Sigma J Eng Nat Sci, Vol. 42, No. 3, pp. 885−899, June, 2024

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2024.00000

Research Article

Lie symmetry analysis of Caputo time-fractional *K*(*m,n*) model equations with variable coefficients

Gülistan İSKENDEROĞLU[1,*], Doğan KAYA[1]

[1]*Department of Mathematics, Istanbul Ticaret University, Istanbul, 34840, Türkiye*

ARTICLE INFO

Article history
Received: 06 October 2022
Revised: 12 March 2023
Accepted: 27 July 2023

Keywords:
Lie Groups; Conservation Laws; Fractional Calculus; Nonlinear Differential Equations

ABSTRACT

In this study, we consider model equations $K(m,n)$ with fractional Caputo time derivatives. By applying the Lie group symmetry method, we determine all symmetries for these equations and present the reduced symmetric equations for the equation $K(m,n)$ with fractional Caputo time derivatives. Furthermore, we obtain the exact solution for $K(1,1)$ with the fractional Caputo time derivative and provide graphs depicting the behavior at different orders of the fractional time derivative. Additionally, by considering the symmetries of the equation, we establish the conservation laws for $K(m,m)$ with the fractional Caputo time derivative.

Cite this article as: İskenderoğlu G, Kaya D. Lie symmetry analysis of Caputo time-fractional *K(m,n)* model equations with variable coefficients. Sigma J Eng Nat Sci 2024;42(3):885−899.

INTRODUCTION

In 1993 Rosenau and Hyman [1] introduced and studied the Korteweg-de Vries (KdV)–type $K(m,n)$ model differential equations with coefficients $A_0 = 1$ and $A_1 = 1$:

$$u_t + A_0(u^m)_x + A_1(u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leq 3. \quad (1)$$

Here and throughout the text, we will denote $u(t,x)$ as a function of two variables as u and the partial derivatives of $u(t,x)$ as $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, and so on.

The $K(m,n)$ model differential equation is a generalization of the KdV equation, describes the evolution of a weakly nonlinear and weakly dispersive wave, and has application in solid-state physics and plasma and fluid physics. These equations have the characteristic that their solitary wave solutions, for certain values of m and n, have a finite core region where they exist and vanish outside of it. For $n = m$ there are solitary waves with a speed λ of propagation of the waves., the so-called compactons in a form [2]:

$$u(t,x) = \left(\frac{2n\lambda}{n+1}\cos\left[\frac{n-1}{4n}(x-\lambda t)\right]\right)^{\frac{2}{(n-1)}}, \quad for \quad |x-\lambda t| \leq \frac{2n\pi}{n-1}, n \geq 1. \quad (2)$$

In particular, Rosenau and Hyman found that solitary waves can compactify under the influence of nonlinear phenomena [1].

Afterward, Charalambous et. al. in [3] studied the symmetry properties of equation (1) with constant $A_0 = \pm 1$ and $A_1 = A_1(t)$ arbitrary nonvanishing function of the variable n,

***Corresponding author.**
*E-mail address: giskenderoglu@ticaret.edu.tr

This paper was recommended for publication in revised form by Editor in-Chief Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey

arbitrary constants m,n ($n \neq 0$) and investigated symmetries of a boundary value problem for $K(m,n)$ model differential equation with an arbitrary constant k of a characterises $k \neq 0,1$ and $k \geq ½$ mod, for $G^{\sim}$ is an invariance group of the equation (1) as below:

$$\begin{cases} u_t + A_0(u^m)_x + t^k(u^n)_{xxx} = 0, & x>0 \;\; t>0, \\ u(0,x) = 0, & x>0, \\ u(t,0) = q(t), \quad u_x(t,0) = u_{xx}(t,0) = 0 & t>0. \end{cases} \tag{3}$$

Kudryashov and Prilipko in [4] introduced a generalized form of (1) as a family of nonlinear partial differential equations (PDE) of order $(2N + 1)$ and depends on $(N + 2)$ parameters denoted by $a_0,..., a_N$, m:

$$u_t + \sum_{k=0}^{N} a_k \frac{\partial^{2k+1} u^m}{\partial x^{2k+1}} = 0, \; N \geq 1, \;\; m \neq 1, \;\; a_k \neq 0. \tag{4}$$

By taking into consideration the traveling wave ansatz they obtained the periodic wave solution and presented exact solutions for $K(1,1)$, $K(2,2)$, $K(3,3)$, and $K(4,4)$ differential equation.

In [5] Bruzon and Gandarias carried out a classification of nonlocal symmetries, which are known potential symmetries of $K(m,n)$ model differential equation:

$$(u_t)^l + au^m u_x + b(u^n)_{xxx} = 0, \quad l,a,b,m,n \neq 0 \; and \; m \neq -1, \tag{5}$$

with generalized evolution term u^m. Here u^m is of considerable interest in mathematical physics.

In addition to the $K(m,n)$ with classical derivatives, there was studied time-fractional $K(m,n)$ differential equation. And the reason is fractional differential equations have gained great significance in physics and mathematics. The theory of fractional derivatives has appeared in many fields of science and likewise has become a meaningful and adequate tool for mathematical modelling. The usefulness of this type of equation lies in the non-local property of fractional derivatives. Proper mathematical modelling of a physical phenomenon depends on the moment and the previous history of time in the form of the memory effect. And this physical phenomenon can be constructed using fractional differentiation. Thus, investigating the solutions of fractional differential equations is essential to understanding the nonlinear process. By understanding the behaviour of these types of equations, researchers can gain insight into the complex dynamics of these systems and develop better models and simulations for real-world applications. The investigation of this type of equation has gained significance and recognition over the last decades, especially because of the huge number of results tested in various seemingly advanced fields of science, applied mathematics, and engineering. These studies include works in physics, biology, dielectric polarization, electromagnetic wave, electrochemistry, numerical finance, and fluid mechanics [6-9].

The Riemann-Liouville fractional derivative [9] is one of the fractional derivative operators in fractional calculus that study fractional integrals and derivatives such as Riemann-Liouville, Caputo, Atangana-Baleanu fractional derivatives [6], and others [8-10].

Wang and Hashemi [11] studied the time-fractional $K(m,n)$ equation:

$$\frac{{}^{RL}\partial^\alpha u}{\partial t^\alpha} + A_0(u^m)_x + A_1(u^n)_{xxx} = 0, \; m,n > 0, \tag{6}$$

with A_0, A_1 constant coefficients and $\frac{{}^{RL}\partial^\alpha u}{\partial t^\alpha}$:

$$\frac{{}^{RL}\partial^\alpha f(t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^\alpha} d\tau, & if \;\; 0 < \alpha < 1, \\ \frac{d}{dt} f(t), & if \;\; \alpha = 1. \end{cases} \tag{7}$$

Riemann-Liouville fractional derivative [9,10] for an arbitrary $f(t)$ function with $\Gamma(\alpha)$ a Gamma function. Here they find two symmetry operators by applying the Lie symmetry method.

As mentioned earlier in the fractional calculus, there are other fractional derivatives like Caputo fractional derivative [9,10] which is defined as:

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha} d\tau, & if \;\; 0 < \alpha < 1, \\ \frac{d}{dt} f(t), & if \;\; \alpha = 1. \end{cases} \tag{8}$$

Here again, $f(t)$ is an arbitrary function, $\Gamma(\alpha)$ is a Gamma function, and $f'(\tau)$ is a derivative of $f(t)$ with respect to t at the point $\tau = t$.

In general, the Caputo and the Riemann-Liouville fractional derivatives do not coincide [10]. And there are two main differences between the two definitions. The first distinction is about the initial conditions required in the process of solving fractional-order differential equations. The initial value for the fractional-order derivative equations with the Caputo derivative is the same as the initial value for integer-order partial derivative equations. While for the Riemann-Liouville derivative, the initial values are fractional order derivatives of the given variables. But the initial condition with an integer order in the physical interpretation is easily solved in real problems. The second distinction is the requirement for $f(t)$, Caputo derivative requires that $f(t)$ is a continuous and k-order differentiable in the interval $(0,t)$ function, but Riemann-Liouville derivative requires just only the continuousness of the function $f(t)$. And in addition to that, the Caputo derivative of a constant function is zero as in the case of the integer-order derivative [10].

In this work, we study symmetries and conservation laws of nonlinear $K(m,n)$ model equation (1) with Caputo time-fractional derivative and variable coefficient:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + A_0(t)(u^m)_x + A_1(u^n)_{xxx} = 0,\ m,n > 0, \tag{9}$$

with boundary and initial conditions

$$\begin{cases} u(0,x) = 0, & x > 0, \\ u(t,0) = q(t), u_x(t,0) = u_{xx}(t,0) = 0, & t > 0. \end{cases} \tag{10}$$

Here $A_0(t)$ is a non-zero and differentiable function, A_1 is a constant, and $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the Caputo fractional derivative. The study of fractional differential equations, such as the time-fractional $K(m,n)$ differential equation, has theoretical significance. The nonlocal property of fractional derivatives is useful for the mathematical modelling of physical phenomena that depend not only on the current moment of time but also on the previous history of time. Thus, solutions of equations of this type can be used to understand and analyse nonlinear processes occurring in various fields of science and technology. In this way, we give a classification of nonlocal symmetries as the infinitesimal operators, such that each of them gives us an invariant equation to our equation. The classification of nonlocal symmetries of the $K(m,n)$ differential equation can provide insight into the underlying physical properties of the wave phenomena described by these equations. This understanding can be useful for the development of new models and techniques for solving them. In this work, we present some solutions with graphs according to small changes in the values of α. So, we show that the solution function grows respectively faster or slower concerning x when α decreases or increases.

