means the cyclic group of order $p \in \{2, 3\}$. Following [19] and [6], we will use the following descriptions of the Fischer spaces of these groups. Let n be an integer and $n \geq 3$. For $p \in \{2, 3\}$, consider the n-dimensional permutational module V of $\mathrm{Sym}(n)$ over $\mathbb{F}_p$. Let e_i,

$i \in \{1, \ldots, n\}$, be a basis of V permuted by $\mathrm{Sym}(n)$. Then the natural semidirect product $V \rtimes \mathrm{Sym}(n)$ is isomorphic to $p \,\mathrm{wr}\, \mathrm{Sym}(n)$. Denote the $(n-1)$-dimensional 'sum-zero' submodule of V by U. Then $Wr(p,n)$ is isomorphic to the natural semidirect product $U \rtimes \mathrm{Sym}(n)$. Note that, for $p = 2$ and even n, U contains a 1-dimensional 'all-one' submodule, which is the center of $Wr(2,n)$. When $p = 3$, U is irreducible. In both cases, U is the unique minimal non-central normal subgroup of $Wr(p,n)$ and $Wr(p,n)/U \simeq \mathrm{Sym}(n)$. Since $\mathrm{Sym}(n)$ does not have proper factor groups containing commuting involutions, we conclude that, up to the center, groups $Wr(p,n)$ have no other factors that are 3-transposition groups. Now we describe the Fischer spaces of these groups.

Assume that $p = 2$. It follows from Lemmas 4.2 and 4.3 that the Fischer space of $Wr(2,n) = U : Sym(n)$ consists of $n(n-1)$ points: $b_{i,j} = (i,j)$ and $c_{i,j} = (e_i+e_j)(i,j)$, for $1 \leq i < j \leq n$; and n^2 lines, where each 'b' line $\{b_{i,j}, b_{i,k}, b_{j,k}\}$, $1 \leq i < j < k \leq n$, is complemented by three 'bc' lines $\{b_{i,j}, c_{i,k}, c_{j,k}\}$, $\{b_{i,k}, c_{i,j}, c_{j,k}\}$, and $\{b_{j,k}, c_{i,j}, c_{i,k}\}$.

Assume that $p = 3$. By Lemma 4.2, for each pair i and j with $1 \leq i < j \leq n$, we have three points: $b_{i,j} = (i,j) = b_{j,i}$, $c_{i,j} = (e_i - e_j)(i,j)$ and $c_{j,i} = (e_j - e_i)(i,j)$. Consequently, the Fischer space has $\frac{3n(n-1)}{2}$ points. By Lemma 4.3, the lines are of several types. First, for each $1 \leq i < j \leq n$, the triple (1) $\{b_{i,j}, c_{i,j}, c_{j,i}\}$ is a line. Secondly, for distinct i, j, and k in $\{1, \ldots, n\}$, the triples (2) $\{b_{i,j}, b_{i,k}, b_{j,k}\}$, (3) $\{b_{i,j}, c_{i,k}, c_{j,k}\}$, (4) $\{b_{j,k}, c_{i,j}, c_{i,k}\}$, and (5) $\{c_{i,j}, c_{j,k}, c_{k,i}\}$ are lines.

Using the descriptions of Fischer spaces, we find bases of radicals for the corresponding Matsuo algebras.

Lemma 4.4. *Let $G = Wr(p,n)$, where $p \in \{2,3\}$ and $n \geq 4$. Denote by D the corresponding 3-transposition set and by M the Matsuo algebra $M_{1/2}(G,D)$. Then* $\dim M^{\perp} = \frac{n(n-3)}{2}$ *and the following assertions hold.*

(i) *If $p = 2$, then $M^{\perp}$ is the span of elements*

$$b_{i,j} - b_{i,l} - b_{j,k} + b_{k,l} + c_{i,j} - c_{i,l} - c_{j,k} + c_{k,l},$$

where i, j, k, l are distinct elements of $\{1, \ldots, n\}$ and i is less than j, k, l.

