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THE ZERO-DIVISOR GRAPH OF A COMMUTATIVE RING WITHOUT IDENTITY

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Abstract. Let R be a commutative ring. The zero-divisor graph of R is the (simple) graph $\Gamma(R)$ with vertices the nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. In this article, we investigate $\Gamma(R)$ when R does not have an identity, and we determine all such zero-divisor graphs with 14 or fewer vertices.

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1. Introduction

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. As in [9], the *zero-divisor graph* of R is the (simple) graph $\Gamma(R)$ with vertices $Z(R) \setminus \{0\}$, and distinct vertices x and y are adjacent if and only if $xy = 0$. This concept is due to Beck [16], who let all the elements of R be vertices and was mainly interested in colorings (also see [2]). The zero-divisor graph of a commutative ring with identity has been studied extensively by many authors; see the two survey articles [5] and [18]. It is now customary to assume that rings have an identity (cf. [1]), but the above definition of $\Gamma(R)$ certainly holds for a commutative ring without identity. The purpose of this article is to investigate the zero-divisor graph of a commutative ring without identity.

Although some results on the zero-divisor graph do not depend on an identity (see Theorem 2.2) and a few results have been given without assuming an identity (cf. [13]), there has been no systematic study for commutative rings without an identity. The concept of zero-divisor graph of a commutative ring was extended to commutative semigroups by DeMeyer, McKenzie, and Schneider in [20]. Let S be a (multiplicative) commutative semigroup with 0 (i.e., $0x = 0$ for every $x \in S$), and let $Z(S) = \{\, x \in S \mid xy = 0 \text{ for some } 0 \neq y \in S \,\}$ be the set of zero-divisors of S. Then the *zero-divisor graph* of S is the (simple) graph $G(S)$ with vertices $Z(S) \setminus \{0\}$, the set of nonzero zero-divisors of S, and two distinct vertices x and y

are adjacent if and only if $xy = 0$. Thus $\Gamma(R) = G(S)$, where $S = R$ or $S = Z(R)$ as a multiplicative semigroup whether or not R has an identity. For a recent survey article on $G(S)$, see [6].

When studying commutative rings R without identity, there are two distinct natural cases, either $R = Z(R)$ or $Z(R) \subsetneq R$. The first case holds when R is finite, and either case may hold when R is infinite. Things are particularly nice for finite commutative rings R without identity since $|V(\Gamma(R))| = |R| - 1$. In the second case, $\Gamma(R)$ is isomorphic to the zero-divisor graph of a commutative ring with identity, namely, $\Gamma(R_S)$, where $S = R \setminus Z(R)$.

In Section 2, we give some properties of $\Gamma(R)$ for a commutative ring R. In Section 3, we study the zero-divisor graph for commutative rings R with $R = Z(R)$; a special case is when R is a finite commutative ring without identity. We pay particular attention to the diameter and girth of $\Gamma(R)$ and also determine when $\Gamma(R)$ is a complete, complete bipartite, or star graph. In Section 4, we study commutative rings R without identity and $Z(R) \subsetneq R$ and briefly discuss the compressed zero-divisor graph $\Gamma_E(R)$. In Section 5, we determine all zero-divisor graphs on 14 or fewer vertices for commutative rings without identity and compare this to commutative rings with identity. In Section 6, we give tables that show which diameters and girths can be realized by zero-divisor graphs of several types of commutative rings.

All rings will be commutative. To avoid confusion, we will use the terminology *commutative ring with identity*, *commutative ring without identity*, and just *commutative ring* for either case. We let $Z(R)$, $Nil(R)$, and $U(R)$ denote the set of zero-divisors, the ideal of nilpotent elements, and set of units (possibly empty) for a commutative ring R. A *regular element* of R is an $x \in R \setminus Z(R)$, and the *total quotient ring* of R is $T(R) = R_S$, where $S = R \setminus Z(R)$ ($T(R) = R$ if $R = Z(R)$). For an additive abelian group G, let G^0 be the induced commutative ring with trivial multiplication. As usual, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, and $\mathbb{F}_q$ will denote the ring of integers, integers modulo n, rational numbers, and the finite field with q elements, respectively.

We next recall some concepts from graph theory; a general reference for graph theory is [17]. Throughout, G will be a simple graph with $V(G)$ its set of vertices, i.e., G is undirected with no multiple edges or loops. We say that G is *connected* if there is a path between any two distinct vertices of G. For vertices x and y of G, define $\mathrm{d}(x, y)$ to be the length of a shortest path from x to y ($\mathrm{d}(x, x) = 0$ and $\mathrm{d}(x, y) = \infty$ if there is no path). The *diameter* of G is $\mathrm{diam}(G) = \sup\{\ \mathrm{d}(x, y) \mid x$ and y are vertices of $G\}$. The *girth* of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycles).

A graph G is *complete* if any two distinct vertices of G are adjacent. The complete graph with n vertices will be denoted by K^n (we allow n to be an infinite cardinal number). A *complete bipartite graph* is a graph G which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices of G are adjacent if and only if they are in distinct vertex sets. If one of the vertex sets is a singleton, then we call G a *star graph*. We denote the complete bipartite graph by $K^{m,n}$, where $|A| = m$ and $|B| = n$ (again, we allow m and n to be infinite cardinals); so a star graph is a $K^{1,n}$.

Let H be a subgraph of a graph G. Then H is an *induced subgraph* of G if every edge in G with endpoints in H is also an edge in H. For a vertex x of a graph G, let $N(x)$ be the set of vertices in G that are adjacent to x and $\overline{N(x)} = N(x) \cup \{x\}$. A vertex x of G is called an *end* if there is only one vertex adjacent to x (i.e., if $|N(x)| = 1$).

Most of the results in this paper are from the the second-named author's PhD dissertation [28] at The University of Tennessee under the direction of the first-named author.

2. Zero-divisor graphs

Let R be a commutative ring (again, "commutative ring" means the ring may or may not have an identity). Associate to R a (simple) graph $\Gamma(R)$ with vertices the nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$. Note that $\Gamma(R)$ is the empty graph if and only if $R = \{0\}$ or $Z(R) = \{0\}$. To avoid any trivialities, we will implicitly assume that $R \neq \{0\}$ and $\Gamma(R)$ is not the empty graph.

We first give two examples of how commutative rings without identity arise naturally from commutative rings with identity.

Example 2.1. (a) *Let A be a commutative ring with identity. If $Z(A)$ is an (necessarily prime) ideal of A, then $R = Z(A)$ is a commutative ring without identity and $R = Z(R) = Z(A)$. Thus $\Gamma(R) = \Gamma(A)$. This would be the case when A is a zero-dimensional (e.g., finite) local ring.*

(b) *Let A be a commutative ring with identity and $x \in A \setminus (Z(A) \cup U(A))$, i.e., x is a non-unit regular element of A. Then $R = xA$ is a commutative ring without identity, $Z(R) \subsetneq R$ since $x \in R \setminus Z(R)$, and $\Gamma(R) \cong \Gamma(A)$ via $xa \leftrightarrow a$. Note that in this case, A, and hence R, is necessarily infinite.*

We next observe that many of the fundamental zero-divisor graph results for commutative rings with identity from [9] also hold for commutative rings without identity.

Theorem 2.2. *Let R be a commutative ring.*

(1) $\Gamma(R)$ *is connected with* $diam(\Gamma(R)) \in \{0, 1, 2, 3\}$.
(2) $gr(\Gamma(R)) \in \{3, 4, \infty\}$.
(3) $\Gamma(R)$ *is complete if and only if* $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $Z(R)^2 = \{0\}$.
(4) *If* $Z(R) \neq \{0\}$, *then* $\Gamma(R)$ *is finite if and only if* R *is finite.*

Proof. (1) The proof in [9, Theorem 2.3] and easier proof in [5, Theorem 2.2] hold for commutative rings without identity (and semigroups). The result for semigroup zero-divisor graphs is given in [20, Theorem 1.2].

(2) The proof in [5, Theorem 2.3] holds for commutative rings without identity (and semigroups), and the result for semigroup zero-divisor graphs is given in [20, Theorem 1.5].

(3) The proof in [9, Theorem 2.8] holds for commutative rings without identity (in the proof of [9, Theorem 2.5], just note that if $a \in R$ is idempotent, then $R = Ra \oplus \{r - ra \mid r \in R\}$). The $R = Z(R)$ case is also observed in [13, Lemma 2.2].

(4) The proof in [9, Theorem 2.2] and easier proof in [5, Theorem 2.1] hold for commutative rings without identity. □

The following results will be needed in later sections.

Theorem 2.3. *Let R be a finite commutative ring.*

(1) *If R does not have an identity, then $R = Z(R)$.*
(2) ([3, Lemma 4.4]) *If R does not have an identity, then either $R = Nil(R)$ or $R = R_1 \times R_2$, where R_1 has an identity and $R_2 = Nil(R_2) \neq \{0\}$.*

Proof. (1) is well known and is also a consequence of (2). □

Theorem 2.4. *Let R be an infinite commutative ring.*

(1) $diam(\Gamma(R)) \neq 0$.
(2) *If* $diam(\Gamma(R)) = 1$, *then* $gr(\Gamma(R)) \neq \infty$ *(i.e.,* $gr(\Gamma(R)) = 3$*).*
(3) *If R has an identity and* $diam(\Gamma(R)) = 3$, *then* $gr(\Gamma(R)) \neq \infty$ *(i.e.,* $gr(\Gamma(R)) \in \{3, 4\}$*).*

Proof. (1) and (2) follow from Theorem 2.2(4) since $\Gamma(R)$ is infinite when R is infinite and $Z(R) \neq \{0\}$.