In addition, we construct conservation laws for the Caputo differential equation with fractional time $K(m,m)$ for special cases $A_0(t)$ This construction of conservation laws in a model differential equation of the KdV-type $K(m,n)$ has both theoretical and managerial significance. As a theoretical consequence, the following can be noted here: conservation laws provide information about the invariance properties of an equation, which gives insight into its fundamental behaviour and properties. The study of conservation laws can lead to the discovery of new solutions such as solitons and compactons, as well as new physical phenomena that can provide a better understanding of the underlying physical processes. And the managerial consequences, in turn, can be as follows: $K(m,n)$ equation has applications in various fields such as solid-state physics, plasma physics, and fluid physics, where conservation laws can provide valuable information for designing and optimizing experiments and systems. Solutions to the $K(m,n)$ equations can be used to model and analyze physical phenomena such as waves and fluid flows, and conservation laws can help predict and control their behavior. Moreover, the study of conservation laws can also lead to the development of numerical methods and algorithms for solving the equation, which can be used for computer simulations.

The present work has organized as follows. In Section 2, we give a brief of literature review. In Section 3, we present brief information on the Lie symmetry method and then apply it to equation (9) with boundary and initial conditions with different values of m, and n. In Section 4, we construct conservation laws for $K(m,n)$ equations in case $m = n$. And we give a conclusion in Section 5.

Literature Review

The $K(m,n)$ model differential equation is a partial differential equation that is used to model a wide range of physical phenomena, including fluid dynamics, heat transfer, and chemical reactions [1, 2]. Based on the literature research presented in the introduction, we can draw the following summary of the literature review of studies of differential equations of the KdV type model $K(m,n)$. The $K(m,n)$ model differential equation is a generalization of the KdV equation that describes the evolution of a weakly nonlinear and weakly dispersive wave and has applications in solid-state physics, plasma physics, fluid physics, and other areas [1, 2].

Rosenau and Hyman in [1] introduced and studied the $K(m,n)$model differential equations in 1993. They found that solitary waves, called compactons, can compactify under the influence of nonlinear phenomena when $m = n$. Charalambous and others in [3] investigated the symmetry properties of the $K(m,n)$ model differential equation and investigated symmetries of a boundary value problem for the $K(m,n)$ model differential equation.

Kudryashov and Prilipko in [4] introduced a generalised form of the $K(m,n)$ model differential equation as a family of nonlinear PDE and obtained the periodic wave solution by taking into consideration the traveling wave ansatz. They presented exact solutions for some types of the $K(m,n)$ differential equations.

Bruzon and Gandarias carried out a classification of nonlocal symmetries, which are known potential symmetries of the $K(m,n)$ model differential equation with generalised evolution term u^m [5].

Furthermore, there have been studies on time-fractional $K(m,n)$ differential equations, as fractional differential equations have gained significance in physics and mathematics [6]. The usefulness of this type of equation lies in the nonlocal property of fractional derivatives. Investigating the solutions of fractional differential equations is essential to understanding the nonlinear process, and it has gained significance and recognition over the last decades, especially because of the large number of results tested in various fields of science, applied mathematics, and engineering. These studies include works in physics, biology, dielectric polarization, electromagnetic wave, electrochemistry, numerical finance, and fluid mechanics (see [6, 8-10] and references therein). The Riemann-Liouville fractional derivative is one of the fractional derivative operators in fractional calculus that study fractional integrals and derivatives such as Riemann-Liouville, Caputo, Atangana-Baleanu fractional derivatives, and the others.

LIE SYMMETRY ANALYSIS OF THE TIME-FRACTIONAL *K*(*M,N*) MODEL EQUATIONS

Lie Symmetry Analysis

In this section, we give short information about the Lie symmetry analysis of fractional PDE. For the first let us talk about the conception of Lie group theory [12,13]. A Lie group is a group that is also a manifold. Continuous symmetry transformations of Lie groups can be described as transformations of independent and dependent variables that depend on some, possibly infinitesimal, parameter ε [12]. So, this infinitesimal transformation of a Riemannian space or pseudo-Riemannian space V^n is given with respect to the coordinates in the below form:

$$\overline{\boldsymbol{X}}_i = x_i + \varepsilon \Xi_i(x_1, x_2, \dots, x_n), \tag{11}$$

where $i = 1, \dots, n$, x_i are the coordinates of a certain point in V^n and $\overline{X}_i$ are the coordinates of its image under the infinitesimal transformation, ε is an infinitesimal parameter not depending on x_i, and Ξ_i is a displacement vector depending on x_i, that defines the generators and which is a basis for the tangent space of the identity element in the group [12, 13].

If a given object F of the space V^n depends on $x \in V^n$ but also the infinitesimal parameter ε, then the principal part of the object F is $F_1(x) + \varepsilon F(x)$ in the Taylor expansion of series with respect to the (small) infinitesimal parameter ε

$$F(x, \varepsilon) = F_0(x) + \varepsilon F_1(x) + \varepsilon^2 F_2(x) + \cdots \tag{12}$$

For our goals, the curves obtained by the infinitesimal transformation of surfaces satisfy the isoperimetric rotation equations, if we have omitted the terms containing the highest powers of the infinitesimal parameter ε, like ε^2, ε^3,….Thus, in the limit of $\varepsilon \to 0$, we can ignore terms of order ε^2 or higher [12].

Now let's consider a time-fractional PDE

$$F\left(t, x, u, \frac{\partial^\alpha u}{\partial t^\alpha}, u_x, u_{xx}, u_{xxx}\right) = 0. \tag{13}$$

The construction of the symmetry group is determined with infinitesimal transformations acting on a space of two independent variables (x,t) and dependent variable u in the form

$$\begin{aligned} \bar{t} &= t + \varepsilon \tau(t, x, u) + O(\varepsilon^2), \\ \bar{x} &= x + \varepsilon \xi(t, x, u) + O(\varepsilon^2), \\ \bar{u} &= u + \varepsilon \eta(t, x, u) + O(\varepsilon^2), \end{aligned} \tag{14}$$

where $\varepsilon > 0$ is an infinitesimal group parameter [12]. And an infinitesimal generator is

$$X = \xi(t, x, u)\frac{\partial}{\partial x} + \tau(t, x, u)\frac{\partial}{\partial t} + \eta(t, x, u)\frac{\partial}{\partial u}, \tag{15}$$

which is a generator of the infinitesimal operators for the differential equation (13) according to $\xi(t, x, u)$, $\tau(t, x, u)$ and $\eta(t, x, u)$.

In general, a definition of the invariance of the differential equations that the reader can find in [12,13] can be given:

Definition 1: *The solution $u = v(t, x)$ of the equation (13) is an invariant solution, resulting under the symmetry (14) with infinitesimal generator (15) if and only if*

- *$u = v(t, x)$ satisfies the equation (13),*
- *$u = v(t, x)$ is an invariance surface under X.*

It states that the solution for the equation (13) $u = v(t, x)$ is an invariant solution, under the infinitesimal generator X with transformation (14) if and only if $u = v(t, x)$ meets below two conditions:

- $X(u - v(t, x)) = 0$ for $u = v(t, x)$, that provides us with

$$\xi(t, x, v(t, x))\frac{\partial v(t, x)}{\partial x} + \tau(t, x, v(t, x))\frac{\partial v(t, x)}{\partial t} = \eta(t, x, v(t, x)), \tag{16}$$

- $F(x, t, v(t, x), D_t^\alpha v(t, x), \partial_x v(t, x), \partial_x^2 v(t, x), \dots,$ $\partial_x^s v(t, x)) = 0$, for $u = v(t, x)$.

Here $D_t^\alpha v(t, x) = \frac{\partial^\alpha v(t,x)}{\partial t^\alpha}$ and $\partial_x^s v(t, x) = \frac{\partial^s v(t,x)}{\partial x^i}$, for $i = 1, \dots, s$.

We would also like to note that according to the infinitesimal transformation (14) a fractional prolongation $pr^{(\alpha,n)}X$ mentioned in [14], of the equation (13) with $E = F(x, t, u, D_t^\alpha u, \partial_x u, \partial_x^2 u, \dots, \partial_x^s u) = 0$ has a form

$$pr^{(\alpha,s)}X(E)|_{E=0} = 0, \tag{17}$$

which we define as invariance criteria and use in the following form

$$pr^{(\alpha,s)}X = X + \eta_\alpha^t \partial_{D_t^\alpha u} + \eta_1^x \partial_{u_x} + \eta_2^x \partial_{\partial_x^2 u} + \dots + \eta_s^x \partial_{\partial_x^s u}. \tag{18}$$

Here the explicit formulas for the extended infinitesimals η_i^x, $i = 1, \dots, s$ are as

$$\eta_1^x = D_x\eta - u_xD_x\xi - u_tD_x\tau,$$
$$\eta_2^x = D_x\eta_1^x - u_{xx}D_x\xi - u_{xt}D_x\tau,$$
$$\eta_3^x = D_x\eta_2^x - u_{xxx}D_x\xi - u_{xxt}D_x\tau, \quad (19)$$
$$\vdots$$
$$\eta_s^x = D_x\eta_{s-1}^x - \partial_x^s uD_x\xi - \partial_x^{s-1}u_tD_x\tau,$$

where D_x is a total derivative in a form

$$D_x = \partial_x + u_x\partial_u + u_{xt}\partial_{u_t} + u_{xx}\partial_{u_x} + \ldots$$

And η_α^t has the following form, which was given by Gazizov, Kasatkin, and. Lukashchuk in [14],

$$\eta_\alpha^t = D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + \tau D_t^\alpha(u_t) - D_t^\alpha(\tau u_t), \quad (20)$$

or

$$\eta_\alpha^t = \frac{\partial^\alpha\eta}{\partial t^\alpha} + (\eta_u - \alpha(\tau_t + u_t\tau_u))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha\eta_u}{\partial t^\alpha} + \mu$$
$$-\sum_{n=1}^{\infty}\binom{\alpha}{n}D_t^n(\xi)\,{}_0I_t^{n-\alpha}(u_x) + \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\frac{\partial^n\eta_u}{\partial t^n} - \binom{\alpha}{n+1}D_t^{n+1}\tau\right]I_t^{n-\alpha}u \quad (21)$$
$$+\frac{(\xi u_x)(0)}{\Gamma(\alpha+1)}t^{-\alpha},$$

here

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k}\binom{\alpha}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}(-u)^r}{\Gamma(n+1-\alpha)}\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}, \quad (22)$$

and

$${}_0I_t^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t\frac{f(\tau)}{(t-\tau)^{1-\beta}}d\tau, \qquad with \quad 0<\beta<1, \quad (23)$$

is a left-sided operator of fractional integration of order β.

