Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2016, 9(3), 178-184
Araştırma Makalesi

Erzincan University
Journal of Science and Technology
2016, 9(3), 178-184
Research Article

Generalizations of Modified Pell and Pell-Lucas Sequences and Their Generating Matrices and Some Sums

Ahmet DAŞDEMİR

Department of Mathematics, Faculty of Arts and Sciences, Kastamonu University, Kastamonu,Turkey

(Geliş Tarihi/Received: 26.02.2016, Kabul Tarihi/Accepted: 03.06.2016)

ABSTRACT

This paper deals with the generalizations of the usual Modified Pell and Pell-Lucas sequences and an investigation of their certain identities. For this purpose, the generalized Modified Pell and Pell-Lucas sequences are created, and appropriate initial conditions for them are presented. Some sum formulae consisting of the generalized Modified Pell and Pell-Lucas sequences are investigated. A very fundamental features corresponding to these new generalizations are obtained by using matrix representations. Especially, an important relationship between the generalized Modified Pell and Pell-Lucas sequence is discovered.

Keywords: Generalized Pell-Lucas sequence, Generalized Modified Pell sequence, Matrix representation, Recurrence relation.

Modifiye Pell ve Pell-Lucas Dizilerinin Genelleştirilmesi ve Onların Üreteç Matrisleri ve Bazı Toplamları

ÖZET

Bu çalışma alışılmış Modifiye Pell ve Pell-Lucas dizilerinin genelleştirilmelerini ve onların bazı özelliklerinin bir incelenmesini ele almaktadır. Bu amaçla genelleştirilmiş Modifiye Pell ve Pell-Lucas dizileri oluşturuluyor ve onlar için uygun başlangıç koşulları sunuluyor. Genelleştirilmiş Pell-Lucas ve Modifiye Pell dizilerinden oluşan bazı toplam formülleri inceleniyor. Matris gösterimleri kullanılarak bu yeni genelleştirmelerle ilgili bazı temel özdeşlikler elde ediliyor. Özellikle genelleştirilmiş Modifiye Pell ve Pell-Lucas dizileri arasında önemli bir bağıntı keşfediliyor.

Anahtar kelimeler: Genelleştirilmiş Pell-Lucas dizisi, Genelleştirilmiş Modifiye Pell dizisi, Matris gösterimi, Rekürans bağıntıları.

1. Introduction

Algebra has been studied over the years since its facilities are important for any person who wants to study advanced mathematics or science. By the general results from linear algebra, the recurrence sequences within the scope of algebra have extensively been studied. They can be approached several points of view, whether essentially numerical or analytical methods are envisaged as the solution method. Therefore, considerable attention is given to these recurrence sequences. As an example of them, Fibonacci sequence can be given. The well-known monographs on the subject are presented by Vajda (1989) and Koshy (2001).

ISSN: 1307-9085
e-ISSN: 2149-4584
*Correspondence to: ahmetdasdemir37@gmail.com
DOI: 10.18185/eufbed.84441

Similarly, Pell sequence is also well-known and is defined by a recurrence relation with the appropriate initial conditions. Pell-Lucas and Modified Pell sequence are defined by the same recurrences relation but with the different initial conditions. The known properties of the Pell-Lucas sequence can thus be written for the Modified Pell sequence. Hence a study of the one involves inevitably familiarity with the other one.

From the inception of the number theory, systematic surveys and analysis on the subject have been under dense study by a great number of researchers. Horadam (1971) investigates the usual Pell sequence and its properties. Ercolano (1994) gives a constructive method for finding all possible matrix generators of the Pell sequence. Kilic and Tasci (2006) extend the results in the above-mentioned and many another related references which are not given here are extended to the sequences of the generalized order- Pell numbers. In the same paper, the authors present their generating matrix, which the consecutive powers of the matrix give matrices whose columns consisting of them. Halici and Daşdemir (2010) and Daşdemir (2011) studied the Pell, Pell-Lucas and Modified Pell sequence and their properties by using matrix approach. Gulec and Taskara (2012) give new generalizations for the classical Pell and Pell-Lucas sequences.

In this paper, the generalizations of the usual Pell-Lucas and Modified Pell sequences are considered. Their matrix representations, some identities and sum formulae are obtained by using matrix approach. It is observed that the sequences obtained in the present paper coincide with the corresponding ones for those which are the certain special cases. In addition, the relationship between the generalized Modified Pell and Pell-Lucas sequence is identified.