(3) If R is reduced, then $\mathrm{gr}(\Gamma(R)) = \infty$ if and only if $\Gamma(R) = K^{1,n}$ [11, Theorem 2.4]. Since $\mathrm{diam}(K^{1,n}) = 2$ (for $n \geq 2$), no such reduced ring has the desired diameter and girth. If R has nonzero nilpotent elements, then $\mathrm{gr}(\Gamma(R)) = \infty$ if and only if either $\Gamma(R)$ is a singleton, $\Gamma(R) = \bar{K}^{1,3}$, or $\Gamma(R) = K^{1,n}$ (where $\bar{K}^{m,3}$ is the graph formed by joining $K^{m,3}$ to the graph $K^{1,m}$ by identifying the center of $K^{1,m}$ with a point in the size 3 partition of $K^{m,3}$) [11, Theorem 2.5]. The first two cases are finite graphs, and the last case has diameter 2. Thus no such nonreduced ring has the desired diameter and girth. $\square$

Although many results hold for $\Gamma(R)$ whether or not R has an identity, there are major differences. We end this section with five examples which illustrate this for finite commutative rings; details will be given in later sections.

Example 2.5. (a) *There are many graphs that can only be realized as the zero-divisor graphs of commutative rings with identity or of commutative rings without identity, but not both. For example, the complete bipartite graph $K^{2,2}$ can only be realized as the zero-divisor graph of a commutative ring with identity (let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$), and the complete graph K^5 can only be realized by a commutative ring without identity (let $R = \mathbb{Z}_6^0$). Moreover, 5 is the fewest number of vertices for which there are graphs that can be realized as zero-divisor graphs only by a commutative ring with identity and other graphs that can be realized as zero-divisor graphs only by a commutative ring without identity.*

(b) *Let $n \geq 1$ be an integer. Then $\Gamma(R) = K^n$ for a finite commutative ring R with identity if and only if $n = p^k - 1$ for a prime p and integer $k \geq 1$* [9, Theorem 2.10]. *However, for every integer $n \geq 1$, $\Gamma(\mathbb{Z}_{n+1}^0) = K^n$.*

(c) *Let $m, n \geq 1$ be integers. Then $\Gamma(R) = K^{m,n}$ for a finite commutative ring R with identity if and only if $m = p^i - 1$ and $n = q^j - 1$ for primes p, q and integers $i, j \geq 1$* [9, Theorem 2.13 and p. 43]. *In particular, $\Gamma(R) = K^{1,n}$ for a finite commutative ring R with identity if and only if $n = p^k - 1$ for a prime p and integer $k \geq 1$. However, for a finite commutative ring R without identity, $\Gamma(R)$ is complete bipartite if and only if $\Gamma(R)$ is either $K^{1,1}$, $K^{1,2}$, or $K^{1,2p^k-2}$ for a prime p and integer $k \geq 1$. So, for example, $K^{1,3}$ can be realized only by $\mathbb{F}_4 \times \mathbb{Z}_2$, $K^{1,14}$ only by $\mathbb{F}_8 \times \mathbb{Z}_2^0$, and $K^{1,2}$ by either $\mathbb{Z}_3 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2^0$.*

(d) *For every integer $n \geq 1$, there is a commutative ring R_n without identity such that $|V(\Gamma(R_n))| = n$, namely, $R_n = \mathbb{Z}_{n+1}^0$ has $\Gamma(R_n) = K^n$. However, for commutative rings with identity, there need not be a zero-divisor graph with n vertices. For example, there are no commutative rings with* 1210, 3342, *or* 5466 *zero-divisors,* *and thus there are no zero-divisor graphs of commutative rings with identity with* 1209*,* 3341*, or* 5465 *vertices* [26]*.*

(e) *There are no finite commutative rings R without identity and either $gr(\Gamma(R)) = 4$ or $diam(\Gamma(R)) = 3$. However, for finite commutative rings with identity, $gr(\Gamma(R)) = 4$ and $diam(\Gamma(R)) = 3$ for $R = \mathbb{Z}_{12}$.*

3. Commutative rings with $R = Z(R)$

In this section, we consider commutative rings where every element is a zero-divisor. Clearly such rings have no identity, and Theorem 2.3(1) gives that every finite commutative ring without identity falls into this category. We begin by concentrating on the girth of $\Gamma(R)$.

Theorem 3.1. *Let R be a commutative ring with $R = Z(R)$.*

(1) *$\Gamma(R)$ is complete if and only if $xy = 0$ for every $x, y \in R$.*
(2) *$gr(\Gamma(R)) \in \{3, \infty\}$.*
(3) *If $diam(\Gamma(R)) = 3$, then $gr(\Gamma(R)) = 3$.*
(4) *If $gr(\Gamma(R)) = \infty$, then $\Gamma(R)$ is either K^1 or a star graph.*
(5) *$\Gamma(R)$ is complete bipartite if and only if it is a star graph.*

Proof. (1) This is a special case of Theorem 2.2(3).

(2) By Theorem 2.2(2), it suffices to show that $\Gamma(R)$ contains a 3-cycle when it has a 4-cycle. So suppose $a — b — c — d — a$ is a 4-cycle for nonzero vertices a, b, c, and d with $ac \neq 0 \neq bd$. Since $a - b \neq 0$, there is a $0 \neq t \in R$ such that $t(a - b) = 0$.

Case 1: $ta = tb \neq 0$. Then $a(ta) = a(tb) = t(ab) = 0$, $(ta)b = t(ab) = 0$, and $ta \notin \{a, b, c, d\}$ since $ac \neq 0 \neq bd$. Thus $a — ta — b — a$ is a 3-cycle.

Case 2: $ta = tb = 0$. If $t \notin \{a, b\}$, then $a — b — t — a$ is a 3-cycle. So suppose $t \in \{a, b\}$. Then either $a^2 = 0$ or $b^2 = 0$. If $a^2 = b^2 = 0$, then $a + b \notin \{0, a, b\}$ ($a + b = 0$ would imply that $a = -b$, and thus $ac = (-b)c = 0$, a contradiction). This would give the 3-cycle $a — a + b — b — a$. So suppose that only one of $a^2 = 0$ or $b^2 = 0$ holds.

Without loss of generality, let $a^2 = 0$ and $b^2 \neq 0$. Repeating the above argument also gives, without loss of generality, that $c^2 = 0$ and $d^2 \neq 0$. Now, $0 \neq ac \in V(\Gamma(R))$ and $a(ac) = a^2c = 0$, $b(ac) = 0$, $c(ac) = ac^2 = 0$, and $d(ac) = 0$. Again, $ac \notin \{a, b, c, d\}$ since it would lead to a similar contradiction as above. Thus $a — ac — d — a$ is a 3-cycle.

Therefore $\mathrm{gr}(\Gamma(R)) = 3$.

(3) Let $x, y \in V(\Gamma(R))$ with $\mathrm{d}(x, y) = 3$ and $x — a — b — y$ a shortest path from x and y. Since $x \neq y$, we have $x - y \neq 0$, and thus $t(x - y) = 0$ for some $0 \neq t \in R$. This implies that $tx = ty \neq 0$ since otherwise $x — t — y$ would be a path in $\Gamma(R)$, and therefore $\mathrm{d}(x, y) = 2$, a contradiction.

We claim that $ta = tb = 0$. To prove this claim, assume $ta \neq 0$. Then $x(ta) = (xa)t = 0$ and $(ta)y = (yt)a = (xt)a = 0$. Thus $x — ta — y$ is a path and $\mathrm{d}(x, y) = 2$, a contradiction. Hence $ta = 0$, and similarly, $tb = 0$.

Next, we show that $t \notin \{0, a, b\}$. Clearly $t \neq 0$. If $t = a$, then $0 = ax = tx = ty = ay$ implying that $x — a — y$ is a path, a contradiction. So $t \neq a$, and similarly, $t \neq b$. Thus $t \in V(\Gamma(R))$ is distinct from a and b, and $t — a — b — t$ is a 3-cycle. Therefore $\mathrm{gr}(\Gamma(R)) = 3$.

(4) By Theorem 2.2(2) and (3) above, $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2\}$. It is easily shown that any connected (simple) graph G with $\mathrm{diam}(G) \in \{0, 1, 2\}$, $\mathrm{gr}(G) = \infty$, and at least 2 vertices must be a star graph.

(5) This is clear since $\mathrm{gr}(\Gamma(R)) \in \{3, \infty\}$ by (2) above. □

We get a sharper result when every element of R is nilpotent, i.e., when $R = Z(R) = Nil(R)$. Note that $Nil(R) \neq \{0\}$ when R is a finite commutative ring without identity by Theorem 2.3(2). However, we may have $Nil(R) = \{0\}$ when $R = Z(R)$ is infinite (let $R = \oplus_{n=1}^{\infty} \mathbb{Z}_2$). The diameter case of the next theorem for commutative rings with $Z(R) = Nil(R)$ has been observed by several authors, and the diameter and girth results were given for semigroup zero-divisor graphs in [19, Theorem 5].