The technical steps of the Lie symmetry analysis to define Lie symmetries are [13]:

Step 1. Construct the (a, s) th prolongation of the vector field X in (18). Here α means, that the equation (13) has a fractional derivative of α with $0 < \alpha < 1$ order and s is the highest order of derivative, that the equation (13) has.

Step 2. Apply the prolonged operator $pr^{(a,s)}$ to equation (13). Here condition $E = 0$ expresses that $pr^{(a,s)}$ vanishes on the solution of the equation (13). So, this condition assures that X is an infinitesimal symmetry generator of the group transformation (14). Hence, $u(t, x)$ is a solution of (13) whenever $\bar{u}(t,\bar{x})$ is one.

Step 3. After expanding the equation in step 2 we get:

$$F\left(t,x,u,\frac{\partial^\alpha u}{\partial t^\alpha},u_x,u_{xx},u_{xxx}\right) = \bar{F}\left(\bar{t},\bar{x},\bar{u},\frac{\partial^\alpha\bar{u}}{\partial\bar{t}^\alpha},\bar{u}_{\bar{x}},\bar{u}_{\bar{x}\bar{x}},\bar{u}_{\bar{x}\bar{x}\bar{x}}\right) +$$
$$\varepsilon Q(\tau,\xi,\eta,t,x,u,\frac{\partial^\alpha u}{\partial t^\alpha},u_x,u_{xx},u_{xxx}) + O(\varepsilon^2), \quad (24)$$

where $Q(\tau,\xi,\eta,t,x,u,\frac{\partial^\alpha u}{\partial t^\alpha},u_x,u_{xx},u_{xxx})$ is some expression of $\tau,\xi,\eta,t,x,u,\frac{\partial^\alpha u}{\partial t^\alpha},u_x,u_{xx},u_{xxx}$. And now, to hold the invariance criteria, we equate the multiplier of ε to zero and then we equate the coefficients of all functionally independent expressions in the remaining derivatives to zero. This will lead to a big number of elementary partial differential equations of the coefficient's functions of the infinitesimal generator (15). By solving them we obtain $\xi(t, x, u)$, $\tau(t, x, u)$ and $\eta(t, x, u)$ and by that we gain the symmetries of the equation (13).

The Lie symmetry method for the PDE gives a transformation that leaves invariant the solution manifold of the equation. In practice, the method reduces the PDE to equations with a fewer number of independent variables for the PDE with the integer derivatives and fractional derivatives. The method provides us with many different types of solutions for the PDE, such as power series solutions, traveling wave and soliton solutions, and so on [15-19].

Application the Symmetry Analysis to Caputo Time-Fractional *K*(*m*,*n*) Equations and Theoretical Implications

According to the infinitesimal transformation (14), we can construct a prolongation formula pr^3X for our equation (9) in a form

$$pr^3X = X + \eta_\alpha^t\partial_{\partial_t^\alpha u} + \eta_2^x\partial_{u_{xx}} + \eta_3^x\partial_{u_{xxx}}, \quad (25)$$

here $\partial_{\partial_t^\alpha u} = \frac{\partial}{\partial\frac{\partial^\alpha u}{\partial t^\alpha}}$.

By following the mentioned technical steps and plugging the infinitesimal transformations (14) into equation (9) we get

$$\frac{\partial^\alpha\bar{u}}{\partial\bar{t}^\alpha} + A_0(\bar{t})(\bar{u}^m)_{\bar{x}} + A_1(\bar{u}^n)_{\bar{x}\bar{x}\bar{x}} = \frac{\partial^\alpha u}{\partial t^\alpha} + A_0(t)(u^m)_x$$
$$+ A_1(u^n)_{xxx} + \varepsilon(\eta_t^\alpha + \tau A'_0(t)u^{m-1}u_x + (m-1)\eta A_0(t)u^{m-2}u_x$$
$$+ A_0(t)\eta_xu^{m-1} + A_1(n(n-1)(n-2)(3\eta_xu^{n-3} + (n-3)\eta u^{n-4}u_x) \quad (26)$$
$$+ 3n(n-1)(\eta_xu^{n-2}u_{xx} + (n-2)\eta u^{n-3}u_xu_{xx} + \eta_{xx}u^{n-2}u_x)$$
$$+ n(\eta_{xxx}u^{n-1} + (n-1)\eta u^{n-2}u_{xxx}))).$$

By considering definition 1, for the invariance of the equation, the factor of ε must be 0, e.i.

$$\eta_t^\alpha + \tau A'_0(t)u^{m-1}u_x + (m-1)\eta A_0(t)u^{m-2}u_x$$
$$+ A_0(t)\eta_xu^{m-1} + A_1(n(n-1)(n-2)(3\eta_xu^{n-3}$$
$$+ (n-3)\eta u^{n-4}u_x) + 3n(n-1)(\eta_xu^{n-2}u_{xx} \quad (27)$$
$$+ (n-2)\eta u^{n-3}u_xu_{xx} + \eta_{xx}u^{n-2}u_x)$$
$$+ n(\eta_{xxx}u^{n-1} + (n-1)\eta u^{n-2}u_{xxx})) = 0.$$

In fact, the above proposition is invariance criteria. By putting the infinitesimals (19) and (21) into equation (27) and equating the multipliers of each derivative of u to zero, we get bellow system of equations:

$$\tau_u = \tau_x = \xi_u = \xi_t = \eta_{uu} = 0, \tag{28}$$

$$\begin{aligned}
&a)\ u^{m-2}(A_0(t)(m^2-m)\eta + 3A_1(m^2-m)\eta_{xx}) \\
&\quad + u^{m-1}(m\tau A'_0(t) + m\alpha\tau_t A_0(t) - mA_0(t)\xi_x \\
&\quad + 3mA_1\eta_{xxu} - mA_1\xi_{xxx}) = 0, \\
&b)\ u^{m-2}(3A_1(m^2-m)\eta_x) + u^{m-1}(3mA_1\eta_{xu} - 3mA_1\xi_{xx}) = 0, \\
&c)\ u^{m-2}A_1(m^2-m)\eta + u^{n-1}(-nA_1\eta_u + n\alpha A_1\tau_t) \\
&\quad + u^{m-1}(mA_1\eta_u - 3mA_1\xi_x) = 0, \\
&d)\ u^{m-3}(A_1(3m^3-9m^2+6m)\eta_x) + u^{m-2}(A_1(6m^2-6m)\eta_{xu} \\
&\quad + A_1(-3m^2+3m)\xi_{xx}) = 0, \\
&e)\ u^{m-4}(A_1(m^4-6m^3+11m^2-6m)\eta) \\
&\quad + u^{n-3}(-n(n-1)(n-2)A_1\eta_u + n(n-1)(n-2)\alpha A_1\tau_t) \\
&\quad + u^{m-3}(3m(m-1)(m-2)A_1(\eta_u - \xi_x)) = 0, \\
&f)\ u^{m-3}(A_1(3m^3-9m^2+6m)\eta) + u^{n-2}(-3n(n-1)A_1\eta_u \\
&\quad + 3n(n-1)\alpha A_1\tau_t) + u^{m-2}(6m(m-1)A_1\eta_u \\
&\quad - 9m(m-1)A_1\xi_x) = 0, \\
&g)\ \eta_{ut} - \frac{1-\alpha}{2}\tau_{tt} = 0.
\end{aligned}$$

Let $m = n \neq 1$ then from equation (c) we have $\eta = \frac{3\xi_x - \alpha\tau_t}{m-1}u$ and from equation (b) we have $\xi_{xx} = 0$, which gives us $\xi = c_1x + c_2$, where c_1 and c_2 are constants. Here and in bellow each c_i, $i = 1, \ldots, 4$ defines an infinitesimal operator in the form (15). Further, we see that, $\eta = \frac{3\xi_x - \alpha\tau_t}{m-1}u$ and $\xi = c_1x + c_2$ satisfy the equations (d), (e), and (f). In another hand, the equation (a) gives us

$$\eta = \frac{A_0(t)\xi_x - \alpha A_0(t)\tau_t - A'_0(t)\tau}{A_0(t)(m-1)}u. \tag{29}$$

Thus, for non-zero and differentiable function $A_0(t)$ we have $\xi = c_2$, $\eta = 0$, and $\tau = 0$, since $\tau A'_0(t) = -2c_1 A_0(t)$ leads to $c_1 = \tau = 0$, and so $\eta = 0$. That means the equation (9) with any non-zero function $A_0(t)$ has only one symmetry according to the infinitesimal operator $X_1 = \frac{\partial}{\partial x}$.

Further, we will consider some special cases of the function $A_0(t)$ that are frequently encountered as coefficients in differential equations. For instance, while $A_0(t) = 1$ we have $c_1 = 0$ and $\tau_{tt} = 0$ with $\alpha \neq \frac{1-m}{1+m}$ from equation (g), which gives us $\tau = c_3t + c_4$, $\eta = -\frac{c_3\alpha}{m-1}u$, and $\xi = c_2$. And relevant operators are in the form:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = t\frac{\partial}{\partial t} + \frac{\alpha}{1-m}u\frac{\partial}{\partial u}. \tag{30}$$

In case $A_0(t) = t^\lambda$, $\lambda \in \mathbb{R}$, $\lambda \neq 0$ we get $\tau = -\frac{2c_1}{\lambda}t$, $\eta = \frac{c_1(3\lambda + 2\alpha)}{\lambda(m-1)}u$, and $\xi = c_1x + c_2$, with infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - \frac{2}{\lambda}t\frac{\partial}{\partial t} + \frac{2\alpha + 3\lambda}{\lambda(m-1)}u\frac{\partial}{\partial u}. \tag{31}$$

And the last case is $A_0(t) = e^t$. In this instance we have $\tau = -2c_1$, $\eta = \frac{3c_1}{m-1}u$, and $\xi = c_1x + c_2$. Regarding c_1 and c_2 we have

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t} + \frac{3}{m-1}u\frac{\partial}{\partial u}. \tag{32}$$

Table 1. The infinitesimal operators.

