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# On the $q$-Cesàro Bounded Double Sequence Space

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**Abstract**
In this article, the new sequence space $\tilde{\mathcal{M}}_u^q$ is acquainted, described as the domain of the 4d (4-dimensional) $q$-Cesàro matrix operator, which is the $q$-analogue of the first order 4d Cesàro matrix operator, on the space of bounded double sequences. In the continuation of the study, the completeness of the new space is given and the inclusion relation related to the space is presented. In the last two parts, the duals of the space are determined and some matrix classes are acquired.

***Keywords:*** *q-analogue, 4d q-Cesàro matrix, Double sequence space, Duals, Matrix transformations*

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## 1. Introduction

Obtaining $q$-analoques of known results has recently been found interesting by researchers. The $q$-analogue of a mathematical expression is the result that contains the parameter $q$ and is more general than that expression, but reduces to the basic expression for $q \to 1$. According to the basic information about $q$-calculus acquired from [1], the $q$-analogue of any nonnegative number $r$ is described as

$$[r]_q = \begin{cases} \dfrac{1-q^r}{1-q} & , \quad q \neq 1, \\ r & , \quad q = 1. \end{cases}$$

A little after the concept of convergence of double series with real terms (convergence in the Pringsheim's sense) introduced by Pringsheim [2], Hardy [3] introduced regular convergence, which also requires convergence according to each index. Zeltser [4] also contributed to these developments by comprehensively examining the topological structure of double sequences. The spaces of all double sequences that are convergent in the Pringsheim's sense ($\mathcal{C}_{\mathcal{P}}$), regularly convergent ($\mathcal{C}_r$), $p$-absolutely summable ($\mathcal{L}_p$) and bounded ($\mathcal{M}_u$) can be given as examples of the most basic double sequence spaces. It is known that a convergent double sequences in the Pringsheim's sense (shortly $\mathcal{P}$-convergent) need not be bounded. The space of bounded and $\mathcal{P}$-convergent double sequences is specifically denoted by $\mathcal{C}_{b\mathcal{P}}$. Additionally, for $p = 1$, the space $\mathcal{L}_p$ [5] is reduced to the space $\mathcal{L}_u$ [6]. The linear space of all double sequences with real terms is represented by $\Omega$.

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If $u = (u_{lm}) \in \Omega$ is $\vartheta$-convergent to a limit point $M$, in that case, it is expressed as $\vartheta - \lim_{l,m\to\infty} u_{lm} = M$ for $\vartheta \in \{\mathcal{P}, b\mathcal{P}, r\}$. Zeltser [6] described the double sequences $e^{rk} = (e^{rk}_{lm})$ by

$$e^{rk}_{lm} := \begin{cases} 1 & , \quad \text{if } (r,k) = (l,m) \\ 0 & , \quad \text{otherwise} \end{cases}$$

and $e$ by $e = \sum_{r,k} e^{rk}$, where $\sum_{r,k} e^{rk} = \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} e^{rk}$. If $b_{rklm} = 0$ for $l > r$ or $m > k$ or both, the 4d matrix $B = (b_{rklm})$ is called as *triangular matrix* and also if $b_{rkrk} \neq 0$, then the 4d triangular matrix $B$ is named as *triangle* for all $r, k, l, m \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, 3, ...\}$.

Consider that $\Psi, \Lambda \in \Omega$, $u = (u_{lm}) \in \Psi$ and the 4d matrix $B = (b_{rklm})$. If $(Bu)_{rk} = \vartheta - \sum_{l,m} b_{rklm}u_{lm}$ (the $B$-transform of $u$) is in $\Lambda$, in that case $B$ is called as a matrix mapping from $\Psi$ into $\Lambda$ and it is denoted by $B : \Psi \to \Lambda$ for all $u = (u_{lm}) \in \Psi$. Moreover, $B \in (\Psi : \Lambda)$ if and only if $B_{rk} \in \Psi^{\beta(\vartheta)}$ and $Bu \in \Lambda$, where $B_{rk} = (b_{rklm})_{l,m\in\mathbb{N}}$, $(\Psi : \Lambda) = \{B = (b_{rklm}) |\ B : \Psi \to \Lambda\}$ for all $r, k \in \mathbb{N}$ and $\Psi^{\beta(\vartheta)}$ is the $\beta(\vartheta)$ dual of $\Psi$.