Theorem 3.2. *Let R be a commutative ring with $R = Nil(R)$. Then diam$(\Gamma(R)) \in \{0, 1, 2\}$ and gr$(\Gamma(R)) \in \{3, \infty\}$. Moreover, if gr$(\Gamma(R)) \neq 3$, then $\Gamma(R)$ is either K^1, $K^2 = K^{1,1}$, or $K^{1,2}$.*

Proof. Let $0 \neq x \in R$ be nilpotent. Assume that for some vertex y, $\mathrm{d}(x, y) = 3$. Let $x — a — b — y$ be a shortest path from x to y. Since x is nilpotent and $xb \neq 0$, we have $x^n b \neq 0$, but $x^{n+1} b = 0$, for some integer $n \geq 1$. Then $x^n b$ is a vertex in our graph and $x — x^n b — y$ is a path, a contradiction. Thus $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2\}$ when $R = Nil(R)$.

By Theorem 3.1(2), $\mathrm{gr}(\Gamma(R)) \in \{3, \infty\}$ when $R = Nil(R)$ $(= Z(R))$. For the "moreover" statement, we consider three cases (this also gives another proof that $\mathrm{gr}(\Gamma(R)) \in \{3, \infty\}$).

Case 1: Let $a^2 = 0$ for every $a \in R$. If $|R| \leq 3$, then $\Gamma(R)$ is K^1 or K^2; so suppose that $|R| \geq 4$. Let $a, b, c \in V(\Gamma(R))$ be distinct such that $a — b —$

c is a path in $\Gamma(R)$. Such a path exists since $\Gamma(R)$ is connected. Consider $d = a+b$. Clearly $d \neq a, b$. If $d = 0$, then $a = -b$, and thus $ac = (-b)c = 0$. In this case, a — b — c — a is a 3-cycle. So suppose that $d \neq 0$. Then $bd = b(a+b) = ba + ab^2 = 0$ and $ad = a(a+b) = a^2 + ab = 0$, implying that a — b — $a+b$ — a is a 3-cycle; so $\mathrm{gr}(\Gamma(R)) = 3$.

Case 2: Suppose there is an $a \in R$ with $a^2 \neq 0$, but $a^3 = 0$. Then $a^2 \in V(\Gamma(R))$ and a^2 — a is a path in $\Gamma(R)$. If a is not an end, then there is a $0 \neq b \in R$ such that $a^2 \neq b$ and b is adjacent to a. Then $ab = 0$, which implies that $a^2 b = 0$. Thus a^2 — a — b — a^2 is a 3-cycle. So suppose that a is an end. Clearly $a^2 + a \in R \setminus \{0, a, a^2\}$; so $a^2 + a$ — a^2 — a is a path. The existence of $a^2 + a$ shows that there is a $b \in R \setminus \{0, a, a^2\}$ adjacent to a^2. If $\mathrm{gr}(\Gamma(R)) \neq 3$, then $\Gamma(R)$ is a star graph (with center a^2) by Theorem 3.1(4).

Case 3: Suppose there is an $a \in R$ with $a^n = 0$, but $a^{n-1} \neq 0$ for some integer $n \geq 4$. Then a^{n-1}, a^{n-2}, and $a^{n-1} + a^{n-2}$ are distinct nonzero elements of R. Thus a^{n-2} — a^{n-1} — $a^{n-1} + a^{n-2}$ — a^{n-2} is a 3-cycle in $\Gamma(R)$ (since $n \geq 4$); so $\mathrm{gr}(\Gamma(R)) = 3$.

Let $\mathrm{gr}(\Gamma(R)) = \infty$ with $\Gamma(R) \neq K^1$ or K^2; we know that $\Gamma(R)$ must be a star graph. By Case 2 above, there is an $a \in R$ such that $a^2 \neq 0$, $a^3 = 0$, and a is an end. To show that $\Gamma(R) = K^{1,2}$, we consider the size of the ring. We have already shown that $\Gamma(R)$ contains the induced subgraph $a^2 + a$ — a^2 — a. We show that $R = \{0, a, a^2, a^2 + a\}$.

Note that a an end forces $a^2 = -a^2$. Suppose there is an end vertex b distinct from a and $a^2 + a$. If $b \neq -a$, then $a + b \neq 0$. We claim that $a + b \notin \{0, a, b, a^2 + a\}$. To see this claim, it is clear that $a + b \neq a$ and $a + b \neq b$. If $a + b = a^2 + a$, then $b = a^2$, a contradiction to b being an end. Thus $a + b \notin \{0, a, b, a^2 + a\}$. Note that $(ab)a = a^2 b = 0$ since b is an end and $\Gamma(R)$ is a star graph with center a^2. This forces $ab = a^2$. Hence $(a+b)(a^2+a) = a^3 + a^2 + a^2 b + ab = 2a^2 = 0$ since $a^2 = -a^2$. Then $a + b$ — $a^2 + a$ — a^2 — $a + b$ is a 3-cycle, a contradiction. So $b = -a$. Thus $|V(\Gamma(R))| = 4$ which implies that $|R| = 5$. Since R does not have an identity, this means that R has trivial multiplication as there are only two nonisomorphic rings of prime order p: $\mathbb{Z}_p$ and $\mathbb{Z}_p^0$. Hence $\Gamma(R) = K^4$, a contradiction. So $b = a$ or $b = a^2 + a$, and thus $R = \{0, a, a^2, a^2 + a\}$. Therefore $\Gamma(R) = K^{1,2}$. □

The next theorem relaxes the condition in Theorem 3.2 that $R = Nil(R)$ to just $R = Z(R)$.

Theorem 3.3. *Let R be a commutative ring with $R = Z(R)$ and $gr(\Gamma(R)) \neq 3$. Then $\Gamma(R)$ is either K^1, $K^2 = K^{1,1}$, $K^{1,2}$, $K^{1,2p^n-2}$, or $K^{1,m}$, where p is prime, $n \geq 1$ is an integer, and m is an infinite cardinal number.*

Proof. Let $R = Z(R)$ and $\mathrm{gr}(\Gamma(R)) = \infty$. Then by Theorems 3.1(2) and 3.1(4), $\Gamma(R)$ is either K^1 or a star graph. If $R = Nil(R)$, then $\Gamma(R)$ is either K^1, $K^2 = K^{1,1}$, or $K^{1,2}$ by Theorem 3.2. Thus we may assume that R is finite with $Nil(R) \subsetneq Z(R) = R$.

By Theorem 2.3(2), we have $R = R_1 \times R_2$, where R_1 has an identity and $R_2 = Nil(R_2) \neq \{0\}$. We consider three cases (this also gives another proof that $\mathrm{gr}(\Gamma(R)) \in \{3, \infty\}$).

Case 1: $|R_2| = 2$ (so $R_2 \cong \mathbb{Z}_2^0$) and R_1 is not an integral domain. Then there are $0 \neq c, d \in R_1$ such that $cd = 0$ and $R_2 = \{0, a\}$ with $a^2 = 0$. Thus $(c, 0)$ — (d, a) — $(0, a)$ — $(c, 0)$ is a 3-cycle; so $\mathrm{gr}(\Gamma(R)) = 3$.

Case 2: $|R_2| = 2$ and R_1 is an integral domain. Thus $R = \{(x, 0) \mid 0 \neq x \in R_1\} \cup \{(x, a) \mid 0 \neq x \in R_1\} \cup \{(0, 0), (0, a)\}$; so $\Gamma(R)$ is a complete bipartite graph with partitions $\{(0, a)\}$ and $\{(x, y) \mid 0 \neq x \in R_1 \text{ and } y \in \{0, a\}\}$. Hence $\mathrm{gr}(\Gamma(R)) = \infty$. In this case, $\Gamma(R)$ is a star graph with center $(0, a)$ and ends $\{(x, y) \mid 0 \neq x \in R_1 \text{ and } y \in \{0, a\}\}$. So $\Gamma(R) = K^{1,2k}$, where $k = |R_1| - 1$. Since R_1 is an integral domain (finite field), $|R_1| = p^n$ for some prime p and integer $n \geq 1$ (so $R_1 \cong \mathbb{F}_{p^n}$). Thus $\Gamma(R) = K^{1,2p^n-2}$.

Case 3: $|R_2| \geq 3$. Theorem 2.2(1) gives $0 \neq a, b \in R_2$ with $ab = 0$ and $a \neq b$. Then $(0, a)$ — $(0, b)$ — $(1, 0)$ — $(0, a)$ is a 3-cycle; so $\mathrm{gr}(\Gamma(R)) = 3$.