$A_0(t)$	$m \neq n$	$m = n$	$m = n = 1$
Arbitrary	$X = \frac{\partial}{\partial x}$	$X = \frac{\partial}{\partial x}$	$X = \frac{\partial}{\partial x}$
1	$X_1 = \frac{\partial}{\partial x}$,	$X_1 = \frac{\partial}{\partial x}$,	$X_1 = \frac{\partial}{\partial x}$,
	$X_2 = \frac{\partial}{\partial t}$,	$X_2 = \frac{\partial}{\partial t}$, $X_3 = t\frac{\partial}{\partial t} + \frac{\alpha}{1-m}u\frac{\partial}{\partial u}$	$X_2 = \frac{\partial}{\partial t}$, $X_3 = u\frac{\partial}{\partial u}$
t^λ	$X = \frac{\partial}{\partial x}$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} - \frac{2}{\lambda}t\frac{\partial}{\partial t} + \frac{2\alpha + 3\lambda}{\lambda(m-1)}u\frac{\partial}{\partial u}$	for $\lambda = -\frac{2}{3}\alpha$ $X_1 = \frac{x}{3}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t}$, $X_2 = u\frac{\partial}{\partial u}$
e^t	$X = \frac{\partial}{\partial x}$	$X_1 = \frac{\partial}{\partial x}$, $X_2 = x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial t} + \frac{3}{m-1}u\frac{\partial}{\partial u}$	$X = \frac{\partial}{\partial x}$

The similar calculations for $m \neq n$, and $m = n = 1$ according to each $A_0(t)$ give us the infinitesimal operators as in the next table.

According to the above table, we see that for all m and n when the coefficient $A_0(t)$ is a non-zero differentiable function, also for $m \neq n$ with $A_0(t) = t^\lambda$, and $A_0(t) = e^t$ we have only one operator that gives us a solution of the equation (9) in a form:

$$u(x,t) = ct^{\alpha-1}. \tag{33}$$

For other cases of the coefficient $A_0(t)$, we have the next situations.

Case 1: m ≠ n

For $A_0(t) = 1$, we have extra two operators X_1 and X_2. X_1 gives us trivial solution $u(x, t) = 0$, and the operator $X_1 + kX_2$, $k \in \mathbb{R}$ gives a characteristic system:

$$\frac{dx}{1} = \frac{dt}{k}, \tag{34}$$

which provides us with a transformation $u(t, x) = \phi(p)$, $p = x + kt$, with a differentiable function $\phi(p)$. So, our equation (9) takes the next form:

$$\begin{aligned} &\frac{1}{k\Gamma(1-\alpha)}\int_x^p \frac{\phi'(\tau)}{(p-\tau)^\alpha}d\tau + m(\phi(p))^{m-1}\phi'(p) \\ &+ A_1(n(\phi(p))^{n-1}\phi'''(p) + 3n(n-1)(\phi(p))^{n-2}\phi'(p)\phi''(p) \\ &+ n(n-1)(n-2)(\phi(p))^{n-3}(\phi'(p))^3) = 0. \end{aligned} \tag{35}$$

Case 2: m = n

For $A_0(t) = 1$, we have a symmetry operator in a general form:

$$X = c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial t} + c_3\left(t\frac{\partial}{\partial t} + \frac{\alpha}{1-m}u\frac{\partial}{\partial u}\right). \tag{36}$$

Let us consider this case for the equation (9) with boundary and initial conditions (10). Since the invariance of the boundary value problem means invariance of the equation with boundary and initial conditions, we have $c_1 = c_2 = 0$. Which gives the invariant criteria for the problem (9)-(10) in the next operator form:

$$X = t\frac{\partial}{\partial t} + \frac{\alpha}{1-m}u\frac{\partial}{\partial u}. \tag{37}$$

This operator provides us the transformation $u = t^{\frac{\alpha}{1-m}}\varphi(x)$, here $\varphi(x)$ a differentiable function. Thus, we can obtain below ordinary differential equation (ODE):

$$\frac{\Gamma\left(1+\frac{\alpha}{1-m}\right)}{\Gamma\left(1-\alpha+\frac{\alpha}{1-m}\right)}\varphi(x) + (\varphi(x)^m)' + A_1(\varphi(x)^m)''' = 0. \tag{38}$$

and by using (16) we get $tq'(t) = \frac{\alpha}{1-m}q(t)$, or $q(t) = k_1 t^{\frac{\alpha}{1-m}}$, where k_1 is an arbitrary constant. By considering the given transformation and the boundary conditions for the transformation we get below Cauchy problem:

$$\begin{cases} \dfrac{\Gamma(1+\frac{\alpha}{1-m})}{\Gamma(1-\alpha+\frac{\alpha}{1-m})}\varphi(x) + (\varphi(x)^m)' + A_1(\varphi(x)^m)''' = 0, \quad x > 0, \\ \varphi(0) = k_1. \end{cases} \tag{39}$$

For $A_0(t) = t^\lambda$, there is a general infinitesimal operator in a form:

$$X = c_4\frac{\partial}{\partial x} + c_5\left(x\frac{\partial}{\partial x} - \frac{2}{\lambda}t\frac{\partial}{\partial t} + \frac{2\alpha+3\lambda}{\lambda(m-1)}u\frac{\partial}{\partial u}\right), \tag{40}$$

where the invariance of the boundary and initial conditions gives us $c_4 = 0$ and

$$x\frac{\partial}{\partial x}q(t) - \frac{2}{\lambda}t\frac{\partial}{\partial t}q(t) = \frac{2\alpha+3\lambda}{\lambda(m-1)}q(t). \tag{41}$$

Thus, we have $q(t) = t^{\frac{2\alpha+3\lambda}{2(1-m)}}k_2$, $k_2 \in \mathbb{R}$. It means that (9)-(10) invariant under $X = x\frac{\partial}{\partial x} - \frac{2}{\lambda}t\frac{\partial}{\partial t} + \frac{2\alpha+3\lambda}{\lambda(m-1)}u\frac{\partial}{\partial u}$ with $u(t,x) = x^{\frac{2\alpha+3\lambda}{\lambda(m-1)}}\psi(z)$, $z = tx^{\frac{2}{\alpha}}$ transformation with the differentiable function $\Psi(z)$, and have below Cauchy problem:

$$\begin{cases} \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^z \frac{\psi'(\tau)}{(p-\tau)^\alpha}d\tau + B_1(\psi(z))^m z^\lambda + B_2(\psi(z))^m \\ +B_3(\psi(z))^{m-3}(\psi'(z))^3 z^3 + B_4(\psi(z))^{m-2}(\psi'(z))^2 z^2 \\ +B_5(\psi(z))^{m-2}\psi'(z)\psi''(z)z^3 + B_6(\psi(z))^{m-1}\psi'(z)z^{1+\lambda} \\ +B_7(\psi(z))^{m-1}\psi'(z)z + B_8(\psi(z))^{m-3}\psi''(z)z \\ +B_9(\psi(z))^{m-3}\psi'''(z)z^2 = 0, \qquad x > 0, \\ \psi(0) = 0. \end{cases} \tag{42}$$

Here B_i, $i = 1,2...,9$ are coefficients in the following forms:

$$\begin{aligned} &B_1 = \frac{m(2\alpha+3\lambda)}{\lambda(m-1)}, \ B_2 = \frac{A_1 n(\lambda+2\lambda m+2\alpha m)(2\lambda+2m+2\alpha m)}{\lambda^3(m-1)^3}, \\ &B_3 = \frac{8A_1 m(m-1)(m-2)}{\lambda^3}, B_4 = \frac{12A_1 m(\lambda+2\lambda m+2m-2+2m\alpha)}{\lambda^3}, \\ &B_5 = \frac{24A_1 m(m-1)}{\lambda^3}, \ \ B_6 = \frac{2m}{\lambda}, \\ &B_7 = \frac{2A_1 m\lambda^2(2+m(14+11m)) + 12A_1 m\lambda(1+2m)(m-1+2m\alpha)}{\lambda^3(m-1)^3} + \\ &\frac{8A_1 m(1+m(m-2+3(m-1)\alpha+3m\alpha^2))}{\lambda^3(m-1)^3}, \\ &B_8 = \frac{6A_1 m(\lambda-2+2m+2m\lambda^2 m\alpha)}{\lambda^3(m-1)^2}, \ \ B_9 = \frac{4A_1 m}{\lambda^3(m-1)}. \end{aligned} \tag{43}$$

Case 3: m = n = 1

In this case, we have $\frac{\partial^\alpha u}{\partial t^\alpha} + A_0(t)u_x + A_1 u_{xxx} = 0$ and for $A_0(t) = 1$ there are three infinitesimal operators:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = u\frac{\partial}{\partial u}. \tag{44}$$

Here X_1 and X_3 give us next characteristic system:

$$\frac{dx}{1/k_4} = \frac{du}{u}, \tag{45}$$

which gives a transformation $u(t, x) = e^{k_4 x}\vartheta(t)$ with a differentiable function $\vartheta(t)$. Thus, our equation via the transformation is reduced to fractional ODE:

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{e^{k_4 x}\vartheta'(\tau)}{(t-\tau)^\alpha}d\tau + k_4 e^{k_4 x}\vartheta(t) + k_4^3 A_1 e^{k_4 x}\vartheta(t) = 0. \tag{46}$$

Hereby, simplifying the above expression, we have:

$$\frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\vartheta'(\tau)}{(t-\tau)^\alpha}d\tau + (k_4 + k_4^3 A_1)\vartheta(t) = 0, \tag{47}$$

with a solution in a form:

$$\vartheta(t) = t^{\alpha-1}E_{\alpha,\alpha}((k_4 + k_4^3 A_1)t^\alpha). \tag{48}$$

Here $E_{\alpha,\beta}(z) = \sum_{j=0}^{\infty}\frac{z^j}{\Gamma(\alpha j+\beta)}$ is a Mittag-Leffler function like the exponential function in fractional analysis. Therefore, for our equation in the case $m = n = 1$ and $A_0(t) = 1$, the solution is:

$$u(t,x) = e^{k_4 x}t^{\alpha-1}E_{\alpha,\alpha}\big((k_4 + k_4^3 A_1)t^\alpha\big). \tag{49}$$

Since $0 < \alpha < 1$, the solution (49) has a singularity at $t = 0$, which we can observe in the graphs below as jumps. We can plot the solution as a surface plot, but we need to use a log scale for the t-axis to visualize the singularity properly. Alternatively, we can plot the solution as a contour plot or a heatmap, which will show us how the solution changes over time and space but without the jumps. While $\alpha = 1$ singularity is disappeared, and the solution function takes the form of the exponential function.

