Some Inequalities for Positive Linear Maps of Operators

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Abstract

Drawing inspiration from Lin [3], we generalize some operator inequalities due to Mond et al. [1] as follows: Let $A$ be positive operator on a Hilbert space with $0 < m \leq A \leq M$. Then for $2 < p < \infty$ and every normalized positive linear map $\Phi$,

$$\Phi^p(A^2) \leq \left( \frac{M^2 + m^2}{4M^p m^p} \right)^2 \Phi(A)^2 p.$$

Let $A$ be positive operator on a Hilbert space with $0 < m \leq A \leq M$. Then for $1 \leq p < \infty$ and every normalized positive linear map $\Phi$,

$$\Phi^p(A^{-2}) \leq \left[ \frac{1}{4(Mm)^p} \left( M + m + \frac{(M - m)^2}{4(M + m)} \right) \right]^2 \Phi(A)^{-2p}.$$

Keywords: Positive Operators, Operator Inequalities, Normalized Positive Linear Maps.
Operatörlerin Pozitif Lineer Dönüşümü için Bazı Eşitsizlikler

Özet

Lin’in [3] teki çalışmasından ilham alarak, Mond ve Pecaric’in [1] deki çalışmasında verilen bazı operatör eşitsizliklerinin genelleştirilmesi şu şekilde yapıldı: A, Hilbert uzayı üzerinde $0 < m \leq A \leq M$ şartını sağlayan bir pozitif operatör olmak üzere, $2 < p < \infty$ ve her normalize edilmiş $\Phi$ pozitif lineer dönüşümü için

$$\Phi^p(A^2) \leq \left( \frac{(M^2 + m^2)^p}{4M^p m^p} \right)^2 \Phi(A)^{2p}$$

eşitsizliği geçerlidir. Yine $A$, Hilbert uzayı üzerinde $0 < m \leq A \leq M$ şartını sağlayan bir pozitif operatör olmak üzere, $1 < p < \infty$ ve her normalize edilmiş $\Phi$ pozitif lineer dönüşümü için

$$\Phi^p(A^{-2}) \leq \left( \frac{1}{4(Mm)^p} \left( M + m + \frac{(M - m)^2}{4(M + m)} \right)^{2p} \right)^2 \Phi(A)^{-2p}$$

eşitsizliği geçerlidir.

Anahtar Kelimeler: Pozitif Operatörler, Operatör Eşitsizlikleri, Normalize Edilmiş Pozitif Lineer Dönüşümler

1. Introduction

Let $M, m$ be scalars and $I$ be the identity operator. We write $A \geq 0$ to mean that the operator $A$ is positive. If $A - B \geq 0$ ($A - B \leq 0$), then we write $A \geq B$ ($A \leq B$). $A^*$ stands for the adjoint of $A$. Other capital letters are used to denote the general elements of the $C^*$-algebra $L(H)$ of all bounded linear operators acting on a Hilbert space $(H, \langle \cdot, \cdot \rangle)$. $L_+(H)$ is the cone of positive (i.e., non-negative semi-definite) operators. Let $S(\alpha, \beta, H)$ be the totality of all self-adjoint operators on $H$ whose spectral are contained
in an interval \((\alpha, \beta)\). A (non-linear) transformation which maps \(L_\alpha(H)\), the set of positive operators on \(H\), into \(L_\alpha(K)\) will be called positive. The operator norm is denoted by \(\|\cdot\|\). A positive linear map \(\Phi\) preserves order-relation, that is, \(A \leq B\) implies \(\Phi(A) \leq \Phi(B)\), and preserves adjoint operation, that is, \(\Phi(A^*) = \Phi(A)^*\). It is said to be normalized if it transforms \(I_\mu\) to \(I_k\) (we use, in both cases, only \(I\)). If \(\Phi\) is normalized, it maps \(S(\alpha, \beta, H)\) to \(S(\alpha, \beta, K)\).

It is well known that for two positive operators \(A, B\),

\[
A \geq B \Rightarrow A^p \geq B^p \quad \text{for} \quad 0 \leq p \leq 1,
\]

but

\[
A \geq B \Rightarrow A^p \geq B^p \quad \text{for} \quad p > 1.
\]

Let \(0 < m \leq A \leq M\) and \(\Phi\) be normalized positive linear map. Mond and Pecaric [1] proved the following operator inequality:

\[
\Phi(A^2) \leq \frac{(M+m)^2}{4Mm} \Phi(A)^2. \quad (1.1)
\]

Lin [3] obtained

\[
\Phi(A^{-1})^2 \leq \left(\frac{(M+m)^2}{4Mm}\right)^2 \Phi(A)^{-2}. \quad (1.2)
\]

If we replace \(A\) by \(A^{-1}\) in (1.1), we get

\[
\Phi(A^{-2}) \leq \frac{(M+m)^2}{4Mm} \Phi(A^{-1})^2, \quad (1.3)
\]

which is

\[
\frac{4Mm}{(M+m)^2} \Phi(A^{-2}) \leq \Phi(A^{-1})^2. \quad (1.4)
\]

Combining (1.2) and (1.4), we have

\[
\Phi(A^{-2}) \leq \left(\frac{(M+m)^2}{4Mm}\right)^3 \Phi(A)^{-2}. \quad (1.5)
\]
Fujii et al. [2] proved that \( t^2 \) is order preserving in the following sense.