The $\vartheta$-summability domain $\Psi^{(\vartheta)}_B$ of the 4d matrix $B$ is expressed as

$$\Psi^{(\vartheta)}_B := \left\{ u = (u_{lm}) \in \Omega : Bu := \left( \vartheta - \sum_{l,m} b_{rklm}u_{lm} \right)_{r,k\in\mathbb{N}} \text{exists, } Bu \in \Psi \subset \Omega \right\}. \tag{1.1}$$

The 4d matrix that transforms bounded and $\mathcal{P}$-convergent double sequences into $\mathcal{P}$-convergent double sequences with the same limit is called as RH regular [7, 8].

The double series spaces $\mathcal{BS}$ and $\mathcal{CS}_\vartheta$ spaces, whose sequences of partial sums are in the spaces $\mathcal{M}_u$ and $\mathcal{C}_\vartheta$, respectively, are described by Altay and Başar [9]. In addition to other related studies on single and double sequence spaces, some $q$-analogue studies and their references can be also expressed as [10–34].

Recently, Erdem and Demiriz [35] constructed a new double sequence space using the domain in $\mathcal{L}_p$ space of the 4d $q$-Cesàro matrix operator ($q$-analogue of the ordinary 4d Cesàro matrix) presented by Çinar and Et [36] and examined some algebraic and topological properties of this space.

As a continuation of the studies mentioned above, this article aims to acquaint the new double sequence space $\tilde{\mathcal{M}}^q_u$ as the domain of the 4d $q$-Cesàro matrix on the space $\mathcal{M}_u$, to examine its completeness, to determine its duals and to present some matrix mappings classes related aforementioned space.

## 2. $q$-Cesàro bounded double sequence space $\tilde{\mathcal{M}}^q_u$

In this section, the space $\tilde{\mathcal{M}}^q_u \in \Omega$ is constructed and we obtain that $\tilde{\mathcal{M}}^q_u$ is Banach space and linearly isomorphic to $\mathcal{M}_u$. Finally, an inclusion relation is presented about the space $\tilde{\mathcal{M}}^q_u$.

The 4d Cesàro matrix $C = (c_{rklm})$ of order one is given by

$$c_{rklm} := \begin{cases} \dfrac{1}{(r+1)(k+1)} & , \quad 0 \le l \le r\ ,\ 0 \le m \le k, \\ \\ 0 & , \quad \text{otherwise,} \end{cases} \tag{2.1}$$

for all $r, k, l, m \in \mathbb{N}$. The 4d $q$-Cesàro matrix $C_{(1,1)}(q) = (c_{zntk}(q))$ that is the $q$-analogue of the matrix $C$ and presented by Çinar and Et [36], is in the form below:

$$c_{rklm}(q) := \begin{cases} \dfrac{q^{l+m}}{[r+1]_q[k+1]_q} & , \quad 0 \le l \le r\ ,\ 0 \le m \le k, \\ \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.2}$$

In the same study, the authors showed that $C_{(1,1)}(q)$ is RH-regular for $q \ge 1$. The inverse $(C_{(1,1)}(q))^{-1}$ of the $C_{(1,1)}(q)$ is presented by

$$c^{-1}_{rklm}(q) := \begin{cases} (-1)^{r+k-(l+m)} \dfrac{[l+1]_q[m+1]_q}{q^{r+k}} & , \quad r-1 \le l \le r\ ,\ k-1 \le m \le k, \\ \\ 0 & , \quad \text{otherwise.} \end{cases} \tag{2.3}$$

From the mentioned above, it can be seen that the $C_{(1,1)}(q)$-transform of a $u=(u_{tk})\in\Omega$ is denoted by

$$\nu_{rk} := (C_{(1,1)}(q)u)_{rk} = \frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k} q^{l+m}u_{lm}, \quad (r,k\in\mathbb{N}). \tag{2.4}$$

It can be said that for the case $q\to 1$, $C_{(1,1)}(q)$ will be reduced to $C$.

Now, it is acquainted the set $\tilde{\mathcal{M}}_u^q$ of all $q$-Cesàro bounded double sequences by

$$\tilde{\mathcal{M}}_u^q = \left\{u=(u_{lm})\in\Omega : \sup_{r,k}\left|\frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k} q^{l+m}u_{lm}\right|<\infty\right\}.$$

Thus, $\tilde{\mathcal{M}}_u^q$ can be rephrased as $\tilde{\mathcal{M}}_u^q = (\mathcal{M}_u)_{C_{(1,1)}(q)}$ with the impression (1.1) and it can be called as $q$-Cesàro bounded double sequence space.