In Figure 1, we depict the exact solution of the fractional $K(1,1)$ equation described by the Caputo fractional derivative in the space coordinates and the time, when A_1, k_2, and α take some values. The first graph (Figure 1) shows the case of $A_1 = 2$, $k_2 = 1$, and $\alpha = 0.7$. We choose such α as a starting point and make small changes to the values of α and observe the changes to the graphs. It is noticeable that the function grows respectively faster or slower with respect to x when α decreases or increases. As mentioned, since $0 < \alpha < 1$, the solution has a singularity at $t = 0$, which appears as a jump in the graph. The magnitude of the jump will depend on the value of α, with larger values of α leading to larger jumps. As t increases, the singularity smooths out, and the solution approaches the exponential function $e^{k_4 x}$ as t goes to infinity. The rate at which the singularity disappears, and the solution approaches the exponential function will also depend on the value of α.

Moreover, for $\alpha = 1$ the solution (49) will take a form $u(t,x) = e^{k_4 x}e^{(k_4 + k_4^3 A_1)t}$ or

$$u(t,x) = e^{k_4(x + (1 + k_4^2 A_1)t)}, \tag{50}$$

and can be shown as in Figure 2 As we see, the singularity here has been removed.

In Figure 3, we show some graphs of solution (49), for different values of α and particular $x = 2$ with $k_4 = 1$, $A_1 = 2$. As we can see from the graph, the solution exhibits jump near $t = 0$ for all values of α, the order of fractional derivative. As α increases, the jumps become smaller and occur at later times. When $\alpha = 1$, the singularity disappears and the solution takes the form of the exponential function $e^{k_4(x+(1+k_4^2 A_1)t)}$, which is smooth for all $t > 0$. Note that the behaviour of the solution also depends on the choice of parameters k_4 and A_1. In the given example, $k_4 = 1$ and $A_1 = 2$, which can affect the location and magnitude of the jumps.

For the case $A_0(t) = t^{-2\alpha/3}$ there is a general infinitesimal operator as below:

$$X = c_6\left(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t}\right) + c_7 u\frac{\partial}{\partial u}. \tag{51}$$

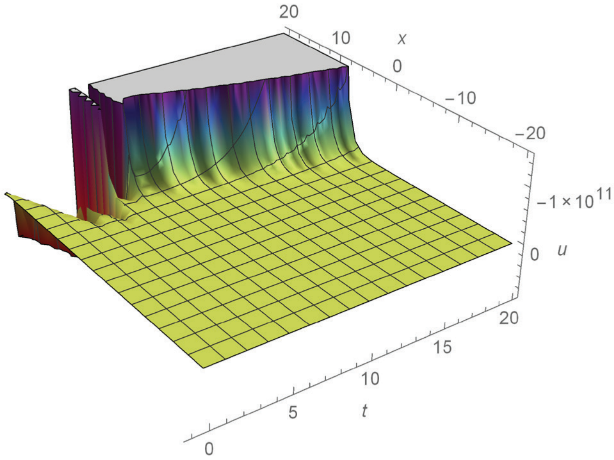
According to the invariance of the equation with the boundary and initial conditions, we get a function $q(t) = d_1 t^{d_2}$, with $d_1, d_2 \in \mathbb{R}^+$. So, our boundary and initial value problem:

$$\begin{cases} \frac{\partial^\alpha u}{\partial t^\alpha} + t^{\frac{-2\alpha}{3}}u_x + A_1 u_{xxx} = 0, & 0 < \alpha < 1, \\ u(0,x) = 0, & x > 0, \\ u(t,0) = d_1 t^{d_2}, \quad u_x(t,0) = u_{xx}(t,0) = 0 & t > 0. \end{cases} \tag{52}$$

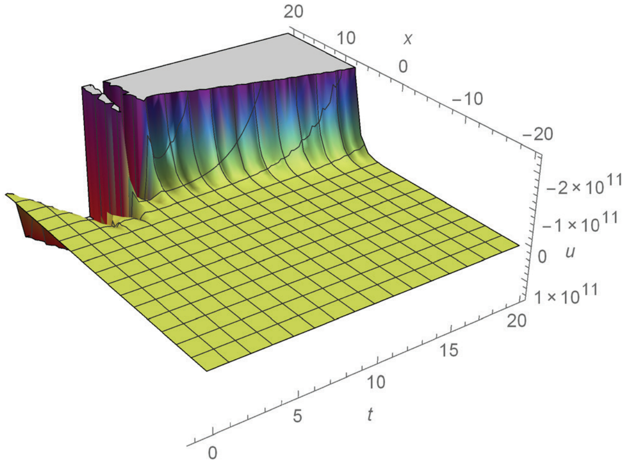
transforms to a problem with an invariant differential equation to our equation under transformation $u(t, x) = x^{3/k_5}\rho(s)$, k_5 is an arbitrary nonzero constant, $s = x^{-3/\alpha}t$ here $\rho(s) = \rho$ is a differentiable function of s:

$$\begin{cases} \frac{\partial^\alpha \rho}{\partial s^\alpha} + C_1\rho + C_2 s^{\frac{-2}{3}}\alpha\rho + C_3 s^{\frac{3-2\alpha}{3}}\rho' + C_4 s\rho' + C_5 s^2\rho'' + C_6 s^3\rho''' = 0, \\ \rho(0) = 0 \end{cases} \tag{53}$$

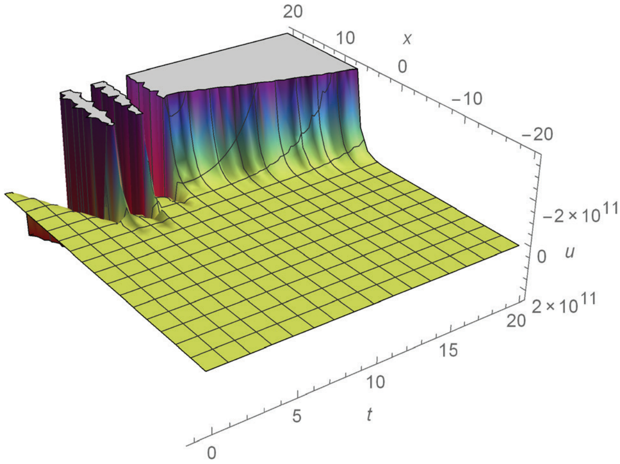
here C_i, $i = 1,2...,6$ are coefficients as below:



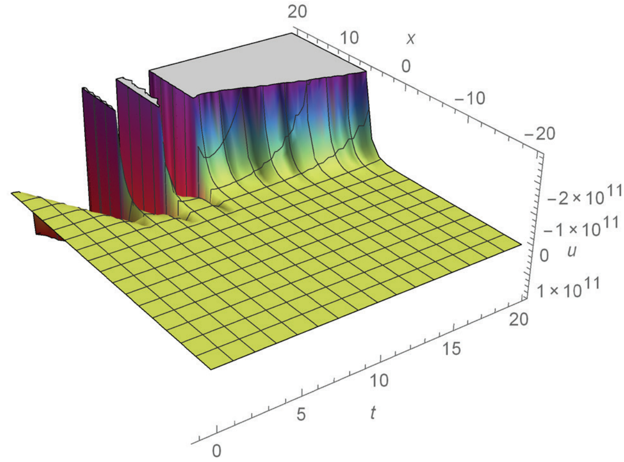
(a) The solution with $A_1 = 2$, $k_4 = 1$, and $\alpha = 0.7$.



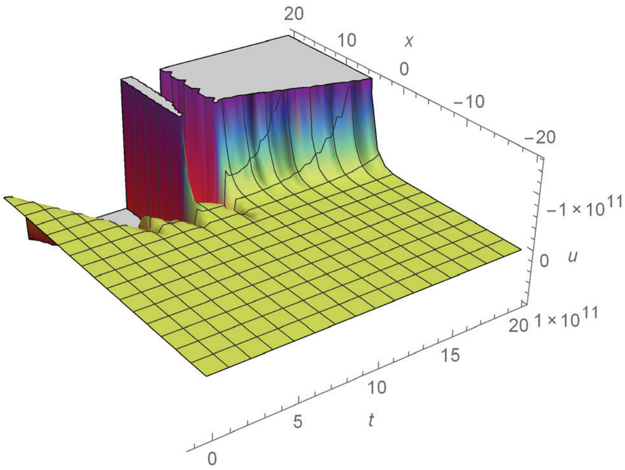
(b) The solution with $A_1 = 2$, $k_4 = 1$, and $\alpha = 0.75$.



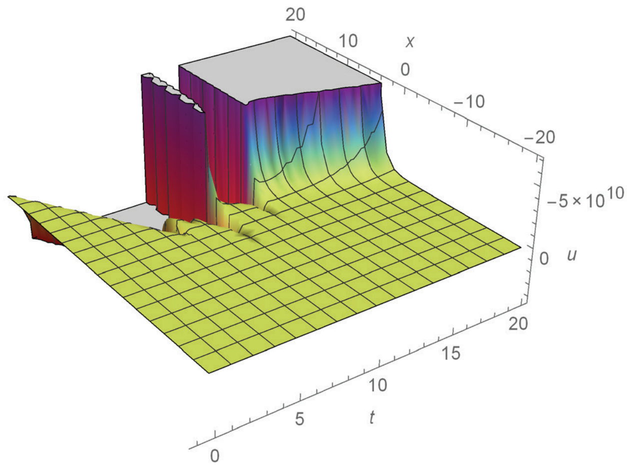
(c) The solution with $A_1 = 2$, $k_4 = 1$ and$\alpha = 0.8$.



(d) The solution with $A_1 = 2$, $k_4 = 1$, and $\alpha = 0.85$.