(ii) *If $p = 3$, then $M^{\perp}$ is the span of elements*

$$b_{i,j} - b_{i,l} - b_{j,k} + b_{k,l} + c_{i,j} - c_{i,l} - c_{j,k} + c_{k,l} + c_{j,i} - c_{l,i} - c_{k,j} + c_{l,k},$$

where i, j, k, l are distinct elements of $\{1, \ldots, n\}$ and i is less than j, k, l.

Proof. By Corollary 3.5, the dimension of $M^{\perp}$ is equal to the multiplicity of -4 in the spectrum of the diagram (D). According to [1, Example PR2], if $p = 2$, then G corresponds to the type **PR2(a)** in Theorem 2.2, while if $p = 3$, then G corresponds to the type **PR2(b)**. In both cases the parameter h equals 1. It follows from Theorem 2.4 that -4 has multiplicity $\frac{n(n-3)}{2}$ in Spec$((D))$. This implies that $\dim M^{\perp} = \frac{n(n-3)}{2}$.

For arbitrary distinct integers i, j, k, l such that $1 \leq i, j, k, l \leq n$ and i is less than j, k, l denote

$$r(i,j)(k,l) = b_{i,j} - b_{i,l} - b_{j,k} + b_{k,l} + c_{i,j} - c_{i,l} - c_{j,k} + c_{k,l} \text{ if } p = 2,$$

and

$$r(i,j)(k,l) = b_{i,j} - b_{i,l} - b_{j,k} + b_{k,l} + c_{i,j} - c_{i,l} - c_{j,k} + c_{k,l} + c_{j,i} - c_{l,i} - c_{k,j} + c_{l,k} \text{ if } p = 3.$$

We claim that each $r(i,j)(k,l)$ belongs to $M^{\perp}$. By symmetry of indices, it suffices to show this for $r(1,2)(3,4)$. Now we verify that each 3-transposition of D is orthogonal to $r(1,2)(3,4)$ with respect to the Frobenius form. Suppose that $p = 3$. Take a 3-transposition $x_{i,j} \in D$, where $x \in \{b, c\}$. First we consider the case $i, j \in \{1, 2, 3, 4\}$. If $x_{i,j} \in \{b_{1,2}, c_{1,2}, c_{2,1}\}$, then

$$(x_{i,j}, b_{1,2} + c_{1,2} + c_{2,1}) = 1 + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}, (x_{i,j}, b_{3,4} + c_{3,4} + c_{4,3}) = 0,$$
$$(x_{i,j}, -b_{1,4} - b_{2,3} - c_{1,4} - c_{4,1} - c_{3,4} - c_{4,3}) = -6 \cdot \frac{1}{4} = -\frac{3}{2}.$$

Therefore, we infer that $(x_{i,j}, r(1,2)(3,4)) = 0$. Similarly, we see that

$$(x_{i,j}, r(1,2)(3,4)) = 0$$

when $x_{i,j} \in \{b_{3,4}, c_{3,4}, c_{4,3}, b_{1,4}, c_{1,4}, c_{4,1}, b_{2,3}, c_{2,3}, c_{3,2}\}$.

Let $x_{i,j} \in \{b_{1,3}, c_{3,1}, c_{3,1}, b_{2,4}, c_{2,4}, c_{4,2}\}$. Then

$$(x_{i,j}, b_{1,2} + c_{1,2} + c_{2,1}) = (x_{i,j}, b_{3,4} + c_{3,4} + c_{4,3}) = \frac{3}{4}, (x_{i,j}, -b_{1,4} - c_{1,4} - c_{4,1})$$
$$= (x_{i,j}, b_{2,3} + c_{2,3} + c_{3,2}) = -\frac{3}{4}.$$

Therefore, we see that $(x_{i,j}, r(1,2)(3,4)) = 0$. Clearly, if $i, j \notin \{1, 2, 3, 4\}$, then $(x_{i,j}, r(1,2)(3,4)) = 0$. So it remains to consider the case when $|\{i,j\} \cap \{1,2,3,4\}| = 1$. Note that for each integer $k \in \{1, 2, 3, 4\}$, exactly six out of the twelve terms in $r(1,2)(3,4)$ contain k as an index, moreover, three of these six are included in the expression with a plus sign and three with a minus sign. This implies that $x_{i,j}$ is orthogonal to $r(1,2)(3,4)$. The case $p = 2$ can be considered in a similar manner.