So, if R is finite with $Nil(R) \subsetneq Z(R)$ and $\mathrm{gr}(\Gamma(R)) \neq 3$, then $\Gamma(R) = K^{1,2p^n-2}$ from Case 2. $\square$

Remark 3.4. *Clearly $\Gamma(R) = K^1$ if and only if $R \cong \mathbb{Z}_2^0$, and $\Gamma(R) = K^2$ if and only if $R \cong \mathbb{Z}_3^0$. The proof of Theorem 3.2 shows that for $R = Nil(R)$, $\Gamma(R) = K^{1,2}$ if and only if $R \cong \{0, 2, 4, 6\} \subseteq \mathbb{Z}_8$. Theorem 2.3(2) and Case 2 of the proof of Theorem 3.3 show that for $R \neq Nil(R)$, $\Gamma(R) = K^{1,2p^n-2}$ if and only if $R \cong \mathbb{F}_{p^n} \times \mathbb{Z}_2^0$. Moreover, for F an infinite field, $\Gamma(F \times \mathbb{Z}_2^0) = K^{1,m}$, where $|F| = m$.*

The next corollary gives a quick summary of possible girths for various diameters of the zero-divisor graphs of commutative rings $R = Z(R)$. Tables 24 and 25 give some examples of the realizable cases.

Corollary 3.5. *Let R be a commutative ring with $R = Z(R)$.*

(1) *If $diam(\Gamma(R)) = 0$, then $gr(\Gamma(R)) = \infty$.*
(2) *If $diam(\Gamma(R)) = 1$, then $gr(\Gamma(R)) \in \{3, \infty\}$.*

(3) *If* $diam(\Gamma(R)) = 2$, *then* $gr(\Gamma(R)) \in \{3, \infty\}$.
(4) *If* $diam(\Gamma(R)) = 3$, *then* $gr(\Gamma(R)) = 3$.

We next turn to the diameter of $\Gamma(R)$. Our goal is to show that $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2\}$ when R is a finite commutative ring without identity (note that we may have $\mathrm{diam}(\Gamma(R)) = 3$ when $R = Z(R)$ is infinite, see Example 6.1).

Theorem 3.6. *Let* $R = R_1 \times R_2$ *be a commutative ring, where* R_1 *has an identity and* $R_2 = Nil(R_2) \neq \{0\}$. *Then* $R = Z(R)$ *and* $diam(\Gamma(R)) = 2$.

Proof. Clearly $R = Z(R)$. For $0 \neq b \in R_2$, let $x = (1, 0)$ and $y = (1, b)$. Then $0 \neq x, y \in Z(R)$ and $xy \neq 0$; so $\mathrm{d}(x, y) \geq 2$. Thus $\mathrm{diam}(\Gamma(R)) \geq 2$. For distinct $x, y \in Z(R) \setminus \{(0, 0)\}$, we consider cases, where $a, c \in R_1$ and $b, d \in R_2$ are all nonzero.

Case 1: Let $x = (a, 0)$ and $y = (0, b)$. Then $xy = 0$ and $\mathrm{d}(x, y) = 1$.

Case 2: Let $x = (a, 0)$ and $y = (c, 0)$. For $z = (0, b)$, then $xz = yz = 0$; so $\mathrm{d}(x, y) \leq 2$.

Case 3: Let $x = (0, b)$ and $y = (0, d)$. For $z = (1, 0)$, then $xz = yz = 0$; so $\mathrm{d}(x, y) \leq 2$.

Case 4: Let $x = (a, 0)$ and $y = (c, d)$. Let $n \geq 2$ be an integer such that $d^n = 0$, but $d^{n-1} \neq 0$. For $z = (0, d^{n-1})$, then $xz = yz = 0$; so $\mathrm{d}(x, y) \leq 2$.

Case 5: Let $x = (0, b)$ and $y = (c, d)$. If $bd = 0$, then $xy = 0$ and $\mathrm{d}(x, y) = 1$. Otherwise, Theorem 3.2 gives a $0 \neq f \in R_2$ distinct from b and d such that $bf = fd = 0$. For $z = (0, f)$, then $xz = yz = 0$; so $\mathrm{d}(x, y) = 2$.

Case 6: Let $x = (a, b)$ and $y = (c, d)$. If $b = d$, then there is an integer $n \geq 2$ with $b^n = 0$ and $b^{n-1} \neq 0$. For $z = (0, b^{n-1})$, then $xz = yz = 0$; so $\mathrm{d}(x, y) \leq 2$. If $b \neq d$, then Theorem 3.2 gives a $0 \neq f \in R_2$ with $bf = fd = 0$. For $z = (0, f)$, then $xz = yz = 0$; so $\mathrm{d}(x, y) \leq 2$.

Thus $\mathrm{d}(x, y) \leq 2$ for every $0 \neq x, y \in R$. Therefore $\mathrm{diam}(\Gamma(R)) \leq 2$, which implies that $\mathrm{diam}(\Gamma(R)) = 2$. □

For a finite commutative ring without identity, we can sharpen Theorem 2.2.

Theorem 3.7. *Let* R *be a finite commutative ring without identity. Then* $diam(\Gamma(R)) \in \{0, 1, 2\}$ *and* $gr(\Gamma(R)) \in \{3, \infty\}$.

Proof. By Theorem 2.3(1), $R = Z(R)$. Thus $\mathrm{gr}(\Gamma(R)) \in \{3, \infty\}$ by Theorem 3.1(2). If $R = Nil(R)$, then $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2\}$ by Theorem 3.2. Otherwise, $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2\}$ by Theorems 2.3(2) and 3.6. □

Corollary 3.8. *Let R be a finite commutative ring with $Nil(R) \subsetneq R = Z(R)$. Then the following statements are equivalent.*

(1) $gr(\Gamma(R)) \neq 3$.
(2) $gr(\Gamma(R)) = \infty$.
(3) $\Gamma(R)$ *is a complete bipartite graph.*
(4) $\Gamma(R)$ *is a star graph.*
(5) $\Gamma(R) = K^{1,2p^n-2}$ *for some prime p and integer $n \geq 1$.*

Proof. This follows directly from Theorem 3.1 and the proof of Theorem 3.3. □

4. Commutative rings with $Z(R) \subsetneq R$

In this section, we study commutative rings R without identity and $Z(R) \subsetneq R$. Such rings are necessarily infinite by Theorem 2.3(1). We also briefly discuss the compressed zero-divisor graph $\Gamma_E(R)$ for R a commutative ring without identity.

The next several theorems will aide in the classification of some of the diameters and girths of zero-divisor graphs in Section 6.

Theorem 4.1. *Let R be a commutative ring with total quotient ring $T(R)$. Then $\Gamma(T(R)) \cong \Gamma(R)$.*

Proof. We may assume that $Z(R) \subsetneq R$. For a commutative ring R without identity, the proof is essentially the same as for commutative rings with identity given in [10, Theorem 2.2], except replace $x/1$ by sx/s for $s \in S = R \setminus Z(R)$. □

Theorem 4.2. *Let R be a commutative ring without identity and $Z(R) \subsetneq R$. Then $T(R)$ has an identity and $\Gamma(R) \cong \Gamma(T(R))$.*

Proof. Let R be a commutative ring without identity and a regular element $s \in S = R \setminus Z(R)$. Then $T(R) = R_S$ has identity s/s, and $\Gamma(R) \cong \Gamma(T(R))$ by Theorem 4.1. □

If $R = Z(R)$, then $\mathrm{gr}(\Gamma(R)) \in \{3, \infty\}$ and $\mathrm{gr}(\Gamma(R)) = 3$ when $\mathrm{diam}(\Gamma(R)) = 3$ by Theorem 3.1. When R does not have an identity and $Z(R) \subsetneq R$, we may have $\mathrm{gr}(\Gamma(R)) = 4$. For example, $R = 2\mathbb{Z} \times \mathbb{Z}_4$ has $\mathrm{diam}(\Gamma(R)) = 3$ and $\mathrm{gr}(\Gamma(R)) = 4$.

Theorem 4.3. *Let R be a commutative ring without identity and $Z(R) \subsetneq R$. If $diam(\Gamma(R)) = 3$, then $gr(\Gamma(R)) \in \{3, 4\}$.*

Proof. By Theorem 4.2, $\Gamma(R) \cong \Gamma(T(R))$; so $\mathrm{diam}(\Gamma(T(R))) = 3$. Moreover, $T(R)$ is an infinite ring with identity since R is infinite; so $\mathrm{gr}(\Gamma(T(R))) \in \{3, 4\}$ by Theorem 2.4(3). Thus, if $\mathrm{diam}(\Gamma(R)) = 3$, then $\mathrm{gr}(\Gamma(R)) \in \{3, 4\}$. □

These results show that the zero-divisor graph of a commutative ring with a non-unit regular element can be realized by both a ring with identity and a ring without identity. Note that since these rings have a non-unit regular element, they are necessarily infinite. Specifically, let R be a commutative ring with a non-unit regular element x. If R has an identity, use Example 2.1(b); if R does not have an identity, use Theorem 4.2. Thus commutative rings without identity and $Z(R) \subsetneq R$ behave very much like commutative rings with identity.

We next briefly consider the compressed zero-divisor graph $\Gamma_E(R)$ for R a commutative ring. Define a congruence relation $\sim$ on R by $x \sim y$ if and only if $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$, and let $R_E = R/\sim = \{ [x]_\sim \mid x \in R \}$ (from now on, denote $[x]_\sim$ by $[x]$). Then R_E is a semigroup under the multiplication $[x][y] = [xy]$, and the *compressed zero-divisor graph* of R is $\Gamma_E(R) = G(R_E)$. For commutative rings with identity, this graph was defined in [23] (using different notation) and has been further studied in [7], [8], [18], and [27].