**Proposition 1.1** Let \( 0 < A \leq B \) and \( 0 < m \leq A \leq M \). Then the following inequality holds:

\[
A^2 \leq \frac{(M + m)^2}{4Mm} B^2.
\]

A quick use of the above proposition and (1.1) give the following preliminary result.

**Proposition 1.2** Let \( 0 < m \leq A \leq M \). Then for normalized positive linear map \( \Phi \):

\[
\Phi(A^2)^2 \leq \frac{(M^2 + m^2)^2}{4M^2m^2} \left( \frac{(M + m)^2}{4Mm} \right)^2 \Phi(A)^4.
\]

It is interesting to ask whether \( t^p \) ( \( p \geq 1 \)) for the inequalities (1.1) and (1.5) is order preserving. This is a main motivation for the present paper.

In this paper, we give \( p \)-power \((p > 2)\) of inequality (1.1) and present an operator inequality which is refinement of (1.5). Furthermore, we achieve a generalization of the refinement inequality.

2. Main Results

We give some lemmas before we give the main theorems of this paper:

**Lemma 2.1** [6] Let \( A \) and \( B \) be positive operators. Then for \( 1 \leq r < \infty \)

\[
\|A^r + B^r\| \leq \|(A + B)^r\|.
\]

(2.1)

**Lemma 2.2** [5] Let \( A, B > 0 \). Then the following norm inequality holds:

\[
\|AB\| \leq \frac{1}{4} \|A + B\|^2.
\]

(2.2)

**Lemma 2.3** [4, p. 41] Let \( A > 0 \) and \( \Phi \) be normalized positive linear map. Then

\[
\Phi(A)^{-1} \leq \Phi(A^{-1}).
\]

(2.3)

**Lemma 2.4** Let \( 0 < m \leq A \leq M \). Then for normalized positive linear map \( \Phi \):
\[ \Phi(A^{-2})^{\frac{1}{2}} \leq \Phi(A^{-1}) + \frac{(M - m)^2}{4Mm(M + m)}. \] (2.4)

**Proof:** In [1, (14)], we replace \( A \) by \( A^{-1} \) and have the result.

Now we prove the first main result in the following theorem.

**Theorem 2.5** Let \( 0 < m \leq A \leq M \). Then for every normalized positive linear map \( \Phi \),

\[ \Phi(A^2)^p \leq \left( \frac{(M^2 + m^2)^p}{4M^p m^p} \right)^2 \Phi(A)^{2p}, \quad 2 < p < \infty. \] (2.5)

**Proof:** The operator inequality (2.5) is equivalent to

\[ \left\| \Phi(A^2)^{\frac{p}{2}} \Phi^{-p}(A) \right\| \leq \left( \frac{(M^2 + m^2)^p}{4M^p m^p} \right)^{\frac{2}{p}}. \] (2.6)

Compute

\[ \left\| \Phi(A^2)^{\frac{p}{2}} (Mm)^p \Phi^{-p}(A) \right\| \leq \frac{1}{4} \left\| \Phi(A^2)^{\frac{p}{2}} + (M^2 m^2 \Phi(A)^{-2})^{\frac{p}{2}} \right\|^2 \] (by (2.2))

\[ \leq \frac{1}{4} \left\| \Phi(A^2) + M^2 m^2 \Phi(A)^{-2} \right\|^p \] (by (2.1))

\[ = \frac{1}{4} \left\| \Phi(A^2) + M^2 m^2 \Phi(A)^{-2} \right\|^p \]

\[ \leq \frac{1}{4} \left\| (M + m)\Phi(A) - mMI + M^2 m^2 \Phi(A)^{-2} \right\|^p. \] (by [1, (10)])

Note that

\[(M - \Phi(A))(m - \Phi(A))\Phi(A)^{-2} \leq 0,\]

then

\[ Mm\Phi(A)^{-2} + I \leq (M + m)\Phi(A)^{-1}. \] (2.7)

Thus
\[
\left\| \Phi(A^2)^p \right\| \leq \frac{1}{4} \left\| (M + m) \Phi(A) - mMI + M^2m^2 \Phi(A)^{-2} \right\|^p \\
\leq \frac{1}{4} \left\| (M + m) \Phi(A) - mMI + Mm(\Phi(A)^{-1} - I) \right\|^p \quad \text{(by (2.7))}
\]
\[= \frac{1}{4} \left\| (M + m)(\Phi(A) + Mm\Phi(A)^{-1}) - 2mMI \right\|^p \]
\[\leq \frac{1}{4} \left\| (M + m)(M + m)I - 2mMI \right\|^p \quad \text{(by (2.3) and [3, (2.3)])}
\]
\[= \frac{1}{4} \left( M^2 + m^2 \right)^p. \]

That is
\[
\left\| \Phi(A^2)^p \right\| \leq \frac{\left( M^2 + m^2 \right)^p}{4M^p m^p}.
\]

Thus (2.5) holds.