Now we present $\frac{n(n-3)}{2}$ linearly independent elements among $\{r(i,j)(k,l)\}$. Consider two sets of elements of D: $\mathcal{B}_1 = \{r(i,j)(n-1,n) \mid 1 \leq i < j < n-1\}$ and

$\mathcal{B}_2 = \{r(1, n-1)(i, n) \mid i \neq 1, n-1, n\}$. Suppose that the set $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly dependent in M. Note that if (i, j) is a pair with $1 \leq i < j < n-1$, then $r(i, j)(n-1, n)$ is the only element of $\mathcal{B}_1 \cup \mathcal{B}_2$ including $b_{i,j}$ in its expression. It follows that if a non-trivial linear combination of elements of $\mathcal{B}_1 \cup \mathcal{B}_2$ is equal to 0, then only elements from $\mathcal{B}_2$ have non-zero coefficients. On the other hand, if $i \neq 1, n-1, n$, then $r(1, n-1)(i, n)$ is the only element in $\mathcal{B}_2$ including $b_{i,n}$ in its expression and hence $\mathcal{B}_2$ is linearly independent; we arrive at a contradiction. Thus, the set $\mathcal{B}_1 \cup \mathcal{B}_2$ is linearly independent. Since $|\mathcal{B}_1| = \frac{(n-2)(n-3)}{2}$, $|\mathcal{B}_2| = n-3$, and $\mathcal{B}_1 \cap \mathcal{B}_2 = \varnothing$, we find $\frac{n(n-3)}{2}$ linearly independent elements in $M^{\perp}$. This implies that the set $\{r(i, j)(k, l)\}$ spans the radical of M and as a basis we can take the elements of $\mathcal{B}_1 \cup \mathcal{B}_2$. □

5. Proof of the main theorem

In this section, we prove Theorem 1. Throughout, we suppose that $\mathbb{F}$ is a field of characteristic zero. First we consider the case when the parameter η in Matsuo algebra is not equal to $\frac{1}{2}$.

Lemma 5.1. *Suppose that $\eta \neq \frac{1}{2}$ and $M = M_\eta(G, D)$ is the Matsuo algebra for a finite connected 3-transposition group (G, D). A factor of M by an ideal $I \neq M$ is a Jordan algebra if and only if one of the following statements holds.*

(i) *G is the cyclic group of order 2 and $I = (0)$;*

(ii) *$\eta = 2$, the product of every two distinct elements of D has order 3, I is the span of elements $d - e$, where d and e run over D. In this case M/I is one-dimensional.*

Proof. Clearly, if $D = \{d\}$, then G is the cyclic group of order 2 and M is generated by d. So M is associative and one-dimensional. Therefore, we can assume that $|D| \geq 2$.

Assume that M/I is a Jordan algebra. Take any $c \in D$. Since (D) is connected and $|D| \geq 2$, there exists $d \in D$ such that $|cd| = 3$. If $x \in M$, then denote by $\overline{x}$ the image of x in M/I. Note that $\overline{c}$ is an idempotent in M/I. Denote by e the third point on the line through c and d in the Fischer space $\Gamma(G, D)$. Then $e \cdot (c - d) = \frac{\eta}{2}(e + c - d - e - d + c) = \eta(c - d)$ and hence $\overline{e} \cdot (\overline{c} - \overline{d}) = \eta(\overline{c} - \overline{d})$. It follows from Lemma 3.2 that $\overline{c} = \overline{d}$. Since c is an arbitrary element of D and (D) is connected, we infer that M/I is one-dimensional and spanned by $\overline{d}$ for each $d \in D$. Suppose that there exist $d, e \in D$ such that $|de| = 2$. Then $0 = \overline{d \cdot e} = \overline{d} \cdot \overline{e} = \overline{d}^2 = \overline{d}$ and hence $d \in I$. All elements of D are conjugated in G, so this is true for all elements of D; a contradiction. Therefore, the product of any two distinct elements of D has order 3. It remains to show that $\eta = 2$. Suppose that c and d are distinct elements in D. By Proposition 3.3, $I \subseteq M^{\perp}$ and hence $c - d \in M^{\perp}$. On the other hand, $(c, c-d) = 1 - \frac{\eta}{2}$ and hence $\eta = 2$.

