Yüzüncü Yıl Üniversitesi Fen Bilimleri Enstitüsü Dergisi/ Journal of the Institute of Natural & Applied Sciences 22 (2): 153-156, 2017

Araştırma Makalesi / Research Article

A Note on Piecewise Endomorsphims

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Abstract: Let $C=\{G_\alpha\}$ be a cover of G by maximal cylelic R-submodules of G and $N=\{f\in M_R(G)| f|_{G_\alpha}$ can be extended to an endomorphismof $G\}$ a subnear-ring of $M_R(G)$. We call N the near-ring of piecewise endomorhisms determined by (R,G,C). From [1] ask when is $N=M_R(G)$? from [2], it follows that if D is a PID and G is free D-module of finite rank n, than $M_D(D^n)=N$. In general this is not the case In addition in [1] it is an open problem whather the orbitranily modules on PID are true or not. In this work, we investigate two cases such that is not N is in $M_R(G)$.
Keywords: Near-rings, Piecewise endomorhisms.

Kısmi Endomorfizmalar Üzerine Bir Not

Özet: $C=\{G_\alpha\}, G$ nin maksimal devirli $R-$alt modellerinden oluşan bir örtüsü ve $N=\{f\in M_R(G)| f|_G, G$ nin bir endomorfizmasına genişletilebilir} olsun. N, $M_R(G)$ nin bir yakın alt halkası olup, (R,G,C) üçlüsü tarafından belirlenen kısmi endomorfizmalar yakın halkalarıdır. [1] de N nin hangi durumlarda $M_R(G)$ ye eşit olduğu sorulmaktadır. [2] de D nin bir temel ideal bölgesi ve G nin sonlu $n-$ranklı bir serbest D mevcut olması durumunda $M_D(D^n)=N$ eşitliği gösterilmiş ve [1] de temel ideal bölgesi üzerinde keyfi modüller için sonucun doğru olup olmadığı bir açık problemdir. Bu çalışmada, N nin $M_R(G)$ ye eşit olup – olmadığını araştıracağız.
Anahtar Kelimeler: Yakın halkalar, Kısmi endomorfizmalar.

Introduction

If R is a ring with identity and G is a unitary ringht R-module, the set

$$M_R(G)=\{f:G\to G| f(ar)=(fa)r,\ a\in G, r\in R\}$$

is a near-ring under the operations of function addition and function composition, called the centralizer near-ring determined by (R,G). with respect to various pairs (R,G), the structure of these near-rings has been investigated in (Maxon and Smith, 1980; Maxon, 1990; Maxon et al., 1991a, 1991b, 1992).

We recall that a cover for an R-module G is a collection $C=\{G_\alpha\}$ of submodules of such that

- $0\subset G_\alpha\subset G,$
- $G_\alpha\not\subset G_\beta$ for $\alpha\neq\beta,$
- $UG_\alpha=G.$

Now let $R=Z$, the ring of integers, $G=Z^n$ the free Z-module of rank n and let $C=\{G_\alpha\}$ be a cover by maximal cylic submodules. Further let $f\in M_Z(Z^n)$ be determined on

$$G_\alpha = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} Z \quad \text{by} \quad f\left(\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}\right) = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix}.$$

Since G_α is a maximal submodule, we have $\gcd(x_1, x_2, ..., x_n) = 1$ and so $\exists\, a_1, a_2, ..., a_n \in Z$ with $a_1x_1 + a_2x_2 + ... + a_nx_n = 1.$ But f can be represented on G_α by the matrix

$$A = \begin{bmatrix} c_1a_1 & c_1a_2 & ... & c_1a_n \\ c_2a_1 & c_2a_2 & ... & c_2a_n \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ c_na_1 & c_na_2 & ... & c_na_n \end{bmatrix}.$$