2. Modified Pell Sequence

In this section, a generalization of the classical Modified Pell sequence, their matrix representations and some fundamentals identities are presented. To do this, k sequences of the generalized order-k Modified Pell sequence are first defined with the same recurrence relation mentioned by Kilic and Tasci (2006) but with the initial conditions:

$$q_n^i = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n + i = 1, \text{ for } 1 - k \le n \le 0, \\ 0 & \text{otherwise}, \end{cases} \quad (1)$$

where q_n^i is the n th term of the i th sequence and said to be the generalized order-k Modified Pell sequence. Eq. (1) is the first of the desired main results. When $(i,k) = (1,2)$, it reduces to usual Pell sequence. Taking $i = k = 2$, the conventional Modified Pell sequence is obtained. Eq. (1) may also be stated by the vector recurrence relation as follows:

$$u_{n+1}^i = Ru_n^i$$

where u_n^i be $\left[q_n^i \quad q_{n-1}^i \quad \cdots \quad q_{n-k+1}^i\right]^T$, and the matrix R has been introduced by Kilic and Tasci (2006).

A new matrix whose columns consist of the vector u_n^i is now defined as follows:

$$K_n = \left[u_n^1 \quad u_n^2 \quad u_n^3 \quad \cdots \quad u_n^{k-1} \quad u_n^k\right]^T \qquad (2)$$

Considering the matrix R and the definition of generalized order-k Modified Pell sequence, $K_{n+1} = R.K_n$ can readily be written. By the induction method, $K_{n+1} = R^n.K_1$ is obtained. In addition, an k-square auxiliary matrix S is defined such as $s_{1i} = 1$ for $1 \le i \le k$, $s_{ii} = -1$ for $2 \le i \le k$ and 0 otherwise. It should be noted that $K_1 = R.S$. According to all the above-stated, $K_n = R^n.S$ or $K_n = E_n.S$ can be written, where the matrix E_n is defined by Kilic and Tasci (2006). Considering the fundamental matrix identities, this expression can more generally be written in the form $K_{n+m} = E_n.K_m$. Recall that in mathematics, an involutory matrix is a matrix that is its own inverse. It is clear that the matrix S is an involutory matrix. Thanks to this excellent characteristics, $E_n = K_n.S$ can also be found. Thus the following theorem can be given.

Theorem 2.1. Let K_n be as in (2). Then,

$$\det K_n = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ (-1)^{n+1} & \text{otherwise.} \end{cases} \qquad (3)$$

Proof. Considering the fundamental matrix identities, computing $\det K_n$ by the Laplace expansion of the determinant with respect to the first column and after some mathematical operations, the proof can immediately be obtained.

From all the above-mentioned statements, the following corollaries are obtained.

Corollary 2.2. Let P_n^k and q_n^k be the generalized order-k Pell and Modified Pell sequences, respectively. Then,

i. $q_n^1 = P_n^1 = P_{n+1}^k$

ii. $q_n^i = P_n^1 - P_{n+1}^i$ for $2 \le i \le k$,

iii. $P_n^i = q_n^1 - q_{n+1}^i$ for $2 \le i \le k$,

iv. $q_{n+m}^i = \sum_{j=1}^{k} P_n^j q_{m-j+1}^i$.

Proof. The proof can easily be seen from the definitions and properties of the matrix K_n, E_n, S and R.

With the above-stated the generalization and investigation of the sequence being considered are thus exhausted.

3. Pell-Lucas Sequence

In this section, the usual Pell-Lucas sequence is considered. Frankly speaking, the works made in previous section can be adapted to the generalized order-k Pell-Lucas sequence. As the previous section, its generalization is defined by the same recurrence relation but with the initial conditions

$$Q_n^i = \begin{cases} 2 & \text{if } n = 0, \\ -2 & \text{if } n+i = 1, \text{ for } 1-k \le n \le 0 \\ 0 & \text{otherwise,} \end{cases} \quad (4)$$

where Q_n^i is the nth term of ith sequence.

When $i = k = 2$, the Pell-Lucas sequence is obtained. Taking $i = 1$ and $k = 2$, a relation such that $Q_n^1 = 2P_{n+1}$ is found. Moreover, there exit a significant relation between the generalized order-k Modified Pell and Pell-Lucas sequence as follows:

$$Q_n^i = 2q_n^i. \quad (5)$$

Let the vector v_n^i be $\left[Q_n^i \quad Q_{n-1}^i \quad \cdots \quad Q_{n-k+1}^i\right]^T$. Then $v_{n+1}^i = R.v_n^i$ can be written. An k-square matrix $L_n = \left[\ell_{ij}\right]$ consisting of the vectors v_n^i is also defined as follows:

$$L_n = \left[v_n^1 \quad v_n^2 \quad v_n^3 \quad \cdots \quad v_n^{k-1} \quad v_n^k\right]^T \quad (6)$$

Thus, $L_{n+1} = R.L_n$. At the same time $L_{n+1} = R^n.L_1$ can be written. An k-square auxiliary matrix T is also defined such as $t_{1i} = 2$ for $1 \le i \le k$, $t_{ii} = -2$ for $2 \le i \le k$ and 0 otherwise. It is clear that $L_1 = R^n.T$. Therefore, $L_n = R^n.T$ or $L_n = E_n.T$ can readily be written. As a result of this expression, $L_{n+m} = E_n.L_m$ is found. Frankly, there exists a relation such as $T = 2.S$. Considering this last equation, the following statement can readily be given:

$$T^n = \begin{cases} 2^n.I & \text{if } n \text{ is even,} \\ 2^n.S & \text{otherwise.} \end{cases} \quad (7)$$

Taking this equality into account, $4.E_n = L_n.T$ is obtained.