Conversely, suppose that $\eta = 2$ and the product of any two elements in D has order 3. We show that for every $c, d \in D$, it is true that $c - d \in M^{\perp}$. First, we see that $(c, c-d) = (d, c-d) = 1 - 1 = 0$. If $e \in D \setminus \{c, d\}$, then $(e, c) = (e, d) = 1$ and hence $(e, c-d) = 0$. Since $c-d$ is orthogonal to all elements in D with respect to the Frobenius form, we infer that $c - d \in M^{\perp}$. This implies that $M/M^{\perp}$ is 1-dimensional and the result follows. □

Matsuo algebras $M_{1/2}(G, D)$, where (G, D) is a 3-transposition group, that are Jordan algebras were classified in [2]. In particular, if (D) is connected, then $G \simeq \mathrm{Sym}(n)$ or has the same central type as the Frobenius group $3^2 : 2$. In view of Theorem 2.2, the symmetric group has type **PR2(a)** and $3^2 : 2$ has type **PR1**. It follows from Corollary 3.5 and Theorem 2.4 that $M_{1/2}(G, D)$ is simple in these cases. To prove Theorem 1 it remains to consider Matsuo algebras for $\eta = \frac{1}{2}$ whose radical is nontrivial.

Proposition 5.2. *Suppose that (G, D) is a finite connected 3-transposition group and the Matsuo algebra $M = M_{1/2}(G, D)$ has nontrivial radical $M^{\perp}$. Then $J = M/M^{\perp}$ is a Jordan algebra if and only if one of the following statements holds.*

1. $G \simeq 2^{\bullet 1} : \mathrm{Sym}(m)$, *where* $m \geq 4$ *and* $\dim J = \frac{m(m+1)}{2}$*;*
2. $G \simeq 3^{\bullet 1} : \mathrm{Sym}(m)$, *where* $m \geq 4$ *and* $\dim J = m^2$*;*
3. $G \simeq O_8^+(2)$ *and* $\dim J = 36$*;*
4. $G \simeq O_6^-(2) \simeq {}^+\Omega_5^+(3)$ *and* $\dim J = 21$*;*
5. $G \simeq Sp_6(2)$ *and* $\dim J = 28$*;*
6. $G \simeq {}^+\Omega_6^-(3)$ *and* $\dim J = 36$*;*
7. $G \simeq 2 \times SU_4(2) \simeq {}^+\Omega_5^-(3)$ *and* $\dim J = 25$*;*
8. $G \simeq SU_5(2)$ *and* $\dim J = 45$*;*
9. $G \simeq 4^{\bullet 1} SU_3(2)'$ *and* $\dim J = 28$*.*

Proof. We sort out possibilities for G from Theorem 2.2. By Corollary 3.5, the dimension of $M^{\perp}$ equals the multiplicity of -4 in the spectrum of the diagram (D). Therefore, we need to find all G such that -4 is in the spectrum of (D). According to Table 1, G does not belong to types **PR1**, **PR2(c)**, **PR7(a, b, c, e)**, **PR8 – PR12**. Now we consider the remaining cases.