When $q$ approaches 1, $\tilde{\mathcal{M}}_u^q$ is reduced to the space $\tilde{\mathcal{M}}_u$ presented in [37]. From now on, any term with a negative index will be ignored and assumed to be $q>1$.

**Theorem 2.1.** *The set $\tilde{\mathcal{M}}_u^q$ is a Banach space with*

$$\|u\|_{\tilde{\mathcal{M}}_u^q} = \|C_{(1,1)}(q)u\|_{\mathcal{M}_u} = \left(\sup_{r,k\in\mathbb{N}}\left|\frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k} q^{l+m}u_{lm}\right|\right). \tag{2.5}$$

*Proof.* It is a known procedure to show that $\tilde{\mathcal{M}}_u^q$ is a normed linear space with (2.5) and it is omitted.

Consider the Cauchy sequence $u^{(n)}=\left(u_{lm}^{(n)}\right)\in\tilde{\mathcal{M}}_u^q$ for $n\in\mathbb{N}$. In that case, $\forall\varepsilon>0$, $\exists M\in\mathbb{N}$ such that

$$\begin{aligned}\|u^{(n)}-u^{(z)}\|_{\tilde{\mathcal{M}}_u^q} &= \left(\sup_{r,k}\left|\frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k} q^{l+m}\left(u_{lm}^{(n)}-u_{lm}^{(z)}\right)\right|\right)\\ &= \left(\sup_{r,k}\left|\left(C_{(1,1)}(q)u^{(n)}\right)_{rk}-\left(C_{(1,1)}(q)u^{(z)}\right)_{rk}\right|\right)<\varepsilon \end{aligned} \tag{2.6}$$

for all $n,z>M$ and it is reached that $\left\{\left(C_{(1,1)}(q)u^{(n)}\right)_{rk}\right\}_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{M}_u$. From the completeness of $\mathcal{M}_u$, $\left\{\left(C_{(1,1)}(q)u^{(n)}\right)_{rk}\right\}_{n\in\mathbb{N}}$ converges and it can be written that $\left\{\left(C_{(1,1)}(q)u^{(n)}\right)_{rk}\right\}_{n\in\mathbb{N}}\to\left(C_{(1,1)}(q)u\right)_{rk}$ for $n\to\infty$. In that case, we may define the sequence $\left(C_{(1,1)}(q)u\right)_{rk}$. After all of these, it must be proven that $\left(C_{(1,1)}(q)u\right)_{rk}\in\mathcal{M}_u$. From $\left\{\left(C_{(1,1)}(q)u^{(n)}\right)_{rk}\right\}_{n\in\mathbb{N}}\in\mathcal{M}_u$, it is obtained that $\left(\sup_{r,k}\left|\left(C_{(1,1)}(q)u^{(n)}\right)_{rk}\right|\right)<\infty$. So, we see that $\left(C_{(1,1)}(q)u\right)_{rk}\in\mathcal{M}_u$ from

$$\begin{aligned}\|\left(C_{(1,1)}(q)u\right)_{rk}\|_{\mathcal{M}_u} &= \left(\sup_{r,k}\left|\left(C_{(1,1)}(q)u\right)_{rk}\right|\right)\\ &\le \left(\sup_{r,k}\left|\left(C_{(1,1)}(q)u^{(n)}\right)_{rk}-\left(C_{(1,1)}(q)u\right)_{rk}\right|\right)\\ &+ \left(\sup_{r,k}\left|\left(C_{(1,1)}(q)u^{(n)}\right)_{rk}\right|\right)<\infty\end{aligned}$$

by applying limit on (2.6) for $z\to\infty$. Consequently, $u\in\tilde{\mathcal{M}}_u^q$ and $\tilde{\mathcal{M}}_u^q$ is complete with $\|.\|_{\tilde{\mathcal{M}}_u^q}$. □

**Theorem 2.2.** *$\tilde{\mathcal{M}}_u^q$ is linearly norm isomorphic to $\mathcal{M}_u$.*

*Proof.* The linearity of the mapping described as $\Upsilon:\tilde{\mathcal{M}}_u^q\to\mathcal{M}_u$, $\Upsilon(u)=C_{(1,1)}(q)u$ is obvious for $u=(u_{lm})\in\tilde{\mathcal{M}}_u^q$. Additionally, from the expression $\Upsilon(u)=0\Rightarrow u=0$, $\Upsilon$ is injective.