Therefore, the function $\hat{f} : Z^n \to Z^n$ is determined by linear maps $\hat{f}(A) = AX$ such that $\hat{f}\big|_{G_\alpha} = f$. That is, $\hat{f}\big|_{G_\alpha}$ can be extended to an endomorphism of $G.$ By this reason, every $f \in M_Z(Z^n)$ homogeneous map of Z^n is piecewise an endomorphism of Z^n as the following meaning: For each $G_\alpha \in C,$ the exists $\delta \in End_Z(Z^n)$ such that $f\big|G_\alpha = \delta.$ That is, $f\big|G_\alpha$ can be extended to an endormorphism of $G.$ By this reason, every $f \in M_Z(Z^n)$ homogeneous function of Z_n is a piecewise endomorhism of Z^n as the fowllowing meaning: For each $G_\alpha \in C,$ the exists $\delta \in End_Z(Z^n)$ such that $f\big|G_\alpha = \delta.$

Proposition 1. Let $R = Z[x], G = (Z[x])^3$ and C be α covering of G by maximal cylic submodules. Then $N \subset M_Z(G).$

Proof. One can verify that

$$A = \begin{bmatrix} 1 \\ x \\ x+2 \end{bmatrix} Z[x] \in C.$$

Further, there exists $f \in M_R(G)$ with

$$f\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{cases} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} k; \text{ if } \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ x+2 \end{bmatrix} k,\ k \in Z[x], \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \text{ if } \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \begin{bmatrix} 1 \\ x \\ x+2 \end{bmatrix} Z[x]. \end{cases}$$

However, we cannot exted the function f to a $\delta \in End_R(G)$ with

$$\delta\begin{bmatrix} 1 \\ x \\ x+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Beacuse, there are not non-constant polynamials that form a matrix $A = [A_{ij}(x)] \in M_3(Z[X])$ such that $x \in (Z[x])^3$ for

$$\delta\begin{bmatrix} 1 \\ x \\ x+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

In that case it follows that $N \subset M_R(G),$ where the domain $Z[x]$ is not a *PID*.

Proposition 2. Let $R = Z$ and $G = (Z[x])^3$ in this case Z is modul over a PID of infinite rank.

Proof. When we consider the maximal cyclic submodules and f homogeneous function defined related to that submodul in Proposition 1, it can be shown that there exists $\delta \in End_Z(G)$ which has the some effect on the submodule and can be determined by the matrix,

$$A=\begin{bmatrix}1&0&0\\1&0&0\\0&0&0\end{bmatrix}.$$

$\delta \in End_Z(G)$ is determined by $\delta(x)=AX$, $\delta \in End_Z(G)$ and δ satisfies

$$\delta\begin{bmatrix}1\\x\\x+2\end{bmatrix}=\begin{bmatrix}1&0&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1\\x\\x+2\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

and

$\delta\big|_{G_\alpha}=f$.

Proposition 3. Let $R=Z$, $G=Z^m$ (free Z-module of infinite rank). Let $C=\{G_\alpha\}$ be a cover of G by maximal cyclic submodules and $f\in M_Z(Z^\infty)$.The expansion of f to Z^∞ is not only unique, It is possible in infinite number.

Proof.

$G_\alpha=(x_1,x_2,...,x_n,...)$ Z maksimal $\Rightarrow \exists i_1,i_2,...,i_n\in N^+$ such that $\gcd(x_{i_1},x_{i_2},...,x_{i_n})=1$; so $a_{i_1},a_{i_2},...a_{i_n}\in Z$ such that $a_{i_1}x_{i_1}+a_{i_2}x_{i_2}+....a_{i_n}x_{i_n}=1$ if