Assume that the type of G is **PR2(a)**. According to Table 1, we see that $-2^{h+1} = -4$ and hence $h = 1$. Therefore, $G = 2^{\bullet 1} : \mathrm{Sym}(m) \simeq Wr(2, m)$,

$|D| = m(m-1)$, $\dim M^\perp = \frac{m(m-3)}{2}$, and $\dim J = \frac{m(m-3)}{2}$. We claim that J is a Jordan algebra in this case. By Lemma 3.1, this is true if and only if all $a, b, c, d \in D$ satisfy the following:

$$w(a, b, c, d) = (a \cdot d, b, c) + (d \cdot c, b, a) + (c \cdot a, b, d) \in M^\perp.$$

We use the description of D as in Section 4, so each $a \in D$ is equal to some $x_{i,j}$, where $1 \leq i \neq j \leq m$ and $x \in \{'b', 'c'\}$. In this notation, expressions for elements a, b, c, d include no more than 8 distinct indices i, j, so we can consider a, b, c, and d as elements of $H_k = Wr(2, k)$ with $k \leq 8$ after renumbering indices in the corresponding elements $x_{i,j}$. Using GAP[2] [7], we verify that the element $w(a, b, c, d)$ for all 3-transpositions a, b, c, d from H_k lies in the radical of the Frobenius form of the Matsuo algebra for H_k, where $4 \leq k \leq 8$. Note that the following enlargement property is true for the radical in these cases: elements from Lemma 4.4 that span $M_k^\perp$ belong to $M_n^\perp$ for all $n \geq k$. This implies that $w(a, b, c, d) \in M^\perp$ for all $a, b, c, d \in D$; as claimed.

Assume that the type of G is **PR2(b)**. Then $-3^h - 1 = -4$ and hence $h = 1$. So $|D| = \frac{3m(m-1)}{2}$, $G = 3^{\bullet 1} : \mathrm{Sym}(m) \simeq Wr(3, m)$ and $\dim M^\perp = \frac{m(m-3)}{2}$. So $\dim J = \frac{3m(m-1)}{2} - \frac{m(m-3)}{2} = m^2$. We verify that J is a Jordan algebra in the same way as in the previous case. Namely, we use GAP to verify the linearized Jordan identity from Lemma 3.1 for all m with $4 \leq m \leq 8$. The general case follows from the description of a basis of $M^\perp$ in Lemma 4.4 since this basis satisfies the enlargement property with increasing m.

Assume that the type of G is **PR2(d)**. Then $-4^h = -4$ and hence $h = 1$. According to [1, Example PR2], G has the same central type as $Wr(\mathrm{Alt}(4), m)$. By the wreath product construction, we can assume that $Wr(\mathrm{Alt}(4), m)$ is a subgroup $Wr(\mathrm{Alt}(4), n)$ if $m \leq n$ and hence there is also an embedding of the corresponding Matsuo algebras. Clearly, if a factor of an algebra A by its ideal is a Jordan algebra, then all subalgebras of A also have factors that are Jordan algebras. Using GAP and Lemma 3.1, we verify that the factor algebra of the Matsuo algebra for $Wr(Alt(4), 4)$ by its radical is not a Jordan algebra. Therefore, this case is impossible.

Assume that the type of G is **PR3**. Recall that $m \geq 3$ and $(m, \epsilon) \neq (3, +)$. If $\epsilon = +$, then $-2^{h+m-2} = -4$. This implies that $h = 0$ and $m = 4$. Then $|D| = 2^7 - 2^3 = 120$, $\dim M^\perp = (2^8 - 4)/3 = 84$, and $\dim J = 36$. If $\epsilon = -$, then $-2^{h+m-1} = -4$, so $h = 0$ and $m = 3$. Therefore, we see that $|D| = 2^5 + 2^2 = 36$,

[2] All verifications in GAP related to this proof can be found at the following link:https://github.com/AlexeyStaroletov/AxialAlgebras/blob/master/JordanFactors/Groups

$\dim M^\perp = (2^3+1)(2^2+1)/3 = 15$, and $\dim J = 21$. Using GAP, we verify that in both cases J is a Jordan algebra.

Assume that the type of G is **PR4**. Then $-2^{h+m-1} = -4$. Since $m \geq 3$, we infer that $h = 0$ and $m = 3$. According to Table 2.4, we find that $|D| = 2^6 - 1 = 63$, $\dim M^\perp = 2^5 + 2^2 - 1 = 35$, $\dim J = 28$. Using GAP, we verify that J is Jordan.