The next theorem relates the compressed zero-divisor graph of a commutative ring with identity to a commutative ring without identity. Note that we may replace $\mathbb{Z}_2^0$ by any ring with trivial multiplication.

Theorem 4.4. *Let R be a commutative ring with identity. Then $T = R \times \mathbb{Z}_2^0$ is a commutative ring without identity, $T = Z(T)$, and $\Gamma_E(T) \cong \Gamma_E(R)$.*

Proof. Clearly $T = Z(T)$ and T has no identity. Define $\varphi : R/\sim \longrightarrow T/\sim$ by $\varphi([x]) = [(x, 0)]$. It is easily verified that $\mathrm{ann}_R(x) = \mathrm{ann}_R(y)$ for $x, y \in R$ if and only if $\mathrm{ann}_T((x, 0)) = \mathrm{ann}_T((y, 0))$, and $[(x, 0)] = [(x, 1)]$ for every $x \in R$. Thus φ is a well-defined bijection. Moreover, φ restricts to a graph isomorphism from $\Gamma_E(R)$ to $\Gamma_E(T)$ since $[(x, 0)][(y, 0)] = [(0, 0)]$ if and only if $[x][y] = [0]$. $\square$

The following theorem is the compressed zero-divisor graph analog of Theorems 4.1 and 4.2.

Theorem 4.5. *Let R be a commutative ring with total quotient ring $T(R)$. Then $\Gamma_E(T(R)) \cong \Gamma_E(R)$. In particular, if R has no identity and $Z(R) \subsetneq R$, then $T(R)$ has an identity and $\Gamma_E(R) \cong \Gamma_E(T(R))$.*

Proof. The proof is essentially the same as for commutative rings with identity in [7, Theorem 3.2]. The "in particular" statement is clear. $\square$

Thus every graph that can be realized as a compressed zero-divisor graph by a commutative ring with identity can be realized by a commutative ring without identity, and every graph that can be realized as a compressed zero-divisor graph by a commutative ring without identity and $Z(R) \subsetneq R$ can be realized by a commutative ring with identity.

Remark 4.6. *The results in this section motivate the following two questions.*

(a) *Let R be an infinite commutative ring with identity. Is there a commutative ring T without identity such that $\Gamma(R) \cong \Gamma(T)$?*

(b) *Let R be a commutative ring with $R = Z(R)$. Is there a commutative ring T with identity such that $\Gamma_E(R) \cong \Gamma_E(T)$?*

5. Classification of small finite commutative rings without identity

In this section, we classify up to isomorphism all commutative rings without identity that have a zero-divisor graph with 14 or fewer vertices. Three papers, [14], [15], and [21] (together with an errata sheet), were helpful in determining the nonisomorphic rings of order 8. In [21], they also reference results for the nonisomorphic rings of other orders.

Recall that if a finite ring R has $|R| = p_1^{n_1} \cdots p_k^{n_k}$ for distinct primes p_i and integers $k_i \geq 1$, then $R = R_1 \oplus \cdots \oplus R_k$ for subrings R_i with $|R_i| = p_i^{n_i}$.

For rings whose underlying abelian group structure is a cyclic group of order k, the commutative rings are $x\mathbb{Z}[x]/(kx, x^2 - k_i x)$, where k_i is a positive divisor of k. These rings are commutative, and the only such ring with an identity is $x\mathbb{Z}[x]/(kx, x^2 - x) \cong \mathbb{Z}_k$. Thus we are interested in the other types of these rings in this paper. The ring $x\mathbb{Z}[x]/(kx, x^2 - kx)$ is the commutative ring of order k with trivial multiplication.

For rings with order p^2 for some prime p, there are eleven nonisomorphic rings: $\mathbb{Z}_{p^2}$, $x\mathbb{Z}[x]/(p^2x, x^2 - px)$, $\mathbb{Z}^0_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_p \times \mathbb{Z}^0_p$, $\mathbb{Z}^0_p \times \mathbb{Z}^0_p$, $\mathbb{F}_{p^2}$, $\mathbb{Z}_p[x]/(x^2)$, $x\mathbb{Z}_p[x]/x^3\mathbb{Z}_p[x]$, $A = \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_p\}$, and $B = \{\left[\begin{smallmatrix} 0 & a \\ 0 & b \end{smallmatrix}\right] \mid a, b \in \mathbb{Z}_p\}$. We will be interested in the rings $x\mathbb{Z}[x]/(p^2x, x^2 - px)$, $\mathbb{Z}^0_{p^2}$, $\mathbb{Z}_p \times \mathbb{Z}^0_p$, $\mathbb{Z}^0_p \times \mathbb{Z}^0_p$, and $x\mathbb{Z}_p[x]/x^3\mathbb{Z}_p[x]$, as these are the rings that are commutative without an identity.

The number of associative rings of order p^3 is given in [21]. We need only worry about the case when $p = 2$ in this paper. Recall that $|V(\Gamma(R))| = |R| - 1$ as it will aid in finding the nonisomorphic classes of rings.

Tables 1, 2, 3, 4, 5, and 6 give the list of nonisomorphic commutative rings without identity, their abelian group structure type, and their zero-divisor graph.

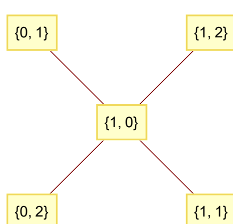
TABLE 1. Rings with graphs on 1, 2, and 3 vertices

Vertices	Group Type	R	Graph
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^0$	K^1
2	$\mathbb{Z}_3$	$\mathbb{Z}_3^0$	K^2
3	$\mathbb{Z}_4$	$\mathbb{Z}_4^0$	K^3
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2$	Figure 1
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0$	K^3
	$\mathbb{Z}_4$	$x\mathbb{Z}[x]/(4x, x^2 - 2x)$	Figure 1
	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$x\mathbb{Z}_2[x]/x^3\mathbb{Z}_2[x]$	Figure 1

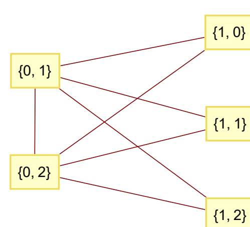
FIGURE 1. 3-vertex graph

TABLE 2. Rings with graphs on 4, 5, and 6 vertices

Vertices	Group Type	R	Graph
4	$\mathbb{Z}_5$	$\mathbb{Z}_5^0$	K^4
5	$\mathbb{Z}_6$	$\mathbb{Z}_6^0$	K^5
	$\mathbb{Z}_6$	$\mathbb{Z}_2^0 \times \mathbb{Z}_3$	Figure 2a
	$\mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_3^0$	Figure 2b
6	$\mathbb{Z}_7$	$\mathbb{Z}_7^0$	K^6



(a)



(b)

FIGURE 2. 5-vertex graphs

Table 3. Rings with graphs on 7 vertices

Vertices	Group Type	R	Graph
7	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Figure 3a
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0 \times \mathbb{Z}_2$	Figure 3b
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0 \times \mathbb{Z}_2^0$	K^7
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^0 \times \mathbb{Z}_4$	Figure 3c
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4^0$	Figure 3b
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^0 \times \mathbb{Z}_4^0$	K^7
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times x\mathbb{Z}[x]/(4x, x^2 - 2x)$	Figure 3a
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^0 \times x\mathbb{Z}[x]/(4x, x^2 - 2x)$	Figure 3b
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{F}_4$	Figure 3d
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2[x]/(x^2)$	Figure 3c
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times x\mathbb{Z}_2[x]/x^3\mathbb{Z}_2[x]$	Figure 3a
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2^0 \times x\mathbb{Z}_2[x]/x^3\mathbb{Z}_2[x]$	Figure 3b
	$\mathbb{Z}_8$	$\mathbb{Z}_8^0$	K^7
	$\mathbb{Z}_8$	$x\mathbb{Z}[x]/(8x, x^2 - 2x)$	Figure 3c
	$\mathbb{Z}_8$	$x\mathbb{Z}[x]/(8x, x^2 - 4x)$	Figure 3b
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Max Ideal of $\mathbb{Z}_4[x]/(x^2)$	Figure 3e
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Max Ideal of $\mathbb{Z}_4[x]/(x^2 - 2x)$	Figure 3f
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Max Ideal of $\mathbb{Z}_4[x]/(x^2 - 2)$	Figure 3c
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Ideal $\langle x, y \rangle$ of $\mathbb{Z}_4[x,y]/(2x, x^2, xy, y^2 - x)$	Figure 3b
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Ideal $\langle x, y \rangle$ of $\mathbb{Z}_4[x,y]/(2x, x^2 - 2y, xy, y^2)$	Figure 3b
	$\mathbb{Z}_2 \times \mathbb{Z}_4$	Ideal $\langle x, y \rangle$ of $\mathbb{Z}_4[x,y]/(2x, x^2 - 2y, xy, y^2 - 2y)$	Figure 3f
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Max Ideal of $\mathbb{Z}_2[x,y]/(x^3, xy, y^2)$	Figure 3b
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Max Ideal of $\mathbb{Z}_2[x,y]/(x^3, xy, y^2 - x^2)$	Figure 3f
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	Max Ideal of $\mathbb{Z}_2[x,y]/(x^2, y^2)$	Figure 3e
	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$x\mathbb{Z}_2[x]/x^4\mathbb{Z}_2[x]$	Figure 3c