Let us consider the sequences $\nu=(\nu_{lm})\in\mathcal{M}_u$ and $u=(u_{lm})$ as follows:

$$u_{rk}=\frac{1}{q^{r+k}}\sum_{l=r-1}^{r}\sum_{m=k-1}^{k}(-1)^{r+k-(l+m)}[l+1]_q[m+1]_q\nu_{lm}\quad (r,k\in\mathbb{N}). \tag{2.7}$$

Then, from the equality

$$\begin{aligned}
\|u\|_{\tilde{\mathcal{M}}_u^q} &= \left(\sup_{r,k}\left|\frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k}q^{l+m}u_{lm}\right|\right)\\
&= \left(\sup_{r,k}\left|\frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k}q^{l+m}\sum_{i=l-1}^{l}\sum_{j=m-1}^{m}\frac{1}{q^{l+m}}(-1)^{l+m-(i+j)}[i+1]_q[j+1]_q\nu_{ij}\right|\right)\\
&= \left(\sup_{r,k}|\nu_{rk}|\right)=\|\nu\|_{\mathcal{M}_u}<\infty,
\end{aligned}$$

it is seen that $\Upsilon$ is surjective. Finally, since $\|u\|_{\tilde{\mathcal{M}}_u^q}=\|\nu\|_{\mathcal{M}_u}$, in that case $\Upsilon$ is norm keeping. □

**Theorem 2.3.** *The inclusion $\mathcal{M}_u\subset\tilde{\mathcal{M}}_u^q$ holds.*

*Proof.* Consider that $u=(u_{lm})\in\mathcal{M}_u$. In that case, it can be written that $\sup_{l,m}|u_{lm}|<\delta$ for at least positive real number $\delta$. Consequently, it is achieved that

$$\begin{aligned}
\|u\|_{\tilde{\mathcal{M}}_u^q} &= \sup_{r,k}\left|\frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k}q^{l+m}u_{lm}\right|\\
&\leq \sup_{r,k}\left|\frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k}q^{l+m}\right||u_{lm}|\\
&\leq \delta\sup_{r,k}\left|\frac{1}{[r+1]_q[k+1]_q}\sum_{l,m=0}^{r,k}q^{l+m}\right|=\delta
\end{aligned}$$

and thus $\mathcal{M}_u\subset\tilde{\mathcal{M}}_u^q$. □

## 3. Dual spaces

In this section, the $\alpha$-, $\beta(\mathcal{P})$-, $\beta(b\mathcal{P})$-and $\gamma$-duals of $\tilde{\mathcal{M}}_u^q$ are determined. For $\Psi,\Lambda\in\Omega$, the set $D(\Psi:\Lambda)$ is defined by

$$D(\Psi:\Lambda)=\left\{\tau=(\tau_{rk})\in\Omega:\tau u=(\tau_{rk}u_{rk})\in\Lambda\quad\text{for all}\quad(u_{rk})\in\Psi\right\}.$$

Then, $\alpha$-, $\beta(\vartheta)$- and $\gamma$-duals of $\Psi$ are defined as

$$\Psi^{\alpha}=D(\Psi:\mathcal{L}_u),\quad \Psi^{\beta(\vartheta)}=D(\Psi:\mathcal{CS}_{\vartheta})\quad\text{and}\quad\Psi^{\gamma}=D(\Psi:\mathcal{BS}).$$

**Theorem 3.1.** $\left[\tilde{\mathcal{M}}_u^q\right]^{\alpha}=\mathcal{L}_u$.

*Proof.* Consider the sequences $u=(u_{lm})\in\tilde{\mathcal{M}}_u^q$ with $\nu=(\nu_{lm})\in\mathcal{M}_u$ and $\tau=(\tau_{lm})\in\mathcal{L}_u$. In that case, $|\nu_{lm}|<N<\infty$ for at least $N>0$ for all $l,m\in\mathbb{N}$.

