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RESEARCH ARTICLE

GEOMETRIC PATTERN OF POWERS OF THREE VIA SIERPINSKI TRIANGULAR NUMBERS

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Abstract

In this study, we describe a new number sequence based on integers arranged in a fractal like structure to visualize powers of three. Moreover, we associate these new numbers with triangular numbers.

Keywords

Triangular number,
Sierpinski triangular number,
Sierpinski triangle,
Diophantine equation

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1. INTRODUCTION

Following the belief that "everything is number", Pythagoreans regarded any positive integer as a set of points on the plane. In this sense, Pythagoreans of the 6*th* century BC introduced figurate numbers which started a very rich history of efforts to associate geometry and arithmetic with each other. In general, a figurate number is a number that can be represented by a regular geometrical arrangement of equally spaced points. If the arrangement forms a regular polygon, a regular polyhedron or a regular polytope, the number is called a polygonal, polyhedral or polytopic number, respectively.

In general, Fermat–Euler theorem, Gauss–Legendre theorem, and Lagrange theorem, which are related to expressing natural numbers as the sum of squares, are some well-known classical results in number theory. As a continuation of these famous theorems, P. Fermat's claim in 1638 that every natural number can be written as the sum of three triangular numbers was proved by C.F. Gauss in 1796. After proving that any natural number can be written as the sum of two triangular numbers and a square in the 1870s, there are many studies that have survived to the present day on the use of triangular numbers in obtaining a natural number. After such studies, triangular numbers became very popular among polygonal numbers in number theory.

The triangular number counts the number of dots in an equilateral triangle array. The first five triangular numbers are $1, 3, 6, 10, 15$ as shown in Figure 1. n^{th} triangular number T_n (Sloane's A000217, https://oeis.org/A000217) is given by the following formula

$$T_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

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That is, the triangular number T_n is equal to the sum of the natural numbers from 1 to n (we refer the reader to for [1]-[18] a wider treatment).

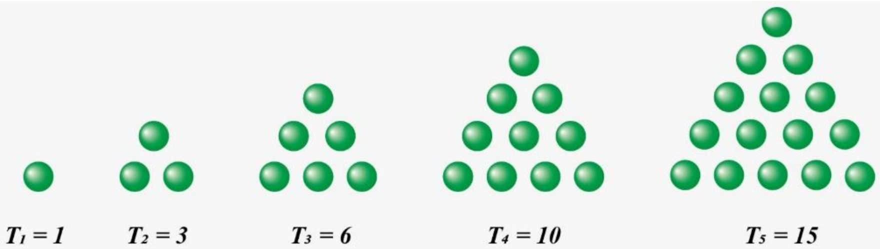

Figure 1. The first five triangular numbers

2. MAIN RESULT

The main motivation for this article is to answer the question "Can powers of three be associated with a geometric model?" In 2000, the equation (I)

$$\sum_{k=0}^{n-1} 3^k = \frac{3^n - 1}{2}$$

for $n \in IN = \{0,1,2,...\}$ given by David Sher [14] in his proof without words about the sums of powers of three can be considered as an answer to this question. In 2016, Charles David Leach [11] gave the relationship between triangular numbers and powers of three with the equation (II)

$$T_{2^k} = 3^n + \sum_{k=1}^{n-1} (3^{n-k-1})(T_{2^k-1}),$$

after his observations in Sher's visual proof. These two equations are identities, which are obtained using the graphical and combinatorial properties of triangular numbers. The main idea that distinguishes our results from the two results above is that we use a new set of numbers to construct powers of three, namely Sierpinski triangular numbers. In many studies in recent years, it seems that powers of three play an important role in the solution of diophantine equations associated with Fibonacci, Pell and Lucas numbers. [3], [4] and [18] are examples of these studies. Therefore, associating the powers of three with a geometric model will effectively contribute to giving visual, understandable and easy proofs in such studies in number theory. Frankly, this situation has been a source of motivation for us to carry out this study.