**Remark 2.6** We can’t get the inequality (1.6) when \( p = 2 \), but we obtain the relation between \( \Phi(A^2)^p \) and \( \Phi(A)^{2p} \) for \( p > 2 \) and moreover the form of the inequality (2.5) is simple.

**Theorem 2.7** Let \( 0 < m \leq A \leq M \). Then for every normalized positive linear map \( \Phi \),
\[
\Phi(\Lambda^{-2}) \leq \frac{1}{4^2 M^2 m^2} \left( M + m + \frac{(M - m)^2}{4(M + m)} \right)^4 \Phi(\Lambda)^{-2}. \quad \text{(2.8)}
\]

**Proof:** The inequality (2.8) is equivalent to
\[
\left\| \Phi(\Lambda^{-2})^{\frac{1}{2}} \Phi(\Lambda) \right\| \leq \frac{1}{4Mm} \left( M + m + \frac{(M - m)^2}{4(M + m)} \right)^2.
\]

Compute
\[
\left\| \frac{1}{Mm} \Phi(\Lambda^{-2})^{\frac{1}{2}} \Phi(\Lambda) \right\| \leq \frac{1}{4} \left\| Mm \Phi(\Lambda^{-2})^{\frac{1}{2}} + \Phi(\Lambda) \right\|^2
\]
\[
\frac{1}{4} \left\| Mm\Phi(A^{-1}) + \frac{(M-m)^2}{4(M+m)} + \Phi(A) \right\|^2 \quad (\text{by Lemma 2.4})
\]
\[
\leq \frac{1}{4} \left( M + m + \frac{(M-m)^2}{4(M+m)} \right)^2.
\quad (\text{by } [3, (2.3)])
\]

That is
\[
\left\| \Phi(A^{-2}) \frac{1}{2} \Phi(A) \right\| \leq \frac{1}{4Mm} \left( M + m + \frac{(M-m)^2}{4(M+m)} \right)^2.
\]

Thus (2.8) holds.

**Remark 2.8** It is easy to compute that \( \frac{1}{4^2 M^2 m^2} \left( M + m + \frac{(M-m)^2}{4(M+m)} \right)^4 \) is smaller than \( \left( \frac{(M+m)^2}{4Mm} \right)^3 \) in the right side of (1.5). Thus (2.8) is a refinement of (1.5).

In the next theorem, we give a generalization of (2.8).

**Theorem 2.9** Let \( 0 < m \leq A \leq M \). Then for every normalized positive linear map \( \Phi \) and \( 1 \leq p < \infty \),
\[
\Phi(A^{-2})^p \leq \frac{1}{4(Mm)^p} \left( M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p} \Phi(A)^{-2p}.
\quad (2.9)
\]

**Proof** : The operator inequality (2.9) is equivalent to
\[
\left\| \Phi(A^{-2}) \frac{p}{2} \Phi(A)^p \right\| \leq \frac{1}{4(Mm)^p} \left( M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p}.
\quad (2.10)
\]

Compute
\[
\left\| (Mm)^p \Phi(A^{-2}) \frac{p}{2} \Phi(A)^p \right\| \leq \frac{1}{4} \left\| mM \Phi(A^{-2})^\frac{1}{2} + \Phi(A)^p \right\|^2 \quad (\text{by } (2.2))
\]
\[
\leq \frac{1}{4} \left\| Mm \Phi(A^{-2})^\frac{1}{2} + \Phi(A)^p \right\|^2 \quad (\text{by } (2.2))
\]

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\[
= \frac{1}{4} \left\| Mm\phi(A^{-1}) + \frac{(M-m)^2}{4(M+m)} + \phi(A) \right\|_p^2 \quad \text{(by Lemma 2.4)}
\]

\[
\leq \frac{1}{4} \left( M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p}. \quad \text{(by [3, (2.3)])}
\]

That is
\[
\left\| \phi(A^{-2})^\frac{p}{2} \phi(A)^p \right\| \leq \frac{1}{4(Mm)^p} \left( M + m + \frac{(M-m)^2}{4(M+m)} \right)^{2p}.
\]

Thus (2.9) holds.

**Remark 2.10** When \( p = 1 \), the inequality (2.9) is (2.8). Thus the inequality (2.9) is a generalization of (2.8).

**Acknowledgements**

The work was supported by the natural science foundation of Hainan Province (No: 114007).

**References**