By using the equality (2.7), it is obtained the inequality

$$
\begin{aligned}
\sum_{l,m}|\tau_{lm}u_{lm}| &= \sum_{l,m}\left|\tau_{lm}\sum_{i=l-1}^{l}\sum_{j=m-1}^{m}\frac{(-1)^{l+m-(i+j)}}{q^{l+m}}[i+1]_q[j+1]_q\nu_{ij}\right| \\
&\leq N\sum_{l,m}|\tau_{lm}|\left|\sum_{i=l-1}^{l}\sum_{j=m-1}^{m}\frac{(-1)^{l+m-(i+j)}}{q^{l+m}}[i+1]_q[j+1]_q\right| \\
&= N\sum_{l,m}|\tau_{lm}|<\infty
\end{aligned}
$$

which gives that $\tau\in\left[\tilde{\mathcal{M}}_u^q\right]^\alpha$ and thus $\mathcal{L}_u\subset\left[\tilde{\mathcal{M}}_u^q\right]^\alpha$.

On the other hand, consider that $\tau\in\left[\tilde{\mathcal{M}}_u^q\right]^\alpha\backslash\mathcal{L}_u$. In that case, $\sum_{l,m}|\tau_{lm}u_{lm}|<\infty$ for all $u=(u_{lm})\in\tilde{\mathcal{M}}_u^q$. For choosing $e\in\tilde{\mathcal{M}}_u^q$, since $\tau e=\tau\notin\mathcal{L}_u$, it is reached the contradiction $\tau\notin\left[\tilde{\mathcal{M}}_u^q\right]^\alpha$. Thus, it should be $\tau\in\mathcal{L}_u$. □

We can express the necessary conditions for the matrix class characterizations that will be used in this and the next section and the matrix classes with the help of a lemma as follows:

$$\sup_{r,k\in\mathbb{N}}\sum_{l,m}|b_{rklm}|<\infty, \tag{3.1}$$
$$\exists a_{lm}\in\mathbb{C}\ni\vartheta-\lim_{r,k\to\infty}b_{rklm}=a_{lm}\quad\text{subsists}, \tag{3.2}$$
$$\forall l\in\mathbb{N},\quad\exists m_0\ni b_{rklm}=0,\quad\forall m>m_0, \tag{3.3}$$
$$\forall m\in\mathbb{N},\quad\exists l_0\ni b_{rklm}=0,\quad\forall l>l_0, \tag{3.4}$$
$$\sup_{r,k,l,m\in\mathbb{N}}|b_{rklm}|<\infty, \tag{3.5}$$
$$\sup_{r,k\in\mathbb{N}}\sum_{l,m}|b_{rklm}|^{p'}<\infty, \tag{3.6}$$
$$\exists a_{lm}\in\mathbb{C}\ni\quad bp-\lim_{r,k\to\infty}\sum_{l,m}|b_{rklm}-a_{lm}|=0, \tag{3.7}$$
$$bp-\lim_{r,k\to\infty}\sum_{l=0}^{r}b_{rklm}\quad\text{subsists},\quad\forall m\in\mathbb{N}, \tag{3.8}$$
$$bp-\lim_{r,k\to\infty}\sum_{m=0}^{k}b_{rklm}\quad\text{subsists},\quad\forall l\in\mathbb{N}, \tag{3.9}$$
$$\sum_{l,m}|b_{rklm}|\quad\text{converges}, \tag{3.10}$$

where $\frac{1}{p}+\frac{1}{p'}=1$.

**Lemma 3.1.** *[6, 7, 38] For $B=(b_{zntk})\in\Omega$, the following statements hold:*

**(i)** *$B\in(\mathcal{M}_u:\mathcal{M}_u)$ iff the condition (3.1) holds.*

**(ii)** *$B\in(\mathcal{M}_u:\mathcal{C}_{\mathcal{P}})$ iff the conditions (3.2), (3.3) and (3.4) hold.*

**(iii)** *$B\in(\mathcal{M}_u:\mathcal{C}_{b\mathcal{P}})$ iff the conditions (3.1), (3.2), (3.7), (3.8), (3.9) and (3.10) hold.*