$f[x_1,x_2,...,x_n...]=f(x)=[c_1,c_2,...,c_n,...]$ for $j=1,2,...,n$;

since

$c_j=a_{i_1}x_{i_1}c_j+a_{i_2}x_{i_2}c_j+....a_{i_n}x_{i_n}c_j$

$$F(x)=\begin{bmatrix}0&...&0&a_{i_1}c_1&0&...&0&a_{i_2}c_1&0&...&0&a_{i_n}c_1&0&...\\0&...&0&a_{i_1}c_2&0&...&0&a_{i_2}c_2&0&...&0&a_{i_n}c_2&0&...\\&.&&&&&&&&&&&&\\&.&&&&&&&&&&&&\\&.&&&&&&&&&&&&\\0&...&0&a_{i_1}c_n&0&...&0&a_{i_2}c_n&0&...&0&a_{i_n}c_n&0&...\\&.&&&&&&&&&&&&\\&.&&&&&&&&&&&&\\&.&&&&&&&&&&&&\end{bmatrix}\begin{bmatrix}.\\.\\.\\x_{i_1}\\x_{i_2}\\.\\.\\.\\x_{i_n}\\.\\.\\.\end{bmatrix}=Ax.$$

If we define $\beta:Z^\infty\to Z^\infty$ as $\beta(x)=AX$ then it is obvious that $\beta\in End_Z(Z^\infty)$ and $\beta\big|_{G_\alpha}=f$. Therefore the expansion of F to Z^∞ can be done indefinitely rather than solely. Because since the components of the vector X which are prime among them can be selected infinitely.

Finally, let us give a proposition which shown that $N=M_R(G)$ may be countless infinite rank case.

Proposition4. Let $R=R$ be real field and $G=R^R$. Since R is a semi-simple ring, then G is a semi-simple R-module such that its rank countless in that case we have $N=M_R(G)$.

Proof. Each $0\neq f\in G$ element generates a maximal cycle R-submodule, which forms a C cover of the maximal cycle submodule G. Since $G_\alpha=R\sin x\in C$ Then $F:G\to G$;

$$F(f(x))=\begin{cases}r\cos; \text{if } f(x)=r\sin x\in G_\alpha\\0 \quad ; \text{if } f\notin G_\alpha\end{cases}$$

where $F\in M_R(G)$ but $F\notin End_R(G)$. $F\big|_{G_\alpha}:G_\alpha\to G_\alpha$ restricted function is an endomorphism of G_α;

$$\begin{aligned}\big|G_\alpha(r_1\sin x+r_2\sin x)&=\big|_{G_\alpha}[(r_1+r_2)\sin x]\\&=(r_1+r_2)\cos x\\&=r_1\cos x+r_2\cos x\\&=F\big|_{G_\alpha}(r_1\sin x)+F\big|_{G_\alpha}(r_2\sin x)\end{aligned}$$

Since G is semi-simple then G has a G' submodule such that $G=R\sin x+G'$

Let us define $\hat{F}:G\to G$ as $F[r\sin x+g(x)]=F(r\sin x)=r\cos x,\ (a\in G')$.

Hence

$\hat{F}[r_1 \sin x + g_1(x) + r_2 \sin x + g_2(x)]$
$= \hat{F}[(r_1 + r_2)\sin x + (g_1 + g_2)(x)]$
$= F[(r_1 + r_2)\sin x]$
$= (r_1 + r_2)(x)]$
$= r_1 \cos x + r_2 \cos x$
$= F(r_1 \sin x) + F(r_2 \sin x)$
$= \hat{F}[r_1 \sin x + g_1(x)] + \hat{F}[r_2 \sin x + g_2(x)].$

Then $\hat{F}$ is an endomorphism and $\hat{F}\big|_{G_\alpha} = F$. Let $\overline{G}$ be the submodule of infinitely derivative functions of G where $G = R\sin x$. Let $\overline{G}$ be a semi-simple and write $\overline{G} = R\sin x + G_1$ as a direct complement of G_1. Let G be a semi-simple and write $\overline{G} = R\sin x + G_1$, where G_1 is a direct complement of $R\sin x$. When we think of $\hat{F}$ above G, Since

$\hat{F}[r\sin x + g(x)] = r\cos x,\ (g(x) \in G_1),$

then $\hat{F}$ extends F to an endomorphism of G.

$$D : G \to G : D(f(x)) = f'(x) = \frac{df(x)}{x}.$$

The derivative mapping also extends F to G. Since

$D[r\sin x + g(x)] = r\cos x + g'(x)$

then $D = F$. This indicates that the expansion may be more then one.

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