The Sierpinski triangle, one of the basic and also popular examples of self-similar sets, is a fractal defined by Polish mathematician Wacław Sierpiński in 1915. It is known that the Sierpinski triangle is created using the following steps; in the first step, any triangle is considered. In the second step, the new triangle formed by considering the midpoints of each side of the first triangle is removed from the original. Thus, three triangles within the original triangle are obtained. Then, the first and second steps are applied again to these three triangles, and the third step is realized. The set of points obtained in this process, which continues by iterating to infinity, is the Sierpinski triangle. By enumerating the total number of vertices of the triangles (Sloane's A000217, https://oeis.org/A067771) that occur in each iteration of the construction of a Sierpinski triangle, we obtain a sequence of numbers which we call "***the sequence of Sierpinski triangular numbers***", or simply "***the Sierpinski triangular numbers***". Let ST_n denote the Sierpinski triangular number n^{th}, then the equalities

$$
\begin{array}{lcll}
ST_1 & = & 3 & \\
ST_2 & = & ST_1 + 3^1 & = ST_1 + 3^1 \\
ST_3 & = & ST_2 + 3^2 & = ST_1 + 3^1 + 3^2 \\
ST_4 & = & ST_3 + 3^3 & = ST_1 + 3^1 + 3^2 + 3^3 \\
ST_5 & = & ST_4 + 3^4 & = ST_1 + 3^1 + 3^2 + 3^3 + 3^4 \\
\vdots & & \vdots & \vdots \\
ST_n & = & ST_{n-1} + 3^{n-1} & = ST_1 + 3^1 + 3^2 + 3^3 + 3^4 + \cdots + 3^{n-1}
\end{array}
$$

can be obtained. That is, the Sierpinski triangular numbers are given by the following explicit formulas:

$$ST_n = 3 + \sum_{k=1}^{n-1} 3^k = \frac{3 + 3^n}{2}$$

for $n \in \{1,2, \dots\}$. The first 5 Sierpinski numbers are $3, 6, 15, 42$ and 123 are visualized in Figure 2.

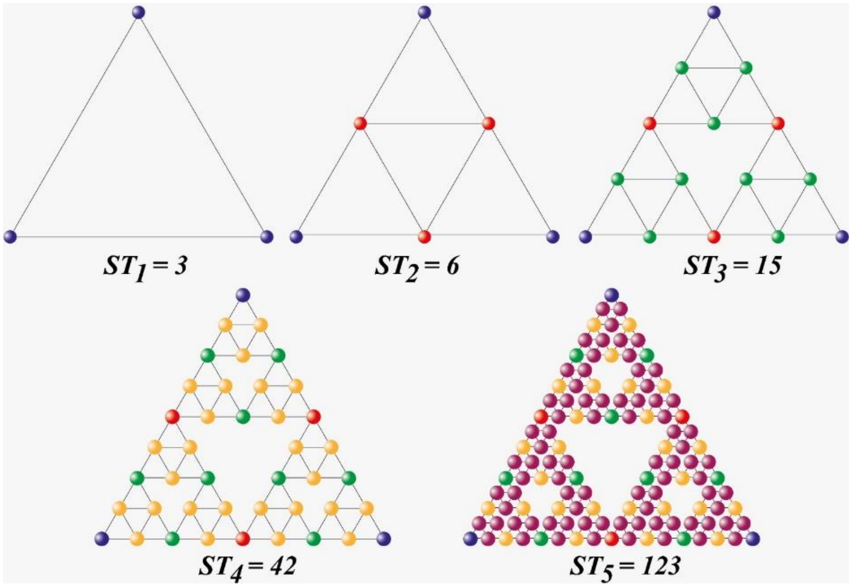

Figure 2

Rewriting the above formula as the equation (III)

$$3^n = 2ST_n - 3,$$

we obtain a characterization of powers of three using the Sierpinski triangular numbers.

Moreover, using equations (I) and (II) we obtain the following relationship between Sierpinski triangular numbers and triangular numbers:

$$2ST_n = 3 + T_{2^n} - \sum_{k=1}^{n-1} (3^{n-k-1})(T_{2^k-1}).$$

Consequently, in the section up to this point, besides characterizing the powers of three with Sierpinski triangular numbers, the relationship between triangular numbers and Sierpinski triangular numbers has been given. By using the combinatorial properties of the Sierpinski triangle, many theorems can be given within the scope of elementary number theory. One such theorem is proved below. But, since it will be out of purpose of our article, we will be content with the following theorem. We should emphasize that this theorem and many others making use of the Sierpinski triangular numbers have an important place in number theory.