**(iv)** *$B\in(\mathcal{C}_{b\mathcal{P}}:\mathcal{M}_u:)$ iff the condition (3.1) holds.*

**(v)** *For $0<p\leq 1$, $B\in(\mathcal{L}_p:\mathcal{M}_u)$ iff the condition (3.5) holds.*

**(vi)** *For $1<p<\infty$, $B\in(\mathcal{L}_p:\mathcal{M}_u)$ iff the condition (3.6) holds.*

It can be given the abbreviations to be used in the next theorem as follows:

$$\begin{aligned}
\Delta_{11}\left(\frac{\tau_{lm}}{q^{l+m}}\right) &= \left(\frac{\tau_{lm}}{q^{l+m}} - \frac{\tau_{l+1,m}+\tau_{l,m+1}}{q^{l+m+1}} + \frac{\tau_{l+1,m+1}}{q^{l+m+2}}\right), \\
\Delta_{10}\left(\frac{\tau_{lk}}{q^{l+k}}\right) &= \left(\frac{\tau_{lk}}{q^{l+k}} - \frac{\tau_{l+1,k}}{q^{l+k+1}}\right), \\
\Delta_{01}\left(\frac{\tau_{rm}}{q^{r+m}}\right) &= \left(\frac{\tau_{rm}}{q^{r+m}} - \frac{\tau_{r,m+1}}{q^{r+m+1}}\right).
\end{aligned} \tag{3.11}$$

**Theorem 3.2.** *Consider that $\Psi \subset \Omega$, $\tau = (\tau_{lm}) \in \Omega$ and the 4d infinite matrix $O = (o_{rklm})$ described by*

$$o_{rklm} := \begin{cases} [l+1]_q[m+1]_q\Delta_{11}\left(\frac{\tau_{lm}}{q^{l+m}}\right) & , \quad 0 \le l \le r-1, \quad 0 \le m \le k-1, \\ [l+1]_q[k+1]_q\Delta_{10}\left(\frac{\tau_{lk}}{q^{l+k}}\right) & , \quad 0 \le l \le r-1, \quad m = k, \\ [r+1]_q[m+1]_q\Delta_{01}\left(\frac{\tau_{rm}}{q^{r+m}}\right) & , \quad 0 \le m \le k-1, \quad l = r, \\ \frac{[r+1]_q[k+1]_q\tau_{rk}}{q^{r+k}} & , \quad m = k, \quad l = r, \\ 0 & , \quad \text{elsewhere} \end{cases} \tag{3.12}$$

*for all $r, k, l, m \in \mathbb{N}$.*

*In that case;*

**(i)** $\left[\tilde{\mathcal{M}}_u^q\right]^{\beta(\vartheta)} = \{\tau = (\tau_{lm}) : O \in (\mathcal{M}_u : C_\vartheta)\}$*, where $\vartheta \in \{\mathcal{P}, b\mathcal{P}\}$.*

**(ii)** $\left[\tilde{\mathcal{M}}_u^q\right]^{\gamma} = \{\tau = (\tau_{lm}) : O \in (\mathcal{M}_u : \mathcal{M}_u)\}$.

*Proof.* **(i)** Consider the sequences $\tau = (\tau_{lm}) \in \Omega$ and $u \in \tilde{\mathcal{M}}_u^q$ with $\nu \in \mathcal{M}_u$ with the relation (2.4). By bearing in mind the equation (2.7), it is reached that

$$\begin{aligned}
\sigma_{rk} &= \sum_{l,m=0}^{r,k} \tau_{lm}u_{lm} = \sum_{l,m=0}^{r,k} \tau_{lm}\left(\frac{1}{q^{l+m}}\sum_{i=l-1}^{l}\sum_{j=m-1}^{m}(-1)^{l+m-(i+j)}[i+1]_q[j+1]_q\nu_{ij}\right) \\
&= \sum_{l=0}^{r-1}[l+1]_q[k+1]_q\Delta_{10}\left(\frac{\tau_{lk}}{q^{l+k}}\right)\nu_{lk} + \sum_{m=0}^{k-1}[r+1]_q[m+1]_q\Delta_{01}\left(\frac{\tau_{rm}}{q^{r+m}}\right)\nu_{rm} \\
&+ \sum_{l=0}^{r-1}\sum_{m=0}^{k-1}[l+1]_q[m+1]_q\Delta_{11}\left(\frac{\tau_{lm}}{q^{l+m}}\right)\nu_{lm} + \frac{[r+1]_q[k+1]_q\tau_{rk}}{q^{r+k}}\nu_{rk} = (O\nu)_{rk}
\end{aligned} \tag{3.13}$$