Finally, the following theorem can be given without the proof because its proof is made by the same way in Theorem 2.1.

Theorem 3.1. Let L_n be as in (6). Then,

$$\det L_n = \begin{cases} 2^k & \text{if } k \text{ is odd,} \\ 2^k(-1)^{n+1} & \text{otherwise.} \end{cases} \quad (8)$$

Before closing this section, it is noted that, considering Eq. (5), Corollary 2.2. is recomposed in accordance with the generalized order-k Pell-Lucas sequence.

4. Sum Formulae

Now, some formulae related to the sums of the terms of the generalized Modified Pell and Pell-Lucas sequences are presented. To do this, the following definitions are first given: for $i \ge 1$,

$$U_n^{(i)} = \sum_{j=\delta}^{n} q_j^i \text{ and } V_n^{(i)} = \sum_{j=\delta}^{n} Q_j^i \quad (9)$$

where $\delta = \begin{cases} 0 & \text{if } i=1, \\ 1 & \text{otherwise.} \end{cases}$

Then the following theorem is started

Theorem 4.1. Let q_n^i be the generalized order-k Modified Pell sequence. Then,

$$q_n^1 = 1 + \sum_{j=1}^{k} U_{n-j}^{(1)}. \tag{10}$$

Proof: (Induction on n) If $n=1$, then

$$q_1^1 = 1 + \sum_{j=1}^{k} U_{1-j}^{(1)}.$$

Since $U_{-i} = 0$,

$$q_1^1 = 1 + U_0^{(1)} + U_{-1}^{(1)} + \cdots + U_{1-k}^{(1)} = 1 + q_0^1 = 2.$$

It appears to be true for $n=1$. Suppose that Eq. (10) holds for n. Then it must be shown that it holds for $n+1$. From the assumption,

$$U_n^{(1)} = 1 + \sum_{j=1}^{k} U_{n-j}^{(1)}$$

is known. Also the following expression can be written:

$$\begin{aligned} U_n^{(1)} &= q_n^1 + U_{n-1}^{(1)} \\ U_{n-1}^{(1)} &= q_{n-1}^1 + U_{n-2}^{(1)} \\ &\vdots \\ U_{n-k+1}^{(1)} &= q_{n-k+1}^1 + U_{n-k}^{(1)} \end{aligned} \tag{11}$$

Adding them side-by-side,

$$\begin{aligned} 1 + \sum_{j=1}^{k} U_{n-j+1}^{(1)} &= 1 + \sum_{j=n-k+1}^{n} q_j^1 + \sum_{j=1}^{k} U_{n-j}^{(1)} \\ &= q_n^1 + q_n^1 + q_{n-1}^1 + \cdots + q_{n-k+1}^1 = q_{n+1}^1 \end{aligned}$$

is found. Thus the proof is completed.

Theorem 4.2. Let q_n^i be the generalized order-k Modified Pell sequence. Then,

$$q_n^2 = 2 + \sum_{j=1}^{k} U_{n-j}^{(2)}. \tag{12}$$

Proof: By the same way in Theorem 4.1, the proof is readily completed.

Actually, the more general case of the above theorems can be given, namely, for $i \geq 1$

$$q_n^i = i + \sum_{j=1}^{k} U_{n-j}^{(i)}, \tag{13}$$

As an example of what could happen, take $n=10$, $i=3$ and $k=5$ in Eq. (13). Then,

$$q_{10}^3 = 3 + U_9^{(3)} + U_8^{(3)} + U_7^{(3)} + U_6^{(3)} + U_5^{(3)} = 4437$$
.