Assume that the type of G is **PR5**. If m is odd, then $-3^{(m-3)/2+h} - 1 = -4$, so $m = 5$ and $h = 0$. According to [1, Example 1.5], it is true that ${}^+\Omega_5^-(3) \simeq 2\times SU_4(2)$ and ${}^+\Omega_5^+(3) \simeq O_6^-(2)$. The algebra J is considered in the corresponding cases for $G \in \{SU_4(2), O_6^-(2)\}$. Suppose that m is even. According to Table 2.4, we see that $\epsilon = -$ and $-3^{(m-4)/2+h} - 1 = -4$. This implies that $m = 6$ and $h = 0$. Then $|D| = (3^5+3^2)/2 = 126$, $\dim M^\perp = (3^6-9)/8 = 90$, and $\dim J = 36$. Using GAP, we verify that J is a Jordan algebra.

Assume that the type of G is **PR6**. If m is even, then $-2^{2h+m-2} = -4$, so $m = 4$ and $h = 0$. Therefore, $|D| = (2^7-1+2^3)/3 = 45$, $\dim M^\perp = 4(2^5-1+7\cdot 2)/9 = 20$, and hence $\dim J = 25$. Using GAP, we see that J is a Jordan algebra. If m is odd, then either $m = 5$ and $h = 0$ or $m = 3$ and $h = 1$. In the first case, we find that $|D| = (2^9 - 1 - 2^4)/3 = 165$, $\dim M^\perp = 8(2^7 - 1 + 2^3)/9 = 120$, and $\dim J = 45$. In the second case, $|D| = 4(2^5 - 1 - 2^2)/3 = 36$, $\dim M^\perp = 8(2^3 - 1 + 2)/9 = 8$, $\dim J = 28$. Using GAP, we see that J is a Jordan algebra in these cases.

Assume that the type of G is **PR7(d)**. In this case, $|D| = 360$, $\dim M^\perp = 252$, and $\dim J = 108$. We use the defining relations of G from the Appendix of [15] to do the calculations with J. Using GAP and Lemma 3.1, we verify that J is not a Jordan algebra in this case. □

Consider a Matsuo algebra $M = M_{1/2}(G, D)$. If we calculate the expression $(xz, y, w)+(zw, y, x)+(wx, y, z)$ from the linearized Jordan identity for all elements $x, y, z, w \in D$ and take the ideal I generated by all obtained elements in M, then I is the smallest ideal of M such that M/I is a Jordan algebra. Proposition 5.2 describes all G such that $M/I \neq 0$. We conclude this section with the following.

Problem 5.1. *In each case of Proposition 5.2 find the smallest ideal I such that M/I is Jordan and identify the corresponding Jordan factors.*

6. Octonion and Albert algebras

Throughout this section we suppose that $\mathbb{F}$ is a field of characteristic not 2 and 3. Recall that an octonion algebra over $\mathbb{F}$ is a composition algebra that has dimension 8 over $\mathbb{F}$. This means that it is a unital non-associative algebra $\mathbb{O}$ over $\mathbb{F}$ with a non-degenerate quadratic form N such that $N(xy) = N(x)N(y)$ for all x and

y in $\mathbb{O}$. For a given field $\mathbb{F}$, there may exist several octonion algebras, but if $\mathbb{F}$ is algebraically closed field, then all octonion algebras over $\mathbb{F}$ are isomorphic. We use the construction of an octonion algebra from [24, Section 4.3.2], which is a generalization of the real octonion algebra, also known as the Cayley numbers.

Take 7 mutually orthogonal square roots of -1, labeled $i_0, \dots, i_6$ (with subscripts understood modulo 7), subject to the condition that for each t, the elements i_t, i_{t+1}, i_{t+3} satisfy the same multiplication rules as i, j, and k (respectively) in the quaternion algebra: $ij = k = -ij, jk = i = -kj, ki = j = -ik$. Their pairwise products can be found in [24, Table 4.18].