for $O = (o_{rklm})$ is defined by (3.12). Thus, by using (3.13), we reach that $\tau u = (\tau_{lm}u_{lm}) \in \mathcal{CS}_\vartheta$ whenever $u = (u_{lm}) \in \tilde{\mathcal{M}}_u^q$ iff $\sigma = (\sigma_{rk}) \in \mathcal{C}_\vartheta$ whenever $\nu \in \mathcal{M}_u$. Consequently, $\tau \in \left[\tilde{\mathcal{M}}_u^q\right]^{\beta(\vartheta)}$ iff $O \in (\mathcal{M}_u : \mathcal{C}_\vartheta)$ for $\vartheta \in \{\mathcal{P}, b\mathcal{P}\}$.

**(ii)** It can be shown to be similar to the first part using the definition of $\gamma$-dual. So, it is omitted. □

## 4. Matrix mappings

This section contains the characterizations of matrix classes $(\tilde{\mathcal{M}}_u^q : \Lambda)$ and $(\Psi : \tilde{\mathcal{M}}_u^q)$, where $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_\mathcal{P}, \mathcal{C}_{b\mathcal{P}}\}$ and $\Psi \in \{\mathcal{M}_u, \mathcal{C}_{b\mathcal{P}}, \mathcal{L}_p\}$ for $0 < p < \infty$.

**Theorem 4.1.** *Consider that 4d matrices $B = (b_{rklm})$ and $H = (h_{rklm})$ with the equality*

$$h_{rklm} \quad = \quad [l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{r+k}}\right). \tag{4.1}$$

*Then, $B \in \left(\tilde{\mathcal{M}}_u^q : \Lambda\right)$ iff $H \in (\mathcal{M}_u : \Lambda)$ and*

$$B_{rk} \in \left(\tilde{\mathcal{M}}_u^q\right)^{\beta(\vartheta)}. \tag{4.2}$$

*Proof.* Suppose that $B \in \left(\tilde{\mathcal{M}}_u^q : \Lambda\right)$. Then, $Bu \in \Lambda$ for all $u \in \tilde{\mathcal{M}}_u^q$ with $\nu = C_{(1,1)}(q)u \in \mathcal{M}_u$. Thus, it is obtained that $B_{rk} \in \left(\tilde{\mathcal{M}}_u^q\right)^{\beta(\vartheta)}$. For the $(i,j)$th partial sums of the series $\sum_{l,m} b_{rklm}u_{lm}$, it is reached that

$$\begin{aligned}
(Bu)_{rk}^{[i,j]} &= \sum_{l,m=0}^{i,j} b_{rklm}u_{lm} \\
&= \sum_{l=0}^{i-1}\sum_{m=0}^{j-1}[l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{l+m}}\right)\nu_{lm} + \sum_{l=0}^{i-1}[l+1]_q[j+1]_q\Delta_{10}^{lj}\left(\frac{b_{rklj}}{q^{l+j}}\right) \\
&+ \sum_{m=0}^{j-1}[i+1]_q[m+1]_q\Delta_{01}^{im}\left(\frac{b_{rkim}}{q^{i+m}}\right) + \frac{[i+1]_q[j+1]_q}{q^{i+j}}b_{rkij}
\end{aligned} \tag{4.3}$$

for all $r,k,i,j \in \mathbb{N}$. Let us define the 4d infinite matrix $H_{rk} = (h_{ijlm}^{[r,k]})$ as

$$h_{ijlm}^{[r,k]} := \begin{cases}
[l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{l+m}}\right) & , \quad 0 \le l \le i-1, \quad 0 \le m \le j-1, \\
[l+1]_q[j+1]_q\Delta_{10}^{lj}\left(\frac{b_{rklj}}{q^{l+j}}\right) & , \quad 0 \le l \le i-1, \quad m = j, \\
[i+1]_q[m+1]_q\Delta_{01}^{im}\left(\frac{b_{rkim}}{q^{i+m}}\right) & , \quad 0 \le m \le j-1, \quad l = i, \\
\frac{[i+1]_q[j+1]_q}{q^{i+j}}b_{rkij} & , \quad m = j, \quad l = i, \\
0 & , \quad \text{otherwise}
\end{cases}$$

the relation (4.3) can be restated as

$$(Bu)_{rk}^{[i,j]} = (H_{rk}\nu)_{[i,j]}. \tag{4.4}$$

Moreover, if we take $\vartheta$-limit on $H_{rk} = \left(h_{ijlm}^{[r,k]}\right)$ for $i,j \to \infty$, it is obtained that

$$\vartheta - lim_{i,j\to\infty}h_{ijlm}^{[r,k]} = [l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{l+m}}\right). \tag{4.5}$$