(e) The solution with $A_1 = 2$, $k_4 = 1$, and $\alpha = 0.9$.



(f) The solution with $A_1 = 2$, $k_4 = 1$, and $\alpha = 0.95$.

Figure 1. The solution of fractional $K(1,1)$ equation.

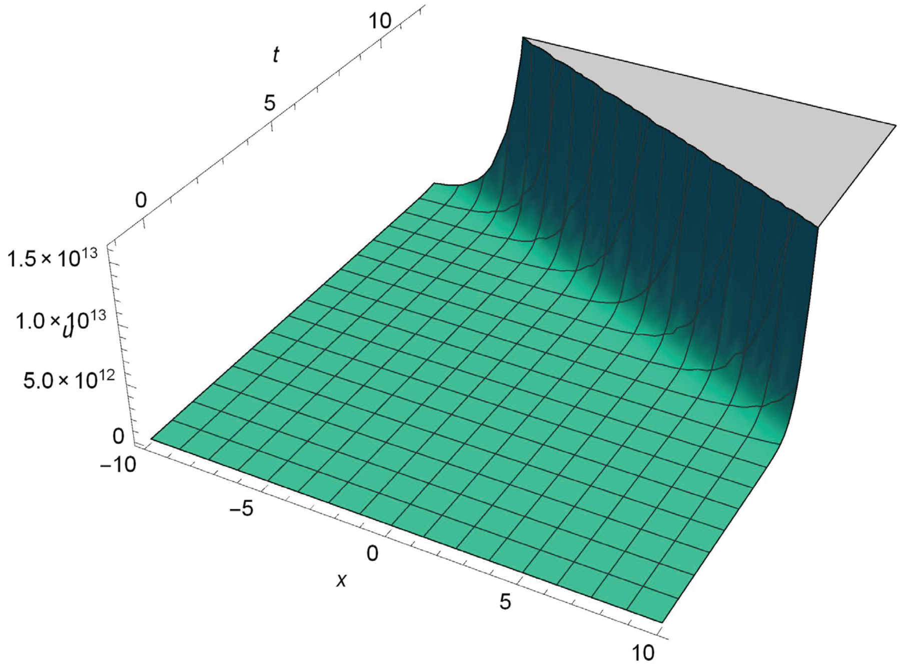


Figure 2. The solution to classical $K(1,1)$; here $A_1 = 2$, $k_4 = 1$ and $\alpha = 1$.

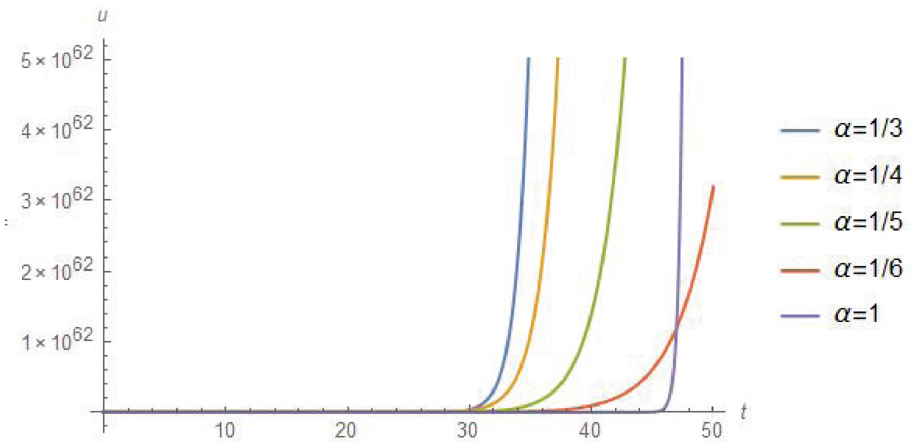


Figure 3. The graphics of $K(1,1)$ for various values of fractional order α with $x = 2$.

$$C_1 = \frac{3(k_5-3)(2k_5-3)A_1}{\alpha^3 k_5^3}, \quad C_2 = \frac{3}{\alpha^3 k_5}, \quad C_3 = -\frac{3}{\alpha},$$
$$C_4 = -\frac{3A_1}{\alpha^3 k_5^2}(9k_5^2 + 9k_5^2\alpha - 27k_5\alpha + 27\alpha^2 + 2k_5^2\alpha^2 - 18k_5\alpha^2), \quad (54)$$
$$C_5 = -\frac{27(3k_5 + \alpha k_5 - 3\alpha)}{k_5\alpha^3}A_1, \quad C_6 = -\frac{27A_1}{\alpha^3}.$$

Thus, here we derive the symmetries of the Lie group of the equation and use these symmetries to obtain invariant solutions. Along with this, we present graphs of the obtained solutions and analyse them. We also get reduced equations that are simpler than the original equation but contain the same information.

CONSERVATION LAWS OF CAPUTO TIME-FRACTIONAL *K*(*M*,*M*) EQUATIONS

Conservation Laws

Knowledge of the conservation laws for PDE gives us an idea of the conserved physical quantities and can be used in the development of stable numerical methods. There are several methods to study the conservation laws, such as partial Noether's approach via Noether's theorem [20] that establishes a connection between conservation laws and symmetries of differential equations like:

- If there is symmetry under space translations, then there exists a conservation of linear momentum. For example, $X = \frac{\partial}{\partial x}$ operator.
- If there is symmetry under time translations, then there exists a conservation of energy. For example, $X = \frac{\partial}{\partial t}$ operator.
- If symmetry under rotations exists, a conservation of angular momentum exists. The infinitesimal operator here can be in the form of the cross-product of spatial variables.
- If there is symmetry under boosts (moving coordinates), then there exists a linear motion of the centre of mass. For example, $X = \frac{\partial}{\partial u} + \frac{t}{\alpha}\frac{\partial}{\partial t}$ operator.
- If there is a symmetry under scaling, then there exists a scaling dimension, which represents how the energy of the system scales with changes in length scale. For example, $X = \frac{x}{3}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t}$ operator.

Emmy Noether formulated three fundamental theorems for analysis and physics. The first theorem states that there is a one-to-one correspondence between the symmetry groups of a variational problem and the conservation laws of its Euler–Lagrange equations [20]. The second theorem says that an infinite-dimensional variational symmetry group depending on an arbitrary function corresponds to a non-trivial differential relation between its Euler–Lagrange equations. And the third is the introduction of higher-order generalized symmetries, which will later play a fundamental role in the discovery and classification of integrable systems and solitons [20].

Mathematically, Noether's theorem can be expressed as:

$$\partial_j C^j = 0, \tag{55}$$

Where C^j is the Noether current, which is a conserved quantity associated with a particular continuous symmetry of the system, ∂_j is a partial derivative with respect to variable j. The index j runs over the space and time coordinates. The equation above states that the divergence of the Noether current is zero, which implies that the total amount of the conserved quantity is conserved over time.

That is to find the conservation law by using Noether's theorem, which states that every continuous symmetry of a system leads to a corresponding conservation law we need do next [12].

Suppose we have a PDE of the form:

$$u_t = G(t, x, u, u_x, u_{xx}, u_{xxx}). \tag{56}$$

To find the conservation law associated with a continuous symmetry of this system, we first identify the infinitesimal generator X of the symmetry. Next, we compute the Lagrangian density of the system. This is a function that describes the dynamics of the system and is given by:

$$L = L(u, u_x, t). \tag{57}$$

Using the generator X, we can construct a conserved current C = (C^t, C^x) where C^t is the charge density, and C^x is the flux density. The charge density is given by:

$$C^t = W_J \frac{\partial L}{\partial u_t}, \tag{58}$$

and the flux density is given by:

$$C^x = W_J \frac{\partial L}{\partial u_x}. \tag{59}$$

Here W_J depends on u, x, t and derivatives of u with respect to x, t.

The conservation laws for a (1+1) dimensional PDE are given by the conservation of the conserved current C. This implies that the integral of the charge density C^t over a spatial domain remains constant in time:

$$\frac{d}{dt}\int_{-\infty}^{\infty} C^t dx = 0, \tag{60}$$

Similarly, the integral of the flux density C^x over a spatial domain remains constant in time:

$$\frac{d}{dt}\int_{-\infty}^{\infty} C^x dx = 0. \tag{61}$$

Another method is the multiplier approach and Ibragimov's method. Since these three methods do not apply to nonlinear PDEs that do not admit a Lagrangian, Ibragimov's method was proposed to overcome these difficulties [21, 22].

In this section, we construct the conservation laws for the time-fractional nonlinear $K(m,n)$ model equations (9) with $m = n$, while for $m \neq n$ the conservation laws can be built similarly. For the formation, we will use Ibragimov's theorem [21, 22] which was applied to fractional differential equations by Gazizov and Lukachshuk [23, 24]. The theorem is based on that, the author by taking the formal Lagrangian uses the variational analysis and symmetries to give formulas for constructing the conservation laws.