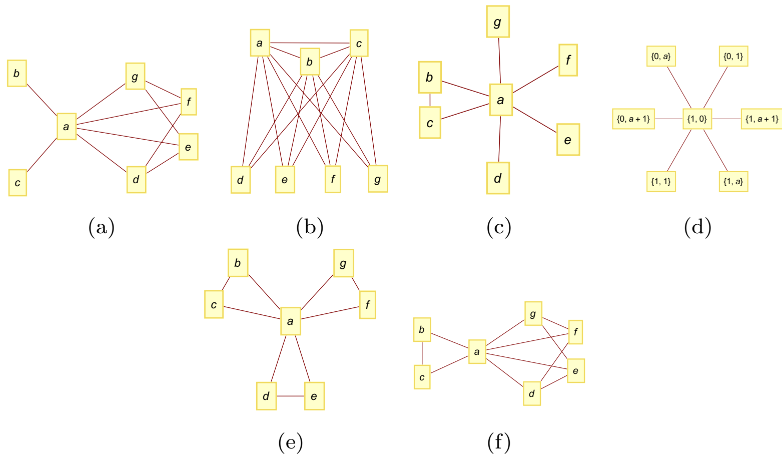


FIGURE 3. 7-vertex graphs

TABLE 4. Rings with graphs on 8, 9, and 10 vertices

Vertices	Group Type	R	Graph
8	$\mathbb{Z}_9$	$\mathbb{Z}_9^0$	K^8
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3^0 \times \mathbb{Z}_3$	Figure 4
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3^0 \times \mathbb{Z}_3^0$	K^8
	$\mathbb{Z}_9$	$x\mathbb{Z}[x]/(9x, x^2 - 3x)$	Figure 4
	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$x\mathbb{Z}_3[x]/x^3\mathbb{Z}_3[x]$	Figure 4
9	$\mathbb{Z}_{10}$	$\mathbb{Z}_{10}^0$	K^9
	$\mathbb{Z}_{10}$	$\mathbb{Z}_2^0 \times \mathbb{Z}_5$	Figure 5a
	$\mathbb{Z}_{10}$	$\mathbb{Z}_2 \times \mathbb{Z}_5^0$	Figure 5b
10	$\mathbb{Z}_{11}$	$\mathbb{Z}_{11}^0$	K^{10}

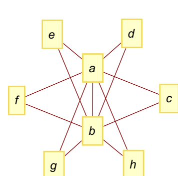
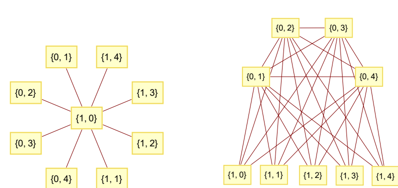


FIGURE 4. 8-vertex graph

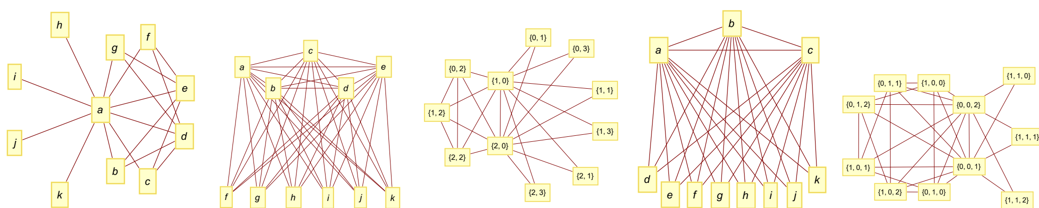


(a) (b)

FIGURE 5. 9-vertex graphs

TABLE 5. Rings with graphs on 11 vertices

Vertices	Group Type	R	Graph
11	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}^0$	K^{11}
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2^0 \times \mathbb{Z}_6$	Figure 6a
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_6^0$	Figure 6b
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2^0 \times \mathbb{Z}_6^0$	K^{11}
	$\mathbb{Z}_{12}$	$\mathbb{Z}_3^0 \times \mathbb{Z}_4$	Figure 6c
	$\mathbb{Z}_{12}$	$\mathbb{Z}_3 \times \mathbb{Z}_4^0$	Figure 6d
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2^0 \times \mathbb{Z}_2^0 \times \mathbb{Z}_3$	Figure 6d
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3^0$	Figure 6e
	$\mathbb{Z}_{12}$	$\mathbb{Z}_3 \times {}^{x\mathbb{Z}[x]}/_{(4x,\, x^2-2x)}$	Figure 6a
	$\mathbb{Z}_{12}$	$\mathbb{Z}_3^0 \times {}^{x\mathbb{Z}[x]}/_{(4x,\, x^2-2x)}$	Figure 6b
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_3 \times {}^{x\mathbb{Z}_2[x]}/_{x^3\mathbb{Z}_2[x]}$	Figure 6a
	$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\mathbb{Z}_3^0 \times {}^{x\mathbb{Z}_2[x]}/_{x^3\mathbb{Z}_2[x]}$	Figure 6b

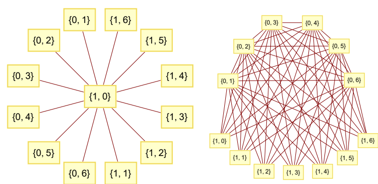


(a) (b) (c) (d) (e)

FIGURE 6. 11-vertex graphs

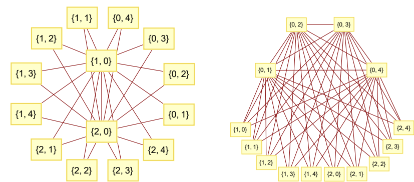
TABLE 6. Rings with graphs on 12, 13, and 14 vertices

Vertices	Group Type	R	Graph
12	$\mathbb{Z}_{13}$	$\mathbb{Z}_{13}^0$	K^{12}
13	$\mathbb{Z}_{14}$	$\mathbb{Z}_{14}^0$	K^{13}
	$\mathbb{Z}_{14}$	$\mathbb{Z}_2^0 \times \mathbb{Z}_7$	Figure 7a
	$\mathbb{Z}_{14}$	$\mathbb{Z}_2 \times \mathbb{Z}_7^0$	Figure 7b
14	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}^0$	K^{14}
	$\mathbb{Z}_{15}$	$\mathbb{Z}_3^0 \times \mathbb{Z}_5$	Figure 8a
	$\mathbb{Z}_{15}$	$\mathbb{Z}_3 \times \mathbb{Z}_5^0$	Figure 8b



(a) (b)

FIGURE 7. 13-vertex graphs



(a) (b)

FIGURE 8. 14-vertex graphs

The work in this section is analogous to Redmond's for commutative rings with identity in [24,25]. As such, it is useful to see a comparison between the zero-divisor graphs of finite commutative rings with identity and finite commutative rings without identity. We look at this comparison in Tables 7 through 20.

TABLE 7. Realizable zero-divisor graphs on 1 vertex

With Identity	
Without Identity	

TABLE 8. Realizable zero-divisor graphs on 2 vertices

With Identity	
Without Identity	

TABLE 9. Realizable zero-divisor graphs on 3 vertices

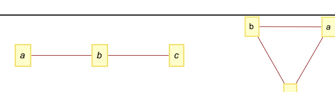
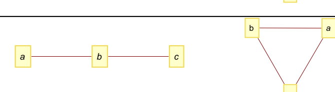
With Identity	
Without Identity	

TABLE 10. Realizable zero-divisor graphs on 4 vertices

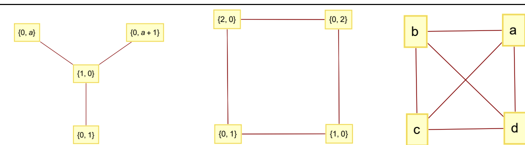
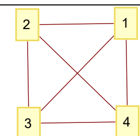
With Identity	
Without Identity	

TABLE 11. Realizable zero-divisor graphs on 5 vertices

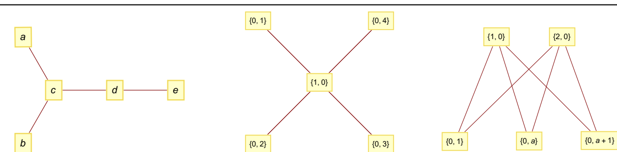
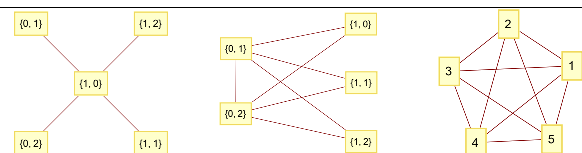
With Identity	
Without Identity	