When $i = j = 2$, Eq. (13) is reduced to

$$q_n^2 = 2 + U_{n-1}^{(2)} + U_{n-2}^{(2)}.$$

So the following well-known result is obtained by Horadam (1994):

$$\sum_{i=1}^{n} q_i = \frac{q_{n+1} + q_n - 2}{2}.$$

Consider the generalized order-k Pell-Lucas sequence. By Eq. (5),

$$Q_n^i = 2i + \sum_{j=1}^{k} V_{n-j}^{(i)} \tag{14}$$

can be written for $i \geq 1$. This result is not to be surprising, because there exists a significant property between the generalized order-k Modified Pell and Pell-Lucas sequences as in (5). Therefore Eq. (14) is a natural result of Eq. (13). When $i = k = 2$, Eq. (14) is also reduced to

$$Q_n^2 = 4 + V_{n-1}^{(2)} + V_{n-2}^{(2)}$$

Then the following well-known result is derived by Horadam (1994):

$$\sum_{i=1}^{n} Q_i = \frac{Q_{n+1} + Q_n - 4}{2}.$$

Some matrix representations will be presented, and the sums for the generalized order-k are obtained by these matrix representations. To do this, define the following matrices.

$$G_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ \vdots & & R & \\ 0 & & & \end{bmatrix} \tag{15}$$

$$M_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ \vdots & & S & \\ 0 & & & \end{bmatrix} \tag{16}$$

$$N_k = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -2 & \cdots & \vdots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & & & \\ \vdots & & T & \\ 0 & & & \end{bmatrix} \tag{17}$$

$$\Phi_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ U_n^{(1)} & & & & \\ U_{n-1}^{(1)} & & K_n & & \\ \vdots & & & & \\ U_{n-k+1}^{(1)} & & & & \end{bmatrix} \tag{18}$$

$$\Psi_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ V_n^{(1)} & & & & \\ V_{n-1}^{(1)} & & L_n & & \\ \vdots & & & & \\ V_{n-k+1}^{(1)} & & & & \end{bmatrix} \tag{19}$$

In order to achieve the other one of the fundamental goals, the following theorems are given.

Theorem 4.3. Let $(k+1) \times (k+1)$ matrices G_k, M_k and Φ_n be as in (15-16,18). Then for $n \geq 1$,

$$\Phi_n = G_k^{\,n}.M_k. \tag{20}$$

Proof: Since $U_{n+1}^{(1)} = q_{n+1}^1 + U_n^{(1)}$ and $K_{n+1} = R.K_n$, the matrix recurrence relation $\Phi_{n+1} = G_k.\Phi_n$ can be written. By the induction method, $\Phi_{n+1} = G_k^n.\Phi_1$ is obtained. From the definition of Φ_n, it is clear that $\Phi_1 = G_k.M_k$. Then, $\Phi_{n+1} = G_k^{n+1}.M_k$. Thus the proof is completed.

The proof of the next theorem is analogous to the proof of Theorem 4.6, so it will be omitted.

Theorem 4.4. Let $(k+1)\times(k+1)$ matrices G_k, N_k and Ψ_n be as in (15,17,19). Then for $n\geq 1$,

$$\Psi_n = G_k{}^n . N_k . \qquad (21)$$

5. Conclusion

In this study, the generalizations of the usual Modified Pell and Pell-Lucas sequences are given with the corresponding initial conditions. Generating matrices and determinantal identities for these sequences are obtained. Furthermore, certain formulae for their sums are presented by matrix methods. A great feature between the generalized order-k Modified Pell and Pell-Lucas is discovered.

Despite all the above-mentioned, many investigations on the subject could not be examined here. These new sequences have the important role in the Fibonacci theory. This is one of the outstanding research areas. The relationships between the generalized order-k Modified Pell and Pell-Lucas sequences and the usual Fibonacci and Lucas sequences can be investigated. Moreover, the matrix theory gives the excellent results for these sequences.

6. References

Daşdemir, A. 2011. On the Pell, Pell-Lucas and Modified Pell Numbers by Matrix Method. Applied Mathematical Sciences, 5(64), 3173-3181.

Ercolano, J. 1994. Matrix Generators of Pell Sequences. Fibonacci Quarterly, 3(1), 34-53.

Gulec, H., Taskara, N. 2012. On the (s,t)-Pell and (s,t)-Pell-Lucas sequences and their matrix representations. Applied Mathematics Letters, 25(10), 1554-1559.

Halici, S., Daşdemir, A. 2010. On some relationships among Pell, Pell-Lucas and Modified Pell Sequences. Sakarya University Journal of Science, 14(2), 141-145.

Horadam, A.F. 1971. Pell identities. Fibonacci Quarterly, 9(3), 245-263.

Horadam, A.F. 1994. Applications of Modified Pell Numbers to Representations. Ulam Quarterly, 3(1) 34-53.

Kilic, E., Tasci, D. 2006. The generalized Binet formula, representation and sums of the generalized order-k Pell Numbers. Taiwanese Journal of Mathematics, 10(6), 1661-1670.

Koshy, T. 2001. Fibonacci and Lucas Numbers with Applications, Wiley-Interscience Publications, Toronto.

Vajda, S. 1989. Fibonacci and Lucas numbers, and the Golden Section, Theory and applications. Dover Publications, New York.