Theorem 1: Positive integer solutions of the quadratic Diophantine equation $(x-y)^2=(2y-3)^2$ for $x>y$ are consecutive Sierpinski triangular numbers.

Proof. Let x and y be consecutive Sierpinski triangular numbers such that $x=ST_{n+1}$ and $y=ST_n$. Then, using the equality (III), we have that

$$(x-y)^2=(ST_{n+1}-ST_n)^2=\left(\frac{3+3^{n+1}}{2}-\frac{3+3^n}{2}\right)^2=(3^n)^2=(2y-3)^2.$$

Conversely, we assume that $(x-y)^2=(2y-3)^2$ for $x>y$. In here, if we make the substitution $x-y=3^u$ and $2y-3=3^v$, then it is obtained the equalities

$$3^u=3^v, \qquad x=\frac{2.3^u+3^v+3}{2} \text{ and } y=\frac{3^v+3}{2} .$$

Using last three equalities, we get that

$$x=\frac{2.3^u+3^v+3}{2}=\frac{2.3^u+3^u+3}{2}=ST_{u+1} \text{ and } y=\frac{3^v+3}{2}=\frac{3^u+3}{2}=ST_u.$$

Thus, x and y are consecutive Sierpinski triangular numbers.

CONFLICT OF INTEREST

The author stated that there are no conflicts of interest regarding the publication of this article.

CRediT AUTHOR STATEMENT

Temel Ermiş: Methodology, Investigation, Formal analysis, Validation, Visualization, Writing – original draft, Writing – review & editing.

REFERENCES

[1] Bruckman P, Dence JB, Dence TP, Young J. Series of reciprocal triangular numbers. The College Math J 2013; 44(3): 177-184.

[2] Caglayan G. Covering a triangular number with pentagonal numbers. Math Intelligencer 2020; 42: 55.

[3] Cagman A. Explicit solutions of powers of three as sums of three Pell numbers based on Baker’s type inequalities. Turkish Journal of Inequalities 2021; 5(1): 93-103.

[4] Coons M, Winning HH. Powers of two modulo powers of three. Journal of Integer Sequences 2015; 18: article 15.6.1.

[5] Charles FM. Proof without words: Square triangular sums. Mathematics Magazine 2019; 92(4): 269.

[6] Edgar T. Proof without words: A recursion for triangular numbers and more. Mathematics Magazine 2017; 90(2): 124-125.

[7] Edgar T. Visual triangular number identities from positional number systems. The College Math J 2021; 52(2): 133-136.

[8] Hopkins B. Proof without words: Products of odd squares and triangular numbers. Mathematics Magazine 2018; 91(1): 42.

[9] Jones MA. Proof without words: The square of a balancing number is a triangular number. The College Math J 2012; 43(3): 212.

[10] Leach CD. Proof without words: Powers of three and triangular numbers. The College Math J 2016; 47: 120.

[11] Matthew JH, Jones MA. Proof without words: Nonnegative integer solutions and triangular numbers. Mathematics Magazine 2002; 75(5): 388.

[12] Nelsen RB. Proof without words: Squares of triangular numbers. Mathematics Magazine 1990; 63(3): 178.

[13] Nelsen RB. Proofs without words II. Classroom Resource Materials. Mathematical Association of America, pp 108, 2000.

[14] Plaza Á. Proof without words: Sum of triangular numbers. Mathematics Magazine 2016; 89(1): 36-37.

[15] Stephen LS. Proof without words: Alternating sum of squares = triangular number. Mathematics Magazine 1992; 65(2): 90.

[16] Tiebekabe P, Diouf I. Powers of three as difference of two Fibonacci numbers. JP Journal of Algebra, Number Theory and Applications 2021; 49(2): 185-196.

[17] Unal H. A visual proof for the sum of the first *n* triangular numbers. Math Intelligencer 2010; 32: 6-7.

[18] Zerger MJ. Proof without words: Sums of triangular numbers. Mathematics Magazine 1990; 63(5): 314.