Sigma J Eng Nat Sci, Vol. 42, No. 3, pp. 900−904, June, 2024

Sigma Journal of Engineering and Natural Sciences
Web page info: https://sigma.yildiz.edu.tr
DOI: 10.14744/sigma.2024.00000

Research Article

The stability analysis of a neural field model with small delay

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ARTICLE INFO

Article history
Received: 11 August 2022
Revised: 03 December 2022
Accepted: 08 February 2023

Keywords:
Neural Field Model; Routh-Hurwitz Criterion; Stability Analysis

ABSTRACT

In this study it is elucidated a mathematical framework in which the stability for the neural field model for two neuron populations with small delay is investigated. The primary purpose of this analysis is to provide a unifying mathematical framework for illustrating the effect of small delay considering the cases in Routh-Hurwitz criterion.

Cite this article as: Özgür B, Demir A. The stability analysis of a neural field model with small delay. Sigma J Eng Nat Sci 2024;42(3):900−904.

INTRODUCTION

Because of the complexity of biological systems in neuroscience, we use mathematical models to understand their general behavior and express the fundamental principles. Two fundamental types of cells in the brain are neurons and glia. Interconnection of neurons is important for living being. Three main parts in connection between neurons are receiving signals from dendrites, sending signals along the axon and creating the response for the signal transmitted. The whole process is based on an electrochemical mechanism [3].

Details about membrane dynamics of a single neuron are introduced in the work of Hodgkin and Huxley [9] by a nonlinear dynamical system. Later, when the case at which neurons are grouped, some researches are made and mathematical models giving the relation on the average pre-synaptic firing rates and the average post-synaptic membrane potentials of neural populations are written. In modelling the brain activity and nonlinear dynamics of populations of neurons, the neural field models are developed by propounding parameters and making estimations on experimental data. For the applications of models of neural field theory in neuroscience, the studies of brain activity, sleep cycles, epilepsy and Alzheimer's disease may be given [3]. For further reading on the neural field models, one can refer to fundamental studies made by Amari [1] and Wilson and Cowan [21]. In the study of Wilson and Cowan, we see excitatory and inhibitory populations of neurons.

These neural field models demonstrate the activity of neurons and consist of integro-differential equations. In brain, neurons have a significant role in receiving and transmiting signals. In communications of neurons, the time for releasing the neurotransmