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Developing the Big Ideas of Number

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ABSTRACT

The mathematical content knowledge (MCK) and pedagogical content knowledge (PCK) of primary and elementary teachers at all levels of experience is under scrutiny. This article suggests that content knowledge and the way in which it is linked to effective pedagogies would be greatly enhanced by viewing mathematical content from the perspective of the 'big ideas' of mathematics, especially of number. This would enable teachers to make use of the many connections and links within and between such 'big ideas' and to make them explicit to children. Many teachers view the content they have to teach in terms of what curriculum documents define as being applicable to the particular year level being taught. This article suggests that a broader view of content is needed as well as a greater awareness of how concepts are built in preceding and succeeding year levels. A 'big ideas' focus would also better enable teachers to deal with the demands of what are perceived to be crowded mathematics curricula. The article investigates four 'big ideas' of number – trusting the count, place value, multiplicative thinking, and multiplicative partitioning – and examines the 'micro-content' that contributes to their development.

Keywords:
mathematical content knowledge, teacher knowledge, pedagogical content knowledge, number

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Introduction

The case for 'Big Idea Thinking'

Teacher content knowledge for teaching mathematics has been the subject of much recent discussion, particularly in Australia (Callingham et al., 2011; Clarke, Clarke & Sullivan, 2012), New Zealand (Anakin & Linsell, 2014) and USA (Thanheiser et al., 2013; Green, 2014). Such discourse has been broad and has encompassed knowledge of teachers at all levels from pre-service teachers (PSTs) and newly graduated teachers to experienced teachers. One key reason for this has been the view that school students in western nations like Australia and USA are not faring as well in high stakes international testing as they might, especially when compared to Asian and Scandinavian nations.

Time for change

Tatto et al. (2008) noted in response to the TEDS-M study that one aspect of the concern was in relation to pedagogies. Many teacher preparation courses focused too much on 'general pedagogies' – non-subject-matter-specific theoretical aspects of teacher education programs – rather than on domain-specific pedagogies needed to effectively teach mathematics. The other aspect of the current dilemma is the mathematical content knowledge of teachers and how this needs to be organised in a more connected way.

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Over the last fifty years or so, many educators and researchers have written about this explicitly and implicitly and this will be discussed later in this paper. However, whilst new curriculum documents for teaching mathematics have been developed in both Australia and USA, they have not stemmed the levels of concern being expressed about teacher knowledge or about how mathematics should be taught. In fact, the Common Core State Standards for Mathematics (NGA Center, 2010) and the Australian Curriculum: Mathematics (ACARA, 2012) are widely viewed as being 'lost opportunities' (Atweh & Goos, 2011; Atweh, Miller & Thornton, 2012; Hurst, 2014a).

In Australia, the concern has been manifest in recent federal government initiatives including a review of the Australian Curriculum and an inquiry into teacher education (Government of Australia, 2013, 2014). These are important initiatives but unless there is a shift in how mathematics is perceived and organised, then nothing is likely to change in terms of teacher knowledge. The notion of 'big ideas' of mathematics is not new but it has, in recent years, been afforded some prominence (Charles, 2005; Clarke, Clarke & Sullivan, 2012; Siemon, Bleckley & Neal, 2012). It is suggested here that a focus on the 'big ideas' of mathematics, in particular of number, is the key to developing teachers' mathematical content knowledge and their capacity to respond effectively to curriculum documents. Charles (2005, p. 10) defines a 'big idea' as "a statement of an idea that is central to the learning of mathematics, one that links numerous mathematical understandings into a coherent whole". He contends that 'big ideas' are important because they enable us to see mathematics as a "coherent set of ideas" that encourage a deep understanding of mathematics, enhance transfer, promote memory and reduce the amount to be remembered (Charles, 2005, p. 10).

Green (2014) noted recently when observing methods for teaching mathematics that "The Americans might have invented the world's best methods for teaching math to children but it was difficult to find anyone actually using them" (p. 2). This comment is clearly related to pedagogies but it is necessarily bound up with content knowledge. One reason for this could be that many teachers see the curriculum as "a mile wide and an inch deep" (NGA Center, 2010, p. 3), a problem reflected in the Australian context as noted by Siemon, Bleckey & Neal (2012) – "A focus on the big ideas is needed to 'thin out' the overcrowded curriculum" (p. 20). The recent curricula developed in Australia and USA have continued to present content in a familiar linear fashion which does little to give teachers reason to consider that mathematics may be more than unconnected 'silos' of information. The view here is that presenting mathematical content knowledge using 'big ideas' as focal points is the way to deepen the understanding of teachers and to have a positive effect on their pedagogies. This view is supported by Gojak (2013) who noted that it is time to change the way in which mathematics education is viewed and that children need to be taught by teachers who deeply understand mathematical concepts.

Developmental, not linear

If change is to occur it needs to be based on a view of the 'big ideas' of number being developmentally linked. This clashes with the traditional linear way of presenting curriculum content. The latter encourages teachers to teach only the content 'designated' to their particular year level without necessarily ensuring that children have the pre-cursor knowledge required to be able to understand it. The situation where children may lack specific knowledge or may develop misconceptions is exacerbated the further they move through school. What needs to happen is for teachers to be encouraged to use 'big ideas' as a series of coherent concepts connected in developmental ways. That is, the foundations for some later concepts are being laid years before full understanding of the concept may manifest itself.

Big . . . little . . . big ideas

If this is to occur, teachers need to understand the 'micro content' that makes up each 'big idea' or key concept. These points of 'micro content' could also be described as 'content descriptors' or 'key learning criteria' for each 'big idea'. If a teacher knows about 'micro content' and can recognise when a child knows it or otherwise, s/he is in a better position to help that child develop a richer understanding of the key concept or 'big idea'. As well, the developmental relationship between the 'big ideas' of number will then help to ensure that the child is building a solid foundation for her/his future learning of the 'big ideas' that follow.

Notwithstanding the importance of such 'micro content', Major (2012) noted how children's misconceptions can be masked by apparent understanding. For example, a teacher could misinterpret the depth of a child's understanding of a key concept (or 'big idea') because the child might demonstrate knowledge of one particular criterion which may lead a teacher to assume that a more complete understanding is present. If the teacher had a deep, rich, and connected understanding of the particular concept, or 'big idea', then s/he would likely be prompted to further investigate and probe the child's thinking.

Conceptual development

The focus on 'big ideas' is not new and can be traced back at least to the work of Bruner (1960) with his emphasis on concepts. Bruner described four essential functions of concepts – they provide structure for a discipline, provide a framework for more easily understanding and recalling details, act as bridges for transfer of learning, and hence provide a structure for on-going learning. These features are quite obviously common to what are called 'big ideas'. Noting Bruner's work, Clark (2011) provided his own definition of a concept:

> *My working definition of "concept" is a big idea that helps us makes sense of, or connect, lots of little ideas. Concepts are like cognitive file folders. They provide us with a framework or structure within which we can file an almost limitless amount of information. One of the unique features of these conceptual files is their capacity for cross-referencing (Clark, 2011, p. 32)*

In 1993, Brooks and Brooks (as cited by Clark, 2011) said that there was a further function of concepts, that being the provision of a framework with which individuals can construct their own understanding. This is inherently linked to the earlier work of Skemp (1976) who described relational understanding as a "building up [of] a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point"(p. 14). More recently Van de Walle, Karp and Bay-Williams (2013) represented Skemp's ideas on a continuum, illustrating relational understanding at one end of the continuum being characterised by multiple connections within and between ideas and instrumental understanding characterised by no or very few such connections.

Connectedness and transfer

It was noted earlier that the 'connectedness' of mathematical content knowledge has been explicitly and implicitly discussed by numerous educators and researchers. In his seminal paper about knowledge growth, Schulman (1986) discussed "substantive structures [as being the] ways in which the basic concepts and principles of the discipline are organized to incorporate its facts" (p. 9). These 'structures' could be said to be akin to the links and connections of 'big ideas' (Hurst, 2014b). Later, Hiebert and Carpenter (1992) noted how understanding depends on a 'network of representations' and Ma (1999) identified 'knowledge packages' where ideas are connected through 'concept knots'. Given the depth and breadth of informed comment about the connected nature of knowledge within a conceptual structure such as 'big ideas', Clark's (2011) comment about transfer of learning is somewhat chilling –"The primary reason that so many adults are unable to transfer what has been learned in one situation to a different situation, is because they have been programmed to think linearly, inductively, and in little boxes" (p. 34). Clark's comment may have been written long before the development of the Common Core State Standards for Mathematics and the Australian Curriculum: Mathematics but it describes the contemporary situation well. Curriculum content is still presented in the same linear fashion as it was in previous curriculum documents and, as a consequence, many teachers continue to teach it in the same unconnected way and inevitably, many children learn it in the same unconnected way.

What are the 'Big Ideas'?

In deciding what 'big ideas' might be and/or look like, it is necessary to consider Charles's (2005) work in which he described twenty one 'big ideas' of mathematics and noted, as did Clarke, Clarke & Sullivan (2012), that it would be unlikely to obtain universal agreement amongst teachers and teacher educators about what precisely such 'big ideas' should be. Siemon, Bleckly & Neal (2012) took a more particular stance in discussing the 'big ideas' of number in terms of how they were presented in the Australian Curriculum: Mathematics and described six 'big ideas of number' which form the basis of the graphic illustration that follows (Figure 1). None of the six ideas presented by Siemon et al. are the same as any of those presented by Charles (2005) apart from *Proportional Reasoning* which Charles termed *Proportionality*. However, the ideas presented by Siemon et al. are embedded in Charles' 'big ideas' in various ways.

Charles' (2005) first 'big idea' is termed *Numbers* and he discusses 'counting numbers' which effectively describes what Siemon et al. (2012) discuss as *Trusting the Count*. Charles' second 'big idea' is *The Base Ten Numeration System* in which he includes what Siemon et al. (2012) have termed *Place Value*. However, as part of his second 'big idea', Charles also discusses the idea that "each place value to the left of another is ten times greater than the one to the right" (2005, p. 13) which is an essential element of the idea of *Multiplicative Thinking* as described by Siemon et al. (2012). In a similar way, Charles has embedded elements of Siemon et al.'s (2012) *Multiplicative Partitioning* in his first 'big idea' of *Numbers* where he discusses fractions and rational numbers and in his fourth 'big idea'(*Comparison*) where he discusses fractions and percent.

A developmental, hierarchical view

Charles' (2005) discussion of the 'big ideas' highlights the important connections that exist within and between the ideas and across various content areas of mathematics. Notwithstanding that, any given number of people might consider the 'big ideas' in a range of ways, the discussion of the 'six big ideas' of number' by Siemon et al. (2012) has one particular strength. It highlights that there is a hierarchical aspect to the development of the six ideas which is presented in a table showing approximate age levels at which it is reasonably expected children would have an understanding of each 'big idea'. This has been adapted to form the graphic that is Figure 1. This also shows how there are elements of each 'big idea' that necessarily develop alongside other ideas. For example, Siemon et al. (2012) note that *Multiplicative Partitioning* should be well developed by the end of Year Six, yet it is clear that many aspects or pre-conditions for its full development are present when children learn about *Trusting the Count, Place Value and Multiplicative Thinking*.

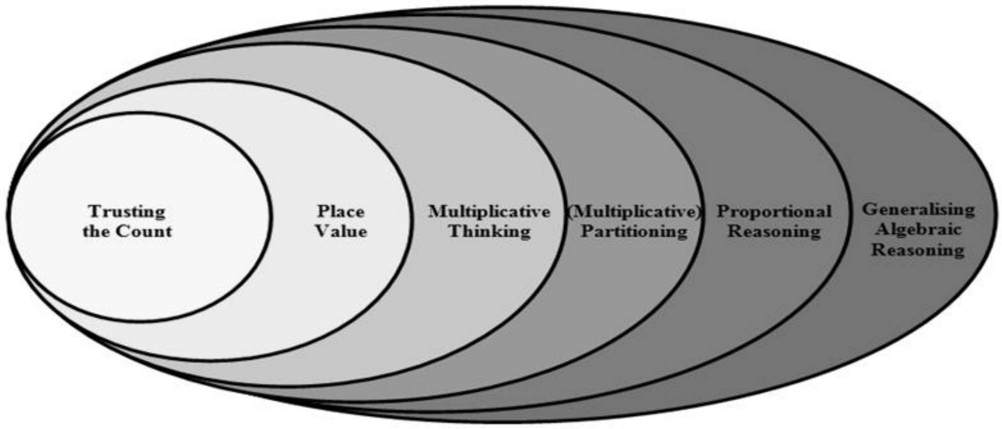

Figure 1. Development of the big ideas of number

The relationship between the 'big ideas' as depicted in Figure 1, should be considered alongside the set of criteria for determining the extent of development of children's understanding of each idea. This is shown later in this article as a series of lists which accompany each 'big idea' and highlights 'landmark' or critical points of development within each of the 'big ideas'. These lists were constructed using the article by Siemon, Bleckley and Neal (2012) as a reference point and is also informed by diagnostic maps from First Steps in Mathematics (Department of Education, Western Australia, 2013b) (FSiM) and by the work of Van de Walle, Karp and Bay-Williams (2013), Siemon, Beswick, Brady, Clark, Faragher and Warren (2011), and Reys et al. (2012).

Of the six 'big ideas', four of them; trusting the count, place value, multiplicative thinking and (multiplicative) partitioning, are firmly rooted in the primary school setting. The final two, proportional reasoning and generalizing algebraic reasoning are developmentally more suitable in secondary school (Siemon et al, 2012). Even so, the rudiments of algebraic reasoning are very much underpinned by an understanding of pattern which constitutes much of essential early number experiences children should have (Siemon et al., 2011). Similarly, many aspects of proportional reasoning are directly attributable to multiplicative thinking and develop simultaneously with it, and multiplicative partitioning, as children work flexibly with fractions. An example of this is how links are made between the concept of equal shares and the fraction construct for division. Connections such as this are explored further later in this article.

The remainder of this paper will examine the 'big ideas' situated in the primary school years, with a particular emphasis on multiplicative thinking. It will briefly examine each of the big ideas, how they are inter-related and how they become apparent and are enacted in the primary classroom. In keeping with the position stated earlier about the central importance of teacher content knowledge, each 'big number idea' has been examined to determine the required 'micro content'. It is essential to identify the key components of each 'big number idea' for teachers to be better positioned to assist students in moving through the trusting the count phase to place value, to multiplicative thinking and to multiplicative partitioning. It needs to be recognized at the outset that this progression through the ideas is not a linear process, but a developmental one.

Making explicit connections: Identifying connecting conduits

In order to develop fully each of the big ideas of number, the myriad connections that exist within and between them need to be identified and understood. This means knowing about underpinning concepts and ideas and how they are linked. As well, certain critical aspects of thinking and where they fit into the developmental sequence also need to be identified. For instance, it is vital that children understand the principles of counting and can count fluently in different ways. However, it is even more important that they move on from an additive approach and begin to think multiplicatively if they are to progress beyond a basic level of mathematical understanding. What then, are the ideas, concepts, and associated experiences that are the conduits by which connections are made? That is, how are trusting the count, place value, and multiplicative thinking connected and related?

It is evident that certain key ideas need to be in place for children to 'trust the count'. The quantification goal of both counting and subitizing depends upon ample early experiences with sorting, classification, grouping and patterns, as children learn about conservation of number (Reys et al., 2012; Department of Education, 2013b). Once children 'trust the count', they are able to understand place value through the entity of the 'ten group'. At this stage, a number of connected ideas loom large and the development of these ideas positions children to think multiplicatively and hence proportionately and algebraically. Numerous mathematics educators and researchers (for example, Jacob & Mulligan, 2014; Young-Loveridge, 2005) have identified the multiplicative array as a key idea in developing children's thinking.

It is not just the construct of the array but the way it links to other ideas that is important. There are obvious connections between it and the notion of equal sharing and grouping, and hence multiplication and division, as well as the commutative and distributive properties of multiplication and the ideas of part-part-whole understanding and flexible partitioning. Jacob & Mulligan (2014) also specifically note how teachers can use the array pattern "to focus students' attention on all three quantities at once . . . the number of groups, the number in each group, and the whole amount, as well as the associated language" (p. 37). Indeed, the use of the terms 'factor' and 'multiple' needs to be an integral part of working and learning with arrays.

Hence it is important for teachers to deeply understand how the 'big number ideas' are inextricably linked through representations such as arrays. This is the essence of 'big idea thinking', part of which is to be able to identify particular points of 'micro content' that underpin to some extent the development of later ideas. Examples of this will be given throughout the next section which deals with four of the 'big number ideas' and their component parts.

Trusting the count

The first of the 'big number ideas' is trusting the count. Originally the term trusting the count was coined by Willis (2002) to highlight how students may not understand that the number said at the end of the counting act represented the total, and was invariant, in that if counted again the same number would be reached. In more recent times the definition of trusting the count has broadened from just being the invariant result, to also mean "…a child's capacity to access flexible mental objects for the numbers 0 -10" (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2011, p.197).

Although a detailed account of trusting the count will not be pursued here it is not to underestimate the importance of trusting the count or the difficulty in the teaching and learning of it. It is however an acknowledgement that there is much research and literature (for example, Department of Education, Western Australia, 2013b; Gelman &Gallistel, 1978) available to guide, particularly the early childhood teacher, through good pedagogical practices to position the students to be able to achieve this particular 'big idea'. As already noted, 'big ideas' are constructed from many 'little ideas' or 'micro content', and so it is with trusting the count. Siemon, Bleckly and Neal (2012) identify a number of such ideas which support the development of trusting the count, and these are further enhanced through a study of First Steps in Mathematics materials (Department of Education, Western Australia, 2013b) and the work of Reys et al. (2012), and Van de Walle, Karp and Bay-Williams (2013). The following list has been developed as an indication of the points of 'micro content' or key understandings that together comprise the 'big number idea' of trusting the count. The addition of the italicised phrases is to illustrate some of the links which exist between the 'big ideas'.

- Early number experiences – Classifying, grouping, ordering, patterns – underpin the development of this idea
- Each object is counted once – one to one correspondence
- Collections can be compared on a one-one basis
- Arrangement of objects in a count does not change the quantity
- In a count, the last number signifies quantity
- Purpose of counting or subitizing is to quantify
- Counting numbers (the number string) are always said in the same order
- Counting on and back can be used to solve simple problems

Other aspects of trusting the count can be shown to directly link to aspects of 'big number ideas' that follow as shown below in the second part of the list.

- Subitizing or instant recognition of small groups can be a means of quantifying – *directly informs the concept of the 'ten group' which underpins place value*
- Small numbers can be seen as the combination of other numbers
- There are multiple ways of seeing grouping of objects
- The part-part-whole relationship can be used as the basis for operating
- Basic addition facts always give the same result irrespective of arrangement – *these four points inform the ideas of flexible partitioning and the distributive property of multiplication*
- Addition and subtraction situations can be considered in terms of a whole and two parts, one of which is unknown or missing
- Additive thinking is employed to solve problems with small numbers – *these points are important precursors to understanding the links between multiplication and division and operating with numbers*
- Skip counting to find the total will give the same result as one-one counting – *this informs the understanding of patterns in the base ten number system and patterns in multiplication facts*
- Share portions from a quantity and know that there more portions there are, the smaller will be the portions – *this informs the understanding of the relationship between multiplication and division, multiplicative partitioning, and proportional reasoning.*

Most teachers, particularly those in the early childhood setting would recognise the elements in the above list, appreciate their place in the development of trusting the count and have a clear understanding of appropriate pedagogy. They would also acknowledge the understandings developed during the trusting the count phase have implicit links, and overlaps, even if not immediately developed, with the second 'big idea' place value and beyond. This is the importance of 'big idea thinking' in that it helps teachers realise the extent to which seemingly simple ideas are the building blocks for other more complex and powerful ideas. Trusting the count underpins the essential element of place value, that is, the 'ten group' which can be counted and manipulated as an entity. At the time of moving the students into numbers beyond ten it is highly likely that such understandings will be emerging and in need of attention in the teaching and learning.

Place Value

What some teachers may find less obvious is the importance of making the connections between trusting the count and place value more explicit. A view that students will intuitively develop an understanding of place value perhaps deserves further scrutiny. Place value is a complex process which is "…subject to considerable inter-individual variability" (Moeller, Pixner, Zuber, Kaufmann & Nuerk, 2011, p. 1839), and the list that follows is an indication of this complexity, showing the variety of criteria which need to be understood. Major (2012) wrote about how this complexity is quite often masked by condensing all of these key criteria into one seemingly simple construct, that of defining place value as a way to say, read and write numbers. Further Major alludes to the fact that because students can achieve the act of saying, reading and writing numbers this can often mask the fact that they are unable to generalize the multiplicative relationships within the place value system, an issue also recognised by other researchers (Irwin, 1996; Kamii, 1986; Thomas, 2004). The following list is a composite of ideas from a range of sources: Department of Education, Western Australia (2013b); Reys et al. (2012); Ross (1989); Siemon, Bleckley & Neal (2012); Siemon, Beswick, Brady, Clark, Faragher and Warren (2011); and Van de Walle, Karp and Bay-Williams (2013).

- Order of digits makes a difference
- Additive property – The quantity represented by the whole numeral is the sum of the values represented by the individual digits
- Positional property – The quantities represented by the individual digits are determined by the position they hold within the whole numeral
- Base ten property – The value of columns or positions increases by a power of ten moving from right to left and decreases by a power of ten moving from left to right – *informs the understanding of the multiplicative relationship in the base ten system*
- Multiplicative property – The value of a number is determined by the product of its face and place values – *informs the understanding of the multiplicative relationship in the base ten system*
- There are patterns in the way we read and say numbers
- There are patterns in the way we write numbers
- Patterns in the number system can help us to build other numbers
- Place value columns have names – *the above four points inform the understanding of the multiplicative nature of the cyclic pattern in the number system*
- Zero can hold a place
- A Ten group is seen as a special entity which can be counted
- The term Ten group can be applied to 'ten tens' or 'ten hundreds' and so on
- We can skip count by ten, hundred both forwards and backwards (in place value parts) – *the above four pointsinform the understanding of the multiplicative relationship in the base ten system*
- Numbers can be partitioned in flexible ways using standard and non-standard partitions – *is linked to the idea of part-part-whole and informs the understanding of the distributive property, and the understanding of the multiplicative situation (division and multiplication), factors and multiples*
- Number partitioning can be shown as indicative of digit value and place value. For example, 26 = 20 + 6 or (2 x 10) + (6 x 1) – *informs the understanding of the distributive property*

Not only does a developing understanding of place value have an impact on the immediate success of students when moving from single to multi-digit numbers, it also has impact on future mathematical attainment. Ketterlin-Geller & Chard (2011) suggest that place value is fundamental to the eventual development of algebraic reasoning, especially a conceptual understanding of the base ten number system and a facility with basic number properties. This is another illustration of the overlap and parallel development between the six 'big number ideas'.

Teacher knowledge of mathematics is an essential component of effective teaching (Ball, Hill & Bass, 2005; Young-Loveridge & Mills, 2009) and effective teaching of place value requires an understanding of the learning progression. There are several ways of viewing the development of place value. One view is described by Ross (1989) who asserted that there were four properties of the numeration system. There are the additive property, whereby the value of a numeral is determined by the sum of the values of individual digits; the positional property, where the position of a digit within a numeral determines its value; the base ten property, where there is a ten times relationship between each place and those to its left and right; and the multiplicative property, where the total value of a digit is determined by the product of its place and face values (Ross, 1989).

Another way of viewing place value development is through three phases. The first phase is unitary value, being the placement of the number in the number string (i.e. 37 is after the number 36). This is a concept which is perhaps not as easy as it might seem, as Moeller et al. (2011) insisted that children must automatically apply place value rules to place the tens and ones in the correct 'bins'; something which according to Gervasoni and Sullivan (2007), 27% of Year 2 students find problematic. Being able to place the numbers into 'bins' is important, as students who are better in determining which of two symbolic numbers is the larger, enjoy higher achievement in mathematics (De Smedt, Noël, Gilmore & Ansari, 2013).

The second phase is quantity value, that is, 36 is 30 + 6. This phase is built on additive thinking and employs standard partitioning along place value lines. Thompson (2009) stated that this understanding of place value is particularly important in employing mental computation strategies.

He concluded that for all of the four operations, the digits in the tens (and hundreds) column are seen as quantities in their own rights, 40 is not seen as four in the tens column or even 4 x 10, but forty. Further, he concludes that this is highly desirable until formal written algorithms are required (Thompson, 2009).

The third and final phase is a column value understanding of place value. That is that 36 is 3 x 10 and 6 x 1, the kind of understanding that is vital in being fluent with many standard written algorithms. Many lower and middle primary school teachers are well versed in the use of trading games and structured and unstructured materials to promote the first and second phases of place value but can at times find the third phase a challenge. This third understanding of place value is an important pre-requisite for developing an understanding of the multiplicative relationship between places in the number system (Thomas, 2004). As stated previously, it should be understood that there is a certain amount of multiplicative thinking which is developing simultaneously with trusting the count, and an increased amount with working towards an understanding of place value (See Figure 1).

The column value understanding of place value relies on a developing understanding of multiplication. There is an argument (Graveiimeijer & van Galen, 2003) to suggest that a combination of procedural (memorisation of basic multiplication and division facts)and a conceptual understanding of multiplication are both required. To move the students through quantity value place value, which is mostly additive in nature, an alternative approach emphasising the significance of the size of the unit and the number of those units in determining quantity is required (Confrey & Maloney, 2010). Larsson (2013) cautioned that if students who use additive thinking are left to practise multiplication facts, algorithms and other procedures, this may not provide them with the opportunity to develop the understanding of multiplication as something more than repeated addition of equal groups. Traditionally, teaching multiplication and division begins with the relationship between repeated addition and multiplication (Confrey & Smith, 1995) an approach which reflects a 'repeated addition' understanding of multiplication. This 'repeated addition' understanding does not necessarily provide the required broader view and the qualitative change in students thinking which is ultimately required (Barmby, Harries, Higgins & Suggate, 2009). This broader view is characterised as requiring: replication (rather than joining as in addition/subtraction); the binary rather than unary nature of multiplication, and the notion of two distinct and different inputs; commutativity for multiplication but not division; and distributivity (Barmby, Harries, Higgins & Suggate, 2009).

One method for trying to build a conceptual understanding of multiplication is the multiplicative array which will be more fully discussed in the next section. Whilst this article focuses on multiplicative arrays, it should be noted that other representations, such as the number-line also need to be employed to develop a rich understanding of column value place value. Moseley (2005) called for the use of multiple representations in mathematics education suggesting that students who experience a broader range of representations have an increased understanding of concepts. Similarly, Young-Loveridge (2005) described the need for children to have access to both counting-based strategies derived from number lines and collection-based strategies using arrays.

Both research (Ma, 1999) and anecdotal evidence would suggest that the complexity of the understandings of place value required to assist the students towards further mathematical understandings is not well understood by many teachers. This rich understanding of the specialised content knowledge (Hill, Ball and Schilling, 2008) of place value seems to elude some teachers. It is suggested here that adopting 'big idea thinking' with its inherent connections may help teachers to articulate both the complexity of place value, and how it is linked to other 'big number ideas'.

Multiplicative thinking

According to Siemon, Bleckly and Neal (2012), the third big idea is multiplicative thinking. In their research Clark & Kamii (1996) found that 52% of fifth grade students were not sound multiplicative thinkers, and the work of Siemon, Breed, Dole, Izard, and Virgona (2006) showed that up to 40% of Year 7 and 8 students performed below curriculum expectations in multiplicative thinking and at least 25% were well below expected level. Further, Siemon et al. declared that the students who are not well established with multiplicative thinking do not have the foundational knowledge and skills needed to participate effectively in further school mathematics, or to access some post-compulsory training opportunities. If we accept, that in order to understand multiplication we need the flexibility which place value affords in dealing with larger numbers, then the progression from trusting the count, through place value, and into multiplicative thinking is a reasonable one.

Multiplicative thinking is fundamental to the development of important mathematical concepts and understandings such as algebraic reasoning, proportional reasoning, rates and ratios, measurement, and statistical sampling (Mulligan & Watson, 1998; Siemon, Izard, Breed & Virgona, 2006). Siegler et al. (2012) advocate that knowledge of division and of fractions (another part of mathematics very much reliant on multiplicative thinking) are unique predictors of later mathematical achievement. However multiplicative thinking is not only a pre-cursor for later important ideas, but the beginnings of multiplicative thinking underpin place value, which in turn informs and underpins multiplicative thinking. This is a strong example for the use of 'big idea thinking'. Teachers need to understand the 'micro content' that connects big number ideas and how such ideas are 'mutually supportive' of one another. For instance, children need to understand that there is a ten times relationship that exists between places in the number system if they are to understand place value and apply it to large numbers and operations. This is the 'base ten property' (Ross, 1989) referred to earlier.

Multiplicative arrays are considered to be powerful ways in which to represent multiplication. They refer to representations of rectangular arrays in which the multiplier and the multiplicand are exchangeable. (Barmby, Harries, Higgins & Suggate, 2009; Young-Loveridge & Mills, 2009). Young-Loveridge (2005) asserted that they have the potential to allow students to visualize commutativity, associativity and distributivity, and added that array representation of multiplication should be employed alongside other representations, to "allow students to develop a deeper and more flexible understanding of multiplication/division and to fully appreciate the two-dimensionality of the multiplicative process" (p. 38-39). Nunes and Bryant's (1995) research supported the strength of arrays in relation to developing a conceptual understanding of commutativity. Wright (2011) states that multiplicative arrays embody the binary nature of multiplication, and contended that as a representation they have value in that they also connect to other mathematical ideas of measurement of area and volume and Cartesian products.

Certainly many of the points listed below as important criteria for indicating multiplicative thinking can be addressed through the use of multiplicative arrays. As with the previous lists for trusting the count and place value, there are some indications of how the specific criteria link to the other 'big number ideas' and the links shown are exemplary rather than exhaustive. The number of important criteria serves to indicate the significance of multiplicative thinking as critical 'big number idea'.

- Cyclical pattern of 100-10-1 is repeated from ones to thousands
- Cyclical pattern of 100-10-1 is repeated beyond 1000s to millions
- Ten times multiplicative relationship exists between places

- The multiplicative relationship extends to numbers less than one, that is to the right of the decimal point
- There is symmetry in the place value number system based around the ones place so that the pattern in naming wholes is reflected in naming decimals – *The above five points are both informed by and underpin place value.*
- Double count by representing one group (e.g., hold up four fingers) and counting repetitions of that group, simultaneously keeping track of the number of groups and the number in each group.
- The multiplicative relationship between quantities is expressed as 'times as many' and 'how many times larger or smaller' a number is than another number
- Numbers move a place each time they are multiplied or divided by 10 – *These two points directly inform the development of multiplicative partitioning and ratio and proportion.*
- Basic number facts to 10 X 10 are recalled and patterns in number facts are investigated
- Number facts can be extended by powers of ten – *These points directly inform the development of mental computation strategies and the understanding of operations.*
- Multiplicative situations can be represented as equal-groups problems, comparison problems, combinations (Cartesian) problems and area/array problems.
- The multiplicative situation is understood factor X factor = multiple with the meanings of the terms clearly understood
- Multiplicative arrays are used to visualize and represent multiplicative situations
- Division and multiplication are known as the inverse of one another
- The commutative property of multiplication is understood and can be shown to be linked to arrays - *This is also an important foundation for algebraic reasoning.*
- Partition division involves finding the size of each group and quotition division involves finding the number of groups and can also be expressed in terms of factors and multiple – *These ideas directly inform the understanding of operations and the use of algorithms.*
- Quotition division can be considered in terms of fractions so that a quantity can be split by 'halving', 'thirding', 'fifthing' etc. - *Informs the understanding of operations, particularly division. It also underpins multiplicative partitioning and the development of proportional reasoning.*
- Prime and composite numbers can be linked to multiplicative arrays – prime numbers can be made only with a single row array - *Informs the understanding of operations, particularly division.*
- Distributive property of multiplication over addition is applied and shown by a multiplicative array - *Informs the understanding of operations, particularly division, as well as the development of mental computation strategies. As well, it is an important foundation for algebraic reasoning.*
- Multiplicative arrays are linked to the concepts of area and volume
- Measurement units have the same multiplicative relationship as the Base Ten Number System – *There are obvious links to understanding measurement concepts which can be used as a context for developing aspects of multiplicative thinking and place value.*
- Cartesian products can be represented symbolically and in tree diagrams - *It underpins multiplicative partitioning and the development of proportional reasoning.*

Multiplicative thinking is not easy to teach or to learn. Whereas most students enter school with informal knowledge that supports counting and early additive thinking (Sophian & Madrid, 2003) students need to re-conceptualise their understanding about number to understand multiplicative relationships (Wright, 2011). Multiplicative thinking is distinctly different from additive thinking even though it is constructed by children following on from their additive thinking processes (Clark & Kamii, 1996). Devlin (2008 a, b, c) also noted that 'multiplication is a tricky concept' and suggested that much of the difficulty can be attributed to teaching it as 'repeated addition'. Devlin discusses what he calls the 'first model phenomenon' in saying

As most math teachers are probably aware, when you teach a new mathematical concept to someone, the way you first introduce it is almost certainly going to be the one the student retains, no matter how much you stress that the concept will later be changed in some way (Devlin, 2008c, p. 3).

The point Devlin makes here is, that if multiplication is taught as 'repeated addition', that is likely how many children will continue to remember it. Unfortunately however, the longer that multiplication as 'repeated addition' lingers as a dominant image, the harder it will be for students to need to understand about ratios, proportions, algebraic relationships, and other 'big number ideas' that follow. Also, Askew & Brown (2003), in citing the work of Hart (1981), pointed out that "understanding multiplication only as repeated addition may lead to misconceptions such as 'multiplication makes bigger' and 'division makes smaller'" (p. 10). This underlines the importance of teachers adopting 'big idea thinking' so that they are able to look beyond the immediate horizon of what they are teaching and see how it connects to and underpins other 'big number ideas' that follow.

Multiplicative thinking is more than the capacity to remember and utilize multiplication facts. What is required is the development of the ability to apply these facts to a variety of situations which are founded on multiplication. Jacob and Willis (2003) proposed five broad stages for the development of multiplicative thinking: One-to-One Counting; Additive Composition; Many-to-One Counting; Multiplicative Relations; and Operating on the Operator.

In the one-to-one counting phase the students are grappling with the basics of counting and do not see the relevance of the many-to-one count, that is, they may know what it means to hand out a given quantity but this is viewed additively and not multiplicatively (Jacob & Willis, 2003). At this point students are not able to use a row by column structure (an array) to work out a number of squares, and they resort to additive strategies (Batista, 1999). Stage 2, additive composition, is when the students understand the principle of trusting the count, that is, that the last number said indicates the quantity. At this stage, through skip-counting, the students can use groups to count more efficiently (Jacob & Willis, 2003). It is important at this stage that the children manipulate materials to facilitate the move to recognising the multiplicative situation, as the materials will help them to: recognise and then count the number in each group, the number of groups and the total; describe multiplicative situations without necessarily finding a total; and transfer these understandings to the division situations (Jacob & Willis, 2003).

The third stage is the development of many-to-one counting. Jacob & Willis suggest that this is a key transitional phase between additive and multiplicative thinking. It is dependent on children trusting the count and understanding that they can keep track of two things simultaneously – the number of groups and the number in each group. "Children know that they can represent one group and count repetitions of that same group" (Jacob & Willis, 2003, p. 5). At this stage they do not necessarily understand the relationship between multiplication and division in that they may not transfer all of the understandings gained with multiplication to the division situation, and they may not consistently employ the inverse relationship between the two operations or the commutative property of multiplication. At the fourth stage, multiplicative relations, the students *are* able to employ the commutative, distributive and inverse properties of multiplication and division (Jacob & Willis, 2003; Mulligan & Watson, 1998). They are also aware that the three aspects of multiplication; the multiplicand, the multiplier and the product, are involved in the multiplicative situation (Jacob & Willis, 2003). It is at this stage that the need for manipulative materials is decreasing, as students need to describe when the operations of multiplication and division became objects of thought rather than actions (Sophian & Madrid, 2003; Wright, 2011). This is the stage which is described by Jacob and Willis (2003) as one in which students treat the numbers in a problem situation as variables, a concept which is quite abstract in nature.

As already noted, the traditional approach has been to facilitate students' multiplicative thinking through a process of making links with repeated addition (Confrey & Smith, 1995). This is an approach which may stand to reinforce additive rather than multiplicative thinking and may be detrimental to the variety of situations to which multiplication needs to be applied (Wright, 2011). This concern has led some researchers to look for alternative constructs to create this bridge (Confrey & Smith, 1995; Sophian & Madrid, 2003). Rather than building from an additive construct, some researchers have advocated the use of "a primitive multiplicative operation", a splitting construct (Confrey and Smith, 1995, p. 66). A splitting construct is where multiple versions of an original are made such as is seen in a tree-diagram or in doubling and halving (Confrey and Smith, 1995). By adopting the splitting construct, teachers may be able alleviate some of the issues where students will wrongly apply additive thinking to multiplicative situations, and in the case of older students, multiplicative thinking (particularly proportional strategies) in additive situations (Van Dooren, De Bock & Verschaffel, 2010).

Siemon et al. (2011) advocate that there needs to be a greater emphasis in the early years of schooling on sharing and splitting as an approach to developing multiplication and division rather than through repeated addition. They argue that 'splitting' is "inherently tied to multiplicative operations of replicating, magnifying, and shrinking" (Siemon et al., 20011, p. 357). This is supported by Downton (2008) who cited earlier work by Sullivan, Clarke, Cheeseman & Mulligan (2001) and Killion & Steffe (2002) in asserting that "the acquisition of an equal-grouping (composite) structure is at the core of multiplicative thinking" (p. 171). Indeed, the multitude of connections between the notion of division/multiplication and other ideas such as fraction, ratio, proportion etc. provide a clear case for utilizing 'big idea thinking' to make such connections explicit.

Multiplicative partitioning

The fourth 'big number idea' of multiplicative partitioning is underpinned by much of what has already been discussed yet aspects of it also inform place value and multiplicative thinking. Again, what is important for teachers is the understanding of the 'micro-content' that constitutes each 'big number idea' and the ways in which the content is connected and linked. It is evident from the list of criteria for multiplicative partitioning that many of the points contained therein develop alongside aspects of the three other ideas so far discussed. This further underlines the importance of teachers adopting 'big idea thinking' to see how these ideas are inextricably linked and how they develop over time.

Siemon, Bleckley & Neal (2012) make the distinction between additive partitioning as characterized by part-part-whole reasoning and multiplicative partitioning which involves the creation of equal parts of a single whole or a collection, or of combinations of wholes and parts. Confrey, Maloney, Nguyen, Mojica & Myers (2009) referred to this as equipartitioning or splitting which was discussed earlier in the context of multiplicative thinking. Equipartitioning is essential when starting to work in the difficult to teach and learn area of rational numbers and their various representation (Anthony & Ding, 2011; Capraro, 2005; Nunes & Bryant, 2009; Usiskin, 2007), and is the foundation of division and multiplication and, ratio and rate (Siemon et al., 2011). As previously noted, children learn about equipartitioning or splitting at a young age when they are exposed to the notion of halving and come to realise that both halves of an object or collection must be the same. Some of the complexity in this 'big number idea' may be illustrated through the fact that in constructing a learning trajectory for equipartitioning, Confrey (2012) outlines 16 levels of cognitive proficiency beginning with equipartitioning collections and single wholes and progressing to equipartitioning of multiple wholes.

The following list of points has again been developed from multiple sources (Department of Education, Western Australia; Siemone et al.; Reys et al.; Van de Walle, Karp & Bay-Williams). The points are indicative of the 'micro-content' that comprises the 'big number idea' of multiplicative partitioning and are not presented in any particular order.

- Objects, quantities and collections can be shared to create equal parts
- There is a relationship between the number of parts and the size and name of the parts and the number of parts increases as the size or share decreases
- Objects, quantities and collections can be repeatedly halved and doubled – e.g., use successive splits to show that one half is equivalent to 2 parts in 4, 4 parts in 8 etc.
- An object, quantity or collection can be partitioned into a number of equal portions to show unit fractions so that say one third is more than one fourth etc.
- The relative magnitude of a fraction is dependent on the relationship between the numerator (how many parts) and denominator (total parts)
- Fractions are renamed as equivalents where the total number of parts (denominator) and required number of parts (numerator) are increased by the same factor
- Fractions with unlike denominators can be compared and ordered
- Common fractions and decimal fractions can be compared, ordered and renamed in conceptual ways
- Construct of fraction as division can be used to produce equal parts (equipartitioning)
- Fractions are used to describe quotients and operators
- Fractions are used to describe part-whole relations
- Fractions are used to describe simple ratios
- Percentages, fractions and decimals express the relationship between two quantities.

- Percentages are special part: whole ratios based on 100.
- Any given percentage can be used as a ratio to generate an infinite number of equivalent fractions (e.g., 50% = ½, 2/4, 3/6 etc.)
- Multiplicative arrays can be used to represent fractions, decimals and percentages
- Benchmark fractions, decimals and percentages, which are the equivalents of one another, can be used to estimate and to solve problems

Charalambous (2010) proposed that "...strong mathematical knowledge for teaching supports teachers in using representations to attach meaning to mathematical procedures..." (p. 273). He further asserted that "...strong MKT [mathematical knowledge for teaching] supports teachers in giving and co-constructing explanations that illuminate the meaning of mathematical procedures" (p. 274). If these propositions are correct then it is not unreasonable to suggest that the reverse may also be true. Weak mathematical knowledge for teaching would impede teachers in using representations to attach meaning to mathematical procedures, and impede teachers in co-constructing explanations that illuminate the mathematics. Indeed, teaching through procedures likely indicates a lack of mathematical knowledge for teaching. Given that research points to teachers having difficulty with the topic of rational numbers (Moseley, Okamoto & Ishida, 2007; Tirosh, 2000; Zhou, Peverly & Xin, 2006) this is problematic. It indicates that the big idea of multiplicative partitioning may not be being taught and learned as effectively as it should. Again this underlines the importance of teachers adopting 'big idea thinking', identifying the key 'micro-content' that comprises each 'big number idea', and understanding and using the myriad connections that exist within each big idea and between it and other big ideas.

Big idea thinking: Making connections

Multiplicative thinking could to some extent be considered the 'biggest' of the 'big number ideas'. While Figure 1 depicts a developmental relationship between the six 'big number ideas', this perhaps only shows part of the picture. Figure 1 also intentionally shows the ellipses for place value and multiplicative thinking stretching back to the beginning of the ellipse for trusting the count indicating that foundation aspects of those two ideas develop simultaneously with aspects of trusting the count. Indeed, as has been suggested earlier, aspects of multiplicative thinking help develop place value understanding.

For instance, as children learn to think additively, they understand and can partition numbers into the hundreds. However, many children initially experience difficulty in moving beyond that, particularly beyond the thousands and a common misconception is that millions follow thousands. This part of place value understanding coincides with the development of multiplicative thinking, specifically that the cyclical pattern in reading and writing numbers continues and that there is a ten times multiplicative relationship between the places in the number system. This is encapsulated in Ross's (1989) notion of the base ten property of the numeration system. If teachers adopt 'big idea thinking', they will be aware of this, and be in a better position to help children develop their understandings of key concepts and ideas. In a similar way, there is considerable overlap between multiplicative thinking and multiplicative partitioning. These links also extend to the next 'big number idea' of proportional reasoning with much of the connectivity centred on the multiplicative array or region. Following is a list of specific ideas that can be demonstrated with a five by three array or region.

- Multiplication facts 5 X 3 = 15, 3 X 5 = 15. Commutativity is shown by rotating the region.
- Division facts 15 ÷ 3 = 5, 15 ÷5 = 3, and inverse relationship.
- For both multiplication and division, the model shows the relationship and terminology of *factor X factor = multiple.*
- The 'times as many' relationship – the total of squares in the region is five times each row of three and three times each column of five.
- Fraction relationship – each row of three is one fifth of the total and each column of five is one third of the total. This can be called 'fifthing' and 'thirding'.
- Equivalent fractions – each row of three is one fifth or five fifteenths of the total and each column is one third or five fifteenths of the total.
- The representation of fraction as part/whole can be shown as a/b.

- Ratio – the relationship between each row of three and the total can be shown as a ratio of 1:5 (ratio of column to total is 1:3).
- Each row of three is increased by a factor of five, and each column of five by a factor of three to produce the total of 15.
- The total can be reduced by a factor of five to show the total in each row and by a factor of three to show the column total.
- Cartesian Products can be demonstrated. Combinations of five shirts (A, B, C, D, E) and three shorts (F, G, H) can be shown as AF, AG, AH, BF, BG, BH etc.
- The area of the region is 5 X 3 = 15 units.
- A larger array or region, say 14 X 6 = 84 can be used to show flexible partitioning of 84, and the distributive property by splitting into 10 X 6 and 4 X 6. The flexible partitioning can be linked to different factor pairs for 84 (84 X 1, 42 X 2, 28 X 3, 21 X 4, 14 X 6, 12 X 7). The distributive property is linked to the formal algorithm for multiplication and later grid representations for multiplication with larger numbers.
- Prime numbers can be demonstrated as arrays/regions with only one row/column.

Conclusion

One of the purposes for developing the Australian Curriculum: Mathematics was to make the curriculum 'deep' rather than 'wide' (National Curriculum Board, 2009). Similarly, it has already been noted that the Common Core State Standards were needed to address a curriculum seen as 'a mile wide and an inch deep' (NGA Centre, 2010). Even so, if each of the content descriptors is taken individually the capacity of any teacher to cover all of the content would be severely strained. What may be of benefit is for teachers to think at more of a 'macro level' in terms of 'big number ideas'. They could then attach to those big ideas the content descriptors, or 'micro-content' as we have termed it here, rather than teach to the content descriptors with the notion that the big ideas will emerge. To do this, teachers need to be given the professional courtesy of being helped towards an understanding of the big ideas and their importance. As noted by Clarke, Clarke and Sullivan (2012), this has significant implications for professional learning initiatives. In this paper we have attempted to give some insight to what the 'big number ideas' may be, what they mean to the classroom practitioner and how they develop through and within each other. As was indicated earlier, the final two big ideas, proportional reasoning and generalizing or algebraic reasoning are developmentally more suitable in secondary school (Siemon et al, 2012) and consequently were not be addressed in this paper.

'Big idea' thinking has the capacity to develop teacher knowledge along the lines of Schulman's (1986) 'substantive structures' and Ma's (1999) 'knowledge packages' and 'concept knots' as described earlier. Such deep and connected knowledge would be likely to lead to more effective concept-based teaching rather than a reliance on teaching procedures, irrespective of where a teacher might be teaching. The focus here has been predominantly the Australian Curriculum: Mathematics (ACARA, 2012) with some reference to the Common Core State Standards for Mathematics (NGA Center, 2010).However, it is suggested that the focus on 'big ideas' with their myriad links and connections would greatly enhance pedagogies for delivering mathematics curricula in any country.

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