From (4.5), it can be defined the 4d matrix $H = (h_{rklm})$ by

$$h_{rklm} = [l+1]_q[m+1]_q\Delta_{11}^{lm}\left(\frac{b_{rklm}}{q^{l+m}}\right). \tag{4.6}$$

If we take $\vartheta$-limit on (4.4) for $i,j \to \infty$, we see that $Bu = H\nu$. Thus, $H\nu \in \Lambda$ while $\nu \in \mathcal{M}_u$ and $H \in (\mathcal{M}_u : \Lambda)$.

Conversely, suppose that $B_{rk} \in \left(\tilde{\mathcal{M}}_u^q\right)^{\beta(\vartheta)}$ and $H \in (\mathcal{M}_u : \Lambda)$. Let $u \in \tilde{\mathcal{M}}_u^q$ with $\nu = C_{(1,1)}(q)u \in \mathcal{M}_u$. In this case, $Bu$ exists. By the $(i,j)$th partial sums of $\sum_{t,k} b_{zntk}u_{tk}$, it is obtained the equality

$$\sum_{l,m=0}^{i,j} b_{rklm}u_{lm} = \sum_{l,m=0}^{i,j} h_{ijlm}^{[r,k]}\nu_{lm}$$

for all $r, k, l, m \in \mathbb{N}$. If we take $\vartheta$-limit as $i, j \to \infty$ on the equation above, we reach that $Bu = H\nu$. Consequently, $B \in \left(\tilde{\mathcal{M}}_u^q : \Lambda\right)$. □

**Corollary 4.1.** *Consider that 4d matrices $B = (b_{rklm})$ and $H = (h_{rklm})$ with (4.1). Then;*

**(i)** $B \in \left(\tilde{\mathcal{M}}_u^q : \mathcal{M}_u\right)$ *iff the condition (3.1) holds with $H$ in place of $B$ and the condition (4.2) holds.*

**(ii)** $B \in \left(\tilde{\mathcal{M}}_u^q : \mathcal{C}_{\mathcal{P}}\right)$ *iff the conditions (3.2), (3.3) and (3.4) hold with $H$ in place of $B$ and the condition (4.2) holds.*

**(iii)** $B \in \left(\tilde{\mathcal{M}}_u^q : \mathcal{C}_{b\mathcal{P}}\right)$ *iff the conditions (3.1), (3.2), (3.7), (3.8), (3.9) and (3.10) hold with $H$ in place of $B$ and the condition (4.2) holds.*

**Lemma 4.1.** *[39] Suppose that $\Psi$,$\Lambda \subset \Omega$, a 4d matrix $B = (b_{rklm})$ and 4d triangle $Y = (y_{rklm})$. Then, $B \in (\Psi : \Lambda_Y)$ iff $YB \in (\Psi : \Lambda)$.*

**Corollary 4.2.** *Consider the 4d matrices $B = (b_{rklm})$ and $W = (w_{rklm})$ with the equality*

$$w_{rklm} = \sum_{l,m=0}^{r,k} c_{rkij}(q) b_{ijlm}.$$

*In that case;*

**(i)** $B \in \left(\mathcal{M}_u : \tilde{\mathcal{M}}_u^q\right)$ *iff the condition (3.1) holds with $W$ instead of $B$.*

**(ii)** $B \in \left(\mathcal{C}_{b\mathcal{P}} : \tilde{\mathcal{M}}_u^q :\right)$ *iff the condition (3.1) holds with $W$ instead of $B$.*

**(iii)** *For $0 < s \leq 1$, $B \in \left(\mathcal{L}_s : \tilde{\mathcal{M}}_u^q\right)$ iff the condition (3.5) holds with $W$ instead of $B$.*

**(iv)** *For $1 < s < \infty$, $B \in \left(\mathcal{L}_s : \tilde{\mathcal{M}}_u^q\right)$ iff the condition (3.6) holds with $W$ instead of $B$.*

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