TABLE 12. Realizable zero-divisor graphs on 6 vertices

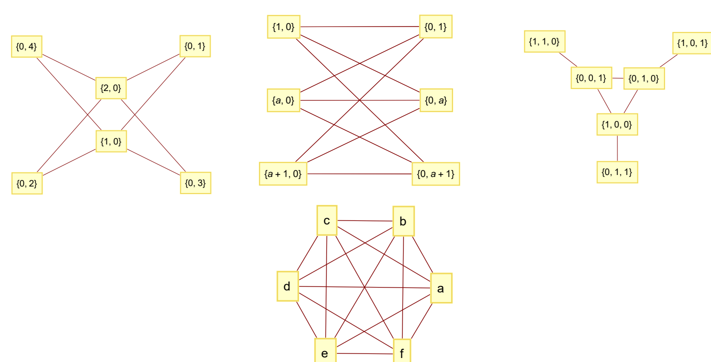
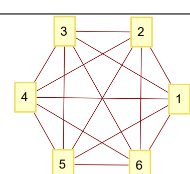
With Identity	
Without Identity	

TABLE 13. Realizable zero-divisor graphs on 7 vertices

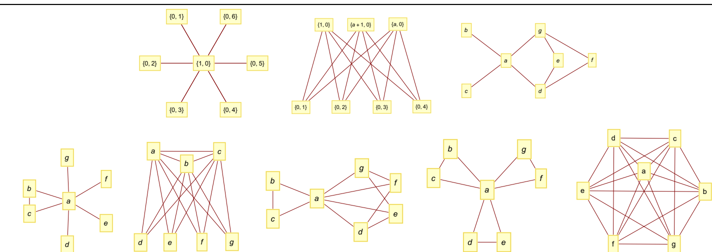
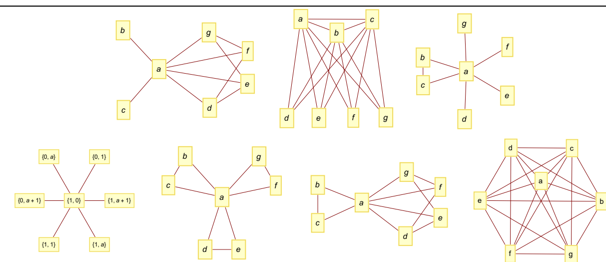
With Identity	
Without Identity	

TABLE 14. Realizable zero-divisor graphs on 8 vertices

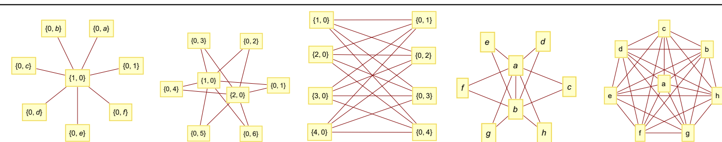
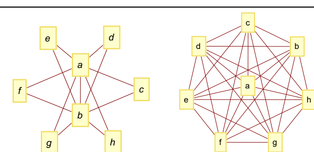
With Identity	
Without Identity	

TABLE 15. Realizable zero-divisor graphs on 9 vertices

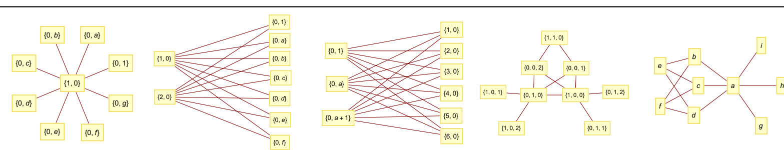
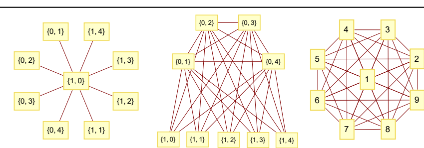
With Identity	
Without Identity	

TABLE 16. Realizable zero-divisor graphs on 10 vertices

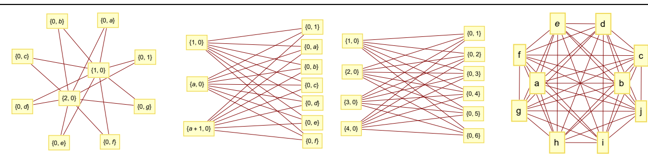
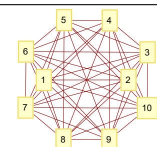
With Identity	
Without Identity	

TABLE 17. Realizable zero-divisor graphs on 11 vertices

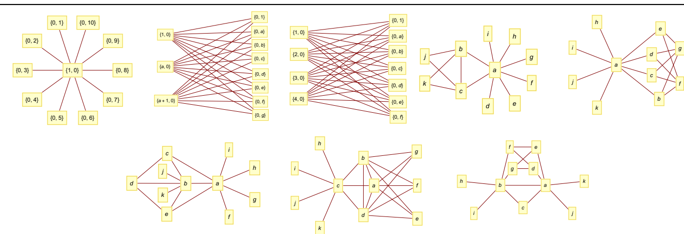
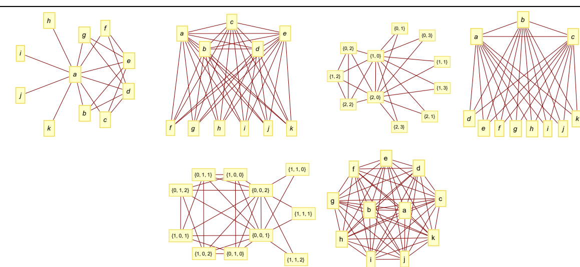
With Identity	
Without Identity	

TABLE 18. Realizable zero-divisor graphs on 12 vertices

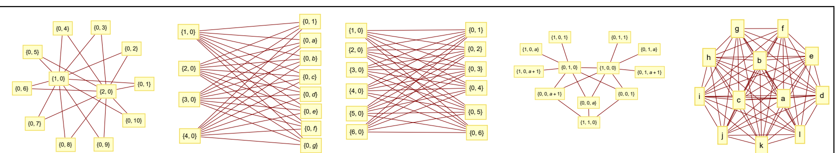
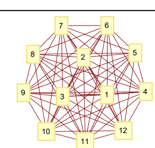
With Identity	
Without Identity	

TABLE 19. Realizable zero-divisor graphs on 13 vertices

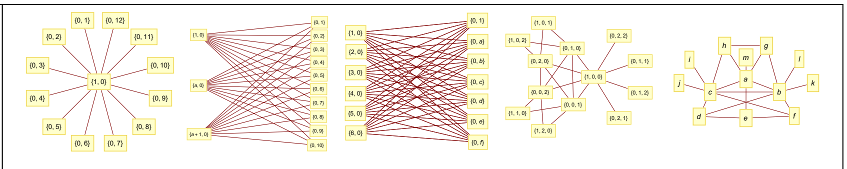
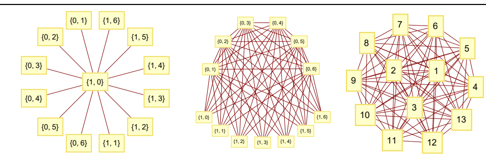
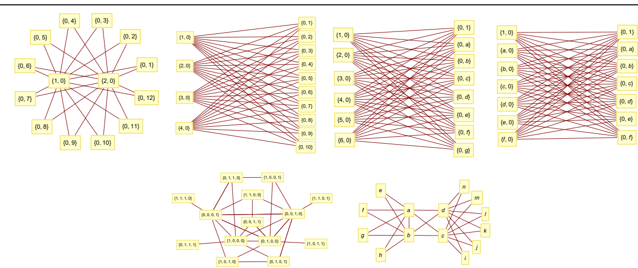
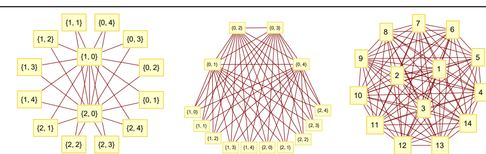
With Identity	
Without Identity	

TABLE 20. Realizable zero-divisor graphs on 14 vertices

With Identity	
Without Identity	

6. Realizable diameters and girths of zero-divisor graphs

Let R be a commutative ring. Then $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\mathrm{gr}(\Gamma(R)) \in \{3, 4, \infty\}$ by Theorem 2.2. In this section, we determine which combinations of girth and diameter can be realized as a zero-divisor graph of some commutative ring. We separate this problem into six different types of rings: finite and infinite rings with identity, where every element is either a zero-divisor or a unit; rings with identity with a non-unit regular element; finite and infinite rings without identity, where every element is a zero-divisor; and rings without identity with a regular element. The following tables give the results, as well as an example of a ring that has the indicated properties.

Table 21 gives the breakdown for finite commutative rings with identity (thus every element is a zero-divisor or a unit). Clearly a zero-divisor graph with diameter 0 must have infinite girth and a zero-divisor graph with diameter 1 cannot have girth 4.

TABLE 21. Finite rings with identity (thus $R = Z(R) \cup U(R)$)

diam \ girth	3	4	∞
0	None	None	$\mathbb{Z}_4$
1	$\mathbb{F}_4[x]/(x^2)$	None	$\mathbb{Z}_2 \times \mathbb{Z}_2$
2	$\mathbb{Z}_{16}$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_6$
3	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2 \times \mathbb{Z}_4$

Table 22 lists the results for infinite rings R, where each element is either a zero-divisor or a unit. Several cases need justification, see Theorem 2.4.

TABLE 22. Infinite rings with identity where $R = Z(R) \cup U(R)$

diam \ girth	3	4	∞
0	None	None	None
1	$\mathbb{Q}[x]/(x^2)$	None	None
2	$\mathbb{Q}[x]/(x^3)$	$\mathbb{Q} \times \mathbb{Q}$	$\mathbb{Z}_2 \times \mathbb{Q}$
3	$\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$	$\mathbb{Z}_4 \times \mathbb{Q}$	None

Table 23 breaks down the case where R is a commutative ring with identity and a non-unit regular element (so R must be infinite). Several cases must be justified here as well and are covered by Theorem 2.4.

TABLE 23. Rings with identity where $Z(R) \cup U(R) \subsetneq R$

diam \ girth	3	4	∞
0	None	None	None
1	$\mathbb{Z}[x]/(x^2)$	None	None
2	$\mathbb{Z}[x]/(x^3)$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}_2 \times \mathbb{Z}$
3	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$	$\mathbb{Z}_4 \times \mathbb{Z}$	None

Next, we give the results for a commutative ring without identity where every element is a zero-divisor. Table 24 lists these results for finite rings. By Theorem 3.7, $\operatorname{diam}(\Gamma(R)) \in \{0, 1, 2\}$ and $\operatorname{gr}(\Gamma(R)) \in \{3, \infty\}$.

TABLE 24. Finite rings without identity (thus $R = Z(R)$)

diam \ girth	3	4	∞
0	None	None	$\mathbb{Z}_2^0$
1	$\mathbb{Z}_4^0$	None	$\mathbb{Z}_3^0$
2	$\mathbb{Z}_2 \times \mathbb{Z}_3^0$	None	$\mathbb{Z}_2^0 \times \mathbb{Z}_2$
3	None	None	None

Table 25 gives the results for infinite rings R with $R = Z(R)$. By Theorem 3.1(2), $\operatorname{gr}(\Gamma(R)) \in \{3, \infty\}$. Theorem 2.4 eliminates the infinite girth for graphs with diameter 0 and 1. By Theorem 3.1(3), there is no such ring $R = Z(R)$ with $\operatorname{diam}(\Gamma(R)) = 3$ and $\operatorname{gr}(\Gamma(R)) = \infty$. For the case where the diameter and girth are both 3, we use a ring that was mentioned in [22, Example 5.1], but where the details were covered in [4, Example 3.13]. As the ring is a bit complicated, we restate [4, Example 3.13] in Example 6.1 to explain the ring (another example is given in [12, Theorem 3.4]).

TABLE 25. Infinite rings without identity where $R = Z(R)$

diam \ girth	3	4	∞
0	None	None	None
1	$\mathbb{Z}^0$	None	None
2	$\mathbb{Z}_4 \times \mathbb{Z}^0$	None	$\mathbb{Z}_2^0 \times \mathbb{Z}$
3	Example 6.1	None	None

Example 6.1 ([4, Example 3.13]). *Let $D = \mathbb{Q}[x, y]_{(x,y)}$, a two-dimensional local UFD with maximal ideal $Q = (x, y)_{(x,y)}$. Let $\mathcal{P}$ be the set of height-one prime ideals of D, and let $I = \mathcal{P} \times \mathbb{N}$. For every $i = (P, n) \in I$, let $Q_i = Q/P$. Let $A = D \times (\oplus_{i \in I} Q_i)$ with coordinate-wise addition and multiplication defined by $(a, (a_i))(b, (b_i)) = (ab, (ab_i + ba_i + a_i b_i))$. Then A is a reduced commutative ring with maximal ideal $M = Q \times (\oplus_{i \in I} Q_i)$ and $Z(A) = M$. Thus $R = M$ is an*

infinite commutative ring with $R = Z(R)$*. Finally, as shown in* [4, Example 3.13], *diam*$(\Gamma(R)) = 3$*, and by Theorem 3.1(3),* *gr*$(\Gamma(R)) = 3$.

Lastly, we consider commutative rings without identity that have a regular element, i.e., $Z(R) \subsetneq R$. Note that R must be infinite; so Theorem 2.4 applies. The last case that needs justification, when the diameter is 3 and girth is infinite, is given in Theorem 4.3. Table 26 lists the results.

TABLE 26. Rings without identity where $Z(R) \subsetneq R$

diam \ girth	3	4	∞
0	None	None	None
1	$x\mathbb{Z}_4[x]$	None	None
2	$x\mathbb{Z}_8[x]$	$\mathbb{Z} \times 2\mathbb{Z}$	$\mathbb{Z}_2 \times 2\mathbb{Z}$
3	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times 2\mathbb{Z}$	$2\mathbb{Z} \times \mathbb{Z}_4$	None

References

[1] D. D. Anderson, *Commutative rngs*, in Multiplicative Ideal Theory in Commutative Algebra: A tribute to the work of Robert Gilmer (J. W. Brewer et al., Eds.), Springer-Verlag, New York, (2006), 1-20.

[2] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring*, J. Algebra, 159(2) (1993), 500-514.

[3] D. D. Anderson and J. Stickles, *Commutative rings with finitely generated multiplicative semigroup*, Semigroup Forum, 60(3) (2000), 436-443.

[4] D. F. Anderson, *On the diameter and girth of a zero-divisor graph, II*, Houston J. Math., 34(2) (2008), 361-371.

[5] D. F. Anderson, M. C. Axtell and J. A. Stickles, Jr., *Zero-divisor graphs in commutative rings*, in Commutative Algebra, Noetherian and Non-Noetherian Perspectives (M. Fontana et al., Eds.), Springer-Verlag, New York, (2011), 23-45.

[6] D. F. Anderson and A. Badawi, *The zero-divisor graph of a commutative semigroup: A survey*, in Groups, Modules, and Model Theory-Surveys and Recent Developments, In Memory of Rüdiger Göbel (M. Droste et al., Eds.), Springer-Verlag, Cham, (2017), 23-39.

[7] D. F. Anderson and J. D. LaGrange, *Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph*, J. Pure Appl. Algebra, 216(7) (2012), 1626-1636.

[8] D. F. Anderson and J. D. LaGrange, *Some remarks on the compressed zero-divisor graph*, J. Algebra, 447 (2016), 297-321.

[9] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217(2) (1999), 434-447.

[10] D. F. Anderson, R. Levy and J. Shapiro, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebras*, J. Pure Appl. Algebra, 180(3) (2003), 221-241.

[11] D. F. Anderson and S. B. Mulay, *On the diameter and girth of a zero-divisor graph*, J. Pure Appl. Algebra, 210(2) (2007), 543-550.

[12] M. Axtell, J. Stickles and W. Trambachls, *Zero-divisor ideals and realizable zero-divisor graphs*, Involve, 2(1) (2009), 17-27.

[13] M. Axtell, J. Stickles and J. Warfel, *Zero-divisor graphs of direct products of commutative rings*, Houston J. Math, 32(4) (2006), 985-994.

[14] R. Ballieu, *Anneaux finis; systèmes hypercomplexes de rang trois sur un corps commtatif*, Annales de la Société Scientific de Bruxelles, Série I, 61 (1947), 222-227.

[15] R. Ballieu, *Anneaux finis à module de type* (p, p^2), Annales de la Société Scientific de Bruxelles, Série I, 63 (1949), 11-23.

[16] I. Beck, *Coloring of commutative rings*, J. Algebra, 116(1) (1988), 208-226.

[17] B. Bollaboás, Graph Theory: An Introductory Course, Springer-Verlag, New York, 1979.

[18] J. Coykendall, S. Sather-Wagstaff, L. Sheppardson and S. Spiroff, *On zero divisor graphs*, in Progress in Commutative Algebra II: Closures, Finiteness and Factorization (C. Francisco et al., Eds.), Walter de Gruyter, Berlin, (2012), 241-299.

[19] F. DeMeyer and L. DeMeyer, *Zero divisor graphs of semigroups*, J. Algebra, 283(1) (2005), 190-198.

[20] F. R. DeMeyer, T. McKenzie and K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum, 65(2) (2002), 206-214.

[21] R. Gilmer and J. Mott, *Associative rings of order* p^3, Proc. Japan Acad., 49 (1973), 795-799.

[22] T. G. Lucas, *The diameter of a zero-divisor graph*, J. Algebra, 301(1) (2006), 174-193.

[23] S. B. Mulay, *Cycles and symmetries of zero-divisors*, Comm. Algebra, 30(7) (2002), 3533-3558.

[24] S. P. Redmond, *On zero-divisor graphs of small finite commutative rings*, Discrete Math., 307(9-10) (2007), 1155-1166.

[25] S. P. Redmond, *Corrigendum to: "On zero-divisor graphs of small finite commutative rings"* [Discrete Math., 307 (2007), 1155-1166], Discrete Math., 307(21) (2007), 2449-2452.

[26] S. P. Redmond, *Counting zero-divisors*, in Commutative Rings: New Research, Nova Sci. Publ., Hauppauge, NY, (2009), 7-12.

[27] S. Spiroff and C. Wickham, *A zero divisor graph determined by equivalence classes of zero divisors*, Comm. Algebra, 39(7) (2011), 2338-2348.

[28] D. Weber, *Various Topics on Graphical Structures Placed on Commutative Rings*, PhD dissertation, The University of Tennessee, 2017.

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