So, we search $C^t = C^t(t, x, u, u_x,...)$ and $C^x = C^x(t, x, u, u_x,...)$ conservation laws for (9) on all its solutions if it satisfies the following conservation equation $D_t(C^t) + D_x(C^x) = 0$, by using the theory of Ibragimov [21, 22]. For the first, let a formal Lagrangian function the equation (9) has a form:

$$L = v\, E, \tag{62}$$

where $v = v(t, x)$ is a new dependent variable and

$$E = \frac{\partial^\alpha u}{\partial t^\alpha} + A_0(t)(u^m)_x + A_1(u^m)_{xxx}, \quad 0 < \alpha < 1. \tag{63}$$

By the definition, given in [12, 21], the Euler–Lagrange operator $\frac{\delta}{\delta u}$ with respect to u is

$$\frac{\delta}{\delta u}=\frac{\partial}{\partial u}+(D_t^\alpha)^*\frac{\partial}{\partial D_t^\alpha u}-D_x\frac{\partial}{\partial u_x}+D_x^2\frac{\partial}{\partial u_{xx}}-D_x^3\frac{\partial}{\partial u_{xxx}}, \quad (64)$$

where $(D_t^\alpha)^*$ is an adjoint operator for D_t^α, which has the form:

$$(D_t^\alpha)^*=(-1)^n\ {}_tI_T^{n-\alpha}(D_t^n), \quad (65)$$

where ${}_tI_T^{n-\alpha}$ is the right-sided operator of fractional integration of order (n - α) defined in Eq. (23). In our case n = 1 and $0 < \alpha < 1$. So, by applying the Euler–Lagrange operator to the formal Lagrangian L we have:

$$\frac{\delta L}{\delta u}=\frac{\partial}{\partial u}vE+(D_t^\alpha)^*\frac{\partial}{\partial D_t^\alpha u}vE-D_x\frac{\partial}{\partial u_x}vE + D_x^2\frac{\partial}{\partial u_{xx}}vE-D_x^3\frac{\partial}{\partial u_{xxx}}vE. \quad (66)$$

Here

$$\frac{\partial}{\partial u}vE=v(m(m-1)A_0(t)u^{m-2}u_x + A_1(m(m-1)(m-2)(m-3)u^{m-4}u_x^3 + 3m(m-1)(m-2)u^{m-3}u_xu_{xx} + m(m-1)u^{m-2}u_{xxx})), \quad (67)$$

$$(D_t^\alpha)^*\frac{\partial}{\partial D_t^\alpha u}vE=(D_t^\alpha)^*v, \quad (68)$$

$$D_x\frac{\partial}{\partial u_x}vE=v_x(mu^{m-1}A_0(t)+A_1(3m(m-1)(m-2)u^{m-3}u_x^2 +3m(m-1)u^{m-2}u_{xx}))+v(m(m-1)A_0(t)u^{m-2}u_x + A_1(3m(m-1)u^{m-2}u_{xxx}+9m(m-1)(m-2)u^{m-3}u_xu_{xx} + 3m(m-1)(m-2)(m-3)u^{m-4}u_x^3)), \quad (69)$$

$$D_x^2\frac{\partial}{\partial u_{xx}}vE=3m(m-1)A_1(2u^{m-2}v_x+(m-2)u^{m-3}u_xu_{xx}v + u_x(u^{m-2}v_{xx}+2(m-2)u^{m-3}u_xv_x + v((m-2)(m-3)u^{m-4}u_x^2+(m-2)u^{m-3}u_{xx}))), \quad (70)$$

$$D_x^3\frac{\partial}{\partial u_{xxx}}vE=mA_1(u^{m-1}v_{xxx}+3(m-1)u^{m-2}u_xv_{xx} + 3v_x((m-1)(m-2)u^{m-3}u_x^2+(m-1)u^{m-2}u_{xx}) + v((m-1)u^{m-2}u_{xxx}+(m-1)(m-2)(m-3)u^{m-4}u_x^3 +3(m-1)(m-2)u^{m-3}u_xu_{xx})). \quad (71)$$

And after the summation of the above derivatives and simplifying we get an adjoint equation of equation (9) in the form:

$$\frac{\delta L}{\delta u}=(D_t^\alpha)^*v-mA_0(t)u^{m-1}v_x-mA_1u^{m-1}v_{xxx}. \quad (72$$

We say that the nonlinear time-fractional $K(m,m)$ model equation is nonlinearly self-adjoint if the adjoint equation (19) is satisfied for all solution $u(t, x)$ of equation (9) with a substitution $v = \varphi(t, x, u)$ and $\varphi(t, x, u) \neq 0$ [23]. This substitution allows us to use the formal Lagrangian as usual classical Lagrangian and obtain the conservation laws.

Thus, x and t-components conservation laws for the equation (9) have the form:

$$C_i^x=\xi_1L+W_i\left(\frac{\partial L}{\partial u_x}-D_x\frac{\partial L}{\partial u_{xx}}+D_x^2\frac{\partial L}{\partial u_{xxx}}\right) + D_x(W_i)\left(\frac{\partial L}{\partial u_{xx}}-D_x\frac{\partial L}{\partial u_{xxx}}\right)+D_x^2(W_i)\left(\frac{\partial L}{\partial u_{xxx}}\right), \quad (73)$$

$$C_i^t=\sum_{k=0}^{m-1}(-1)^kD_t^{\alpha-1-k}(W_i)D_t^k\left(\frac{\partial L}{\partial D_t^\alpha u}\right) -(-1)^mJ\left(W_i,D_t^m\frac{\partial L}{\partial D_t^\alpha u}\right), \quad (74)$$

here $W_i=\eta^i-\xi^iu_x-\tau^iu_t$ and for $n - 1 < \alpha < n$, J is integral in a form:

$$J(f,g)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T\frac{f(x,s)g(x,p)}{(p-s)^{\alpha+1-n}}dpds,\ \ n-1<\alpha<n. \quad (75)$$

Construction of the Conservation Laws of Caputo Time-Fractional $K(m,m)$ Equations

Now we can construct C^x and C^t for our equation (9) with the infinitesimal generators. In the obtained conservation laws in the form of expressions below $J(u_x, v_t)$is integral as in Eq. (75) with $n = 1$, A_1 is arbitrary constant, $m \neq 1$, $\lambda \neq 0$.

For the non-zero and differentiable function $A_0(t)$ we have symmetry under space translation with $W = u_x$, so here we get a conservation of linear momentum and

$$C^t=vD_t^{\alpha-1}(u_x)+J(u_x,v_t), \quad (76)$$

$$C^x=v(\frac{\partial\ u}{\partial t^\alpha}+A_0(t)(u^m)_x+A_1(u^m)_{xxx}) + u_x\left(\frac{\partial L}{\partial u_x}-D_x\frac{\partial L}{\partial u_{xx}}+D_x^2\frac{\partial L}{\partial u_{xxx}}\right) + u_{xx}\left(\frac{\partial L}{\partial u_{xx}}-D_x\frac{\partial L}{\partial u_{xxx}}\right)+u_{xxx}\left(\frac{\partial L}{\partial u_{xxx}}\right). \quad (77)$$

For $A_0(t)$ = 1 we get $W_1 = u_x$, $W_2 = u_t$, and $W_3=tu_t-\frac{\alpha}{1-m}u$. Here we get again conservation of linear momentum, and energy and there is a scaling dimension, which represents how the energy of the system scales with changes in length scale. This scaling dimension is related to the scaling behaviour of the system under the rescaling of the space and time coordinates. So, we can obtain the following conservation laws:

$$C_1^t=vD_t^{\alpha-1}(u_x)+J(u_x,v_t), \quad (78)$$

$$
\begin{aligned}
C_1^x = {} & v(\frac{\partial^\alpha u}{\partial t^\alpha} + (u^m)_x + A_1 (u^m)_{xxx}) \\
& + m(m-1)(m-2) A_1 u^{m-3} u_x^3 v \\
& + m(m-1) A_1 u^{m-2} u_x (3 v u_{xx} - u_x (u_x v_u + v_x)) \\
& + m u^{m-1} (v u_x + A_1 v u_{xxx} + A_1 (u_x^3 v_{uu} - u_{xx} v_x \\
& + 2 u_x^2 v_{xu} + u_x v_{xx})),
\end{aligned} \tag{79}
$$

$$
\begin{aligned}
C_2^t = {} & v\left(\frac{\partial^\alpha u}{\partial t^\alpha} + (u^m)_x + A_1 (u^m)_{xxx}\right) \\
& + v D_t^{\alpha-1}(u_t) + J(u_t, v_t),
\end{aligned} \tag{80}
$$

$$
\begin{aligned}
C_2^x = {} & m(m-1)(m-2) A_1 u^{m-3} u_x^2 u_t v \\
& + m(m-1) A_1 u^{m-2} ((2 v u_x u_{xt} + v u_t u_{xx}) \\
& - u_x u_t (u_x v_u + v_x)) + m u^{m-1} (v (u_t + A_1 u_{xxt}) \\
& + A_1 (u_t u_x^2 v_{uu} - u_{xt} v_x + u_x (2 u_t x_{xu} - u_{xt} v_u) \\
& + u_t (u_{xx} v_u + v_{xx}))),
\end{aligned} \tag{81}
$$

$$
\begin{aligned}
C_3^t = {} & t v(\frac{\partial^\alpha u}{\partial t^\alpha} + A_0 (u^m)_x + A_1 (u^m)_{xxx}) \\
& + v D_t^{\alpha-1}(t u_t - \frac{\alpha}{1-m} u) + J(t u_t - \frac{\alpha}{1-m} u, v_t),
\end{aligned} \tag{82}
$$

$$
\begin{aligned}
C_3^x = {} & m u^{m-3} (\frac{\alpha}{m-1} u + t u_t)[(m-1)(m-2) A_1 u_x^2 v \\
& + (m-1) A_1 u (v u_{xx} - u_x (u_x v_u + v_x)) + \\
& + u^2 (v + A_1 (u_{xx} v_u + u_x^2 v_{uu} + 2 u_x v_{xu} + v_{xx}))] \\
& + m A_1 u^{m-2} \left(\frac{\alpha}{1-m} u_x + t u_{xt}\right) \big(2(m-1) v u_x \\
& - u(u_x v_u + v_x)\big) + m A_1 u^{m-1} v \left(\frac{\alpha}{1-m} u_{xx} + t u_{xxt}\right).
\end{aligned} \tag{83}
$$