Now we define the Albert algebra $A(\mathbb{F})$ corresponding to $\mathbb{O}$. Elements of $A(\mathbb{F})$ are 3×3 Hermitian matrices (i.e., matrices x such that $x^T = \overline{x}$) over the octonion algebra $\mathbb{O}$. For brevity let us define

$$(d, e, f \mid D, E, F) = \begin{pmatrix} d & F & \overline{E} \\ \overline{F} & e & D \\ E & \overline{D} & f \end{pmatrix},$$

where d, e, f lie in $\mathbb{F}$ and $^-$ denotes the octonion conjugation, i.e., the linear map fixing 1 and negating i_n for all n. Multiplication of such matrices makes sense, and the Jordan product $X \circ Y = \frac{1}{2}(XY + YX)$ for every $X, Y \in A(\mathbb{F})$ allows to consider $A(\mathbb{F})$ as a simple Jordan algebra.

Proposition 6.1. *The Albert algebra $A(\mathbb{F})$ is an axial $\mathbb{F}$-algebra of Jordan type $\frac{1}{2}$ generated by four primitive axes a, b, c, d, where*

$$a = \frac{1}{2}(1, 1, 0 \mid 0, 0, i_0) = \frac{1}{2}\begin{pmatrix} 1 & i_0 & 0 \\ -i_0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \frac{1}{2}(1, 0, 1 \mid 0, i_1, 0)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & -i_1 \\ 0 & 0 & 0 \\ i_1 & 0 & 1 \end{pmatrix}, c = \frac{1}{2}(0, 1, 1 \mid i_2, 0, 0) = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & i_2 \\ 0 & -i_2 & 1 \end{pmatrix},$$

$$d = \frac{1}{9}(1, 4, 4 \mid 4i_4, 2i_3, 2i_6) = \frac{1}{9}\begin{pmatrix} 1 & 2i_6 & -2i_3 \\ -2i_6 & 4 & 4i_4 \\ 2i_3 & -4i_4 & 4 \end{pmatrix}.$$

Proof. We claim that as a basis of $A(\mathbb{F})$ we can take the following 27 elements:

$$a, b, c, d, ab, ac, ad, bc, bd, cd, a(bc), b(ac), c(ab), a(bd), a(cd), b(ad), b(cd), \\ c(ad), c(bd), (ab)(cd), (ac)(bd), d(a(bc)), d(b(ac)), a(b(cd)), \\ (ab)(c(ad)), (ab)(c(bd)), (ac)(b(cd)).$$

All calculations are straightforward and can be done by hand or by computer[3].

Now one can write 27×27 matrix of coefficients of these 27 elements with respect to the standard basis of $A(\mathbb{F})$ (i.e., $(1, 0, 0 \mid 0, 0, 0), \dots, (0, 0, 0 \mid 0, 0, i_6)$). Using GAP, we find that the determinant of this matrix equals $\frac{1}{2^{78} \cdot 3^{36}}$ and hence 27 elements form a basis of $A(\mathbb{F})$.

Since $A(\mathbb{F})$ is known to be a Jordan algebra and a, b, c, d are its idempotents, Lemma 3.2 implies that each of these elements gives a Peirce decomposition of the algebra. According to [17, Section 4], an idempotent e in $A(\mathbb{F})$ is a primitive axis iff $Tr(e) = 1$, where Tr means the trace of e, i.e., the sum of elements on its diagonal. Therefore, we infer that a, b, c, d are primitive axes generating $A(\mathbb{F})$. This completes the proof of the proposition. □

Corollary 6.2. *If the characteristic of* $\mathbb{F}$ *equals zero, then* $A(\mathbb{F})$ *is not a factor of any of the Matsuo algebras.*

Proof. Suppose (G, D) is a 3-transposition group and $M = M_\eta(G, D)$ is its Matsuo algebra for $\eta \in \mathbb{F} \setminus \{0, 1\}$ such that $A(\mathbb{F})$ is a factor of M. Since $A(\mathbb{F})$ is simple, we can assume that (D) is connected. Now $\dim_{\mathbb{F}} A(\mathbb{F}) = 27$ and the result follows from Proposition 6.1 and Theorem 1. □

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