For $A_0(t) = t^\lambda$ we have $W_1 = u_x$ and $W_2 = x u_x - \frac{2}{\lambda} t u_t - \frac{2\alpha + 3\lambda}{\lambda(m-1)} u$, according to which we can get conservation of linear momentum and the next conservation laws

$$
C_1^t = v D_t^{\alpha-1}(u_x) + J(u_x, v_t), \tag{84}
$$

$$
\begin{aligned}
C_1^x = {} & v(\frac{\partial^\alpha u}{\partial t^\alpha} + t^\lambda (u^m)_x + A_1 (u^m)_{xxx}) \\
& + m(m-1)(m-2) A_1 u^{m-3} u_x^3 v \\
& + m(m-1) A_1 u^{m-2} u_x (3 v u_{xx} - u_x (u_x v_u + v_x)) \\
& + m u^{m-1} (v t^\lambda u_x + A_1 v u_{xxx} + A_1 (u_x^3 v_{uu} - u_{xx} v_x \\
& + 2 u_x^2 v_{xu} + u_x v_{xx})),
\end{aligned} \tag{85}
$$

$$
\begin{aligned}
C_2^t = {} & -\frac{2}{\lambda} t v \left(\frac{\partial^\alpha u}{\partial t^\alpha} + t^\lambda (u^m)_x + A_1 (u^m)_{xxx}\right) \\
& + v D_t^{\alpha-1}\left(x u_x - \frac{2}{\lambda} t u_t - \frac{2\alpha + 3\lambda}{\lambda(m-1)} u\right) \\
& + \ J\left(x u_x - \frac{2}{\lambda} t u_t - \frac{2\alpha + 3\lambda}{\lambda(m-1)} u, v_t\right),
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
C_2^x = {} & x v(\frac{\partial^\alpha u}{\partial t^\alpha} + t^\lambda (u^m)_x + A_1 (u^m)_{xxx}) \\
& + m u^{m-3} \left(\frac{3\lambda + 2\alpha}{\lambda(m-1)} u - \frac{2}{\lambda} t u_t + x u_x\right) \times ((m-1)(m-2) A_1 u_x^2 v \\
& + (m-1) A_1 u (v u_{xx} - u_x (u_x v_u + v_x)) + u^2 [t^\lambda v + A_1 (u_{xx} v_u + u_x^2 v_{uu} \\
& + 2 u_x v_{xu} + v_{xx})]) + m A_1 u^{m-2} \left(\frac{\lambda(m+2) + 2\alpha}{\lambda(m-1)} u_x - \frac{2}{\lambda} t u_{xt} + x u_{xx}\right) \\
& \times \big(2(m-1) v u_x - u(u_x v_u + v_x)\big) + m A_1 u^{m-1} v \\
& \times \left(\frac{\lambda(2m-5) - 2\alpha}{\lambda(m-1)} u_{xx} - \frac{2}{\lambda} t u_{xxt} + x u_{xxx}\right).
\end{aligned} \tag{87}
$$

In case $A_0(t) = e^t$ we obtain $W_1 = u_x$ and $W_2 = x u_x - 2 u_t - \frac{3}{(m-1)} u$, which gives us the conservation of linear momentum and following conservation laws:

$$
C_1^t = v D_t^{\alpha-1}(u_x) + J(u_x, v_t), \tag{88}
$$

$$
\begin{aligned}
C_1^x = {} & v(\frac{\partial^\alpha u}{\partial t^\alpha} + e^t (u^m)_x + A_1 (u^m)_{xxx}) \\
& + m(m-1)(m-2) A_1 u^{m-3} u_x^3 v \\
& + m(m-1) A_1 u^{m-2} u_x (3 v u_{xx} - u_x (u_x v_u + v_x)) \\
& + m u^{m-1} (v e^t u_x + A_1 v u_{xxx} + A_1 (u_x^3 v_{uu} - u_{xx} v_x \\
& + 2 u_x^2 v_{xu} + u_x v_{xx})),
\end{aligned} \tag{90}
$$

$$
\begin{aligned}
C_2^t = {} & -2v \left(\frac{\partial^\alpha u}{\partial t^\alpha} + e^t (u^m)_x + A_1 (u^m)_{xxx}\right) \\
& + v D_t^{\alpha-1}\left(x u_x - 2 u_t - \frac{3}{(m-1)} u\right) \\
& + \ J\left(x u_x - 2 u_t - \frac{3}{(m-1)} u, v_t\right),
\end{aligned} \tag{91}
$$

$$
\begin{aligned}
C_2^x = {} & x v(\frac{\partial^\alpha u}{\partial t^\alpha} + e^t (u^m)_x + A_1 (u^m)_{xxx}) \\
& + m u^{m-3} \left(\frac{3}{m-1} u + 2 u_t - x u_x\right) \times \Big((m-1)(m-2) A_1 u_x^2 v \\
& + (m-1) A_1 u \big(v u_{xx} - u_x (u_x v_u + v_x)\big)\Big) + m A_1 u^{m-2} \\
& \times \left(\frac{m+2}{m-1} u_x - 2 u_{xt} + x u_{xx}\right) (2(m-1) v u_x - u(u_x v_u + v_x)) \\
& + m u^{m-1} (e^t v + A_1 (u_{xx} v_u + u_x^2 v_{uu} + 2 u_x v_{xu} + v_{xx})) \\
& + m A_1 u^{m-1} v (\frac{2m+1}{m-1} u_{xx} - 2 u_{xxt} + x u_{xxx}).
\end{aligned} \tag{92}
$$

Thus, here we can see that the conservation laws depend on the fractional orders of the equation and the coefficients and have explicit formulas for the conserved quantities.

CONCLUSION

In this paper, we give the Lie group analysis of Caputo time-fractional $K(m,n)$ equations and obtain the infinitesimal operators. Each infinitesimal operator gives us an invariant equation to our equation. So here, by using symmetries, we present some solutions with graphs according to small changes in the values of α. Thus, we show that the solution function grows respectively faster or slower concerning x when α decreases or increases. Here, we would like to note that the solution to the $K(1,1)$ fractional differential equation with $\alpha \in (0,1)$ has a singularity at $t = 0$, which is removed for $\alpha = 1$. Moreover, for $\alpha = 1$ solution takes the form of the exponential function, which the graph given. Also, we construct conservation laws of the Caputo time-fractional $K(m,m)$ differential equation for the special cases of the function $A_0(t)$. We present that the conservation laws depend on the fractional orders of the equation and the coefficients and provide explicit formulas for the conserved quantities.

As a limitation of our work, we can point out the lack of experimental data to support our conclusions.

Our potential directions for future research on $K(m,m)$ fractional differential equations will be focused on studying the nonlocal properties of fractional $K(m,m)$ differential equations and their effects on the dynamics of the system, which can significantly affect its behaviour. Moreover, the question of the study of multidimensional $K(m,m)$ fractional differential equations is also still open. These equations allow studying of complex systems in higher dimensions.

AUTHORSHIP CONTRIBUTIONS

All authors contributed to the study's conception and design. G.I. performed the analysis of the problem and D.K. initiated the research. The first draft of the manuscript was written by G.I., and all authors commented on and added to the following versions of the manuscript. All authors read, reviewed, edited, and approved the final manuscript.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

REFERENCES

[1] Rosenau P, Hyman J. Compactons: Solitons with finite wavelength. Phys Rev Lett 1993;70:564–567. [CrossRef]

[2] Rosenau P, Levy D. Compactons in a class of nonlinearly quintic equations. Phys Lett A 1999;252:297–306. [CrossRef]

[3] Charalambous K, Vaneeva O, Sophocleous C. Group classification of variable coefficient K(m,n) equations. J Geom Symmetry Phys 2014;33:79–90. [CrossRef]

[4] Kudryashov NA, Prilipko SC. Exact solutions of the generalized K(m,m) equation. Commun Sci Simul 2011;16:1107–1113. [CrossRef]

[5] Bruzon MS, Gandarias ML. Classical potential symmetries of the K(m,n) equation with generalized evolution term. WSEAS Trans Math 2010;9:275–284.

[6] Baleanu D, Jajarmi A, Asad J, Blaszczyk T. The motion of a bead sliding on a wire in a fractional sense. Acta Phys Pol A 2017;131:1561–1564. [CrossRef]

[7] Yavuz M, Yokus A. Analytical and numerical approaches to nerve impulse model of fractional-order. Numer Meth Part Differ Equ 2020;36:1348–1368. [CrossRef]

[8] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and Applications of Fractional Differential Equations. Amsterdam: Elsevier Science; 2006.

[9] Miller KS, Ross B. An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: Wiley-Interscience; 1993.

[10] Podlubny I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, Some Methods of their Solution and Some of their Applications. 1st ed. San Diego: Academic Press; 1998.

[11] Wang GW, Hashemi MS. Lie symmetry analysis and soliton solutions of time - fractional K(m,n) equation. Pramana J Phys 2017;88:7. [CrossRef]

[12] Olver PJ. Applications of Lie groups to differential equations. Berlin, Germany: Springer; 2012.

[13] Bluman GW, Anco SC. Symmetry and Integration Methods for Differential Equations. New York: Springer; 2002.

[14] Gazizov RK, Kasatkin AA, Lukashchuk SY. Continuous transformation groups of fractional differential equations. Vestnik USATU 2007;9:125–135. Russian.

[15] Clarkson P, Kruskal M. New similarity reductions of the Boussinesq equation. J Math Phys 1989;30:2201–2213. [CrossRef]

[16] Nucci MC. Lie symmetries of a Painlevé-type equation without Lie symmetries. J Nonlinear Math Phys 2008;15:205–211. [CrossRef]

[17] Iskandarova G, Kaya D. Symmetry solution on fractional equation. Int J Optim Control Theor Appl 2017;7:255–259. [CrossRef]

[18] Khalique ChM, Adeyemo OD. A study of (3+1)-dimensional generalized Korteweg-de Vries-Zakharov-Kuznetsov equation via Lie symmetry approach. Results Phys 2021;18:103197. [CrossRef]

[19] Iskenderoglu G, Kaya, D. On lie group analysis of boundary value problem with Caputo fractional derivative. Sigma J Eng Nat Sci 2019;10:369–376.

[20] Noether E. Invariante variationsprobleme, königliche gesellschaft der wissenschaften zu Göttingen, Nachrichten. Math Phys Klas Heft 1918;2:235–257.

[21] Ibragimov N. A new conservation theorem. J Math Anal Appl 2007;333:311–328. [CrossRef]

[22] Ibragimov N. Nonlinear self-adjointness and conservation laws. J Phys A Math Gen 2011;44:4109–4112. [CrossRef]

[23] Lukashchuk SY. Conservation laws for time-fractional subdiffusion and diffusion-wave equations. Nonlinear Dyn 2015;80:791−802. [CrossRef]
[24] Gazizov RK, Ibragimov NH, Lukashchuk SY. Nonlinear self-adjointness, conservation laws and exact solution of fractional Kompaneets equations. Commun Nonlinear Sci Numer Simul 2015;23:153−163. [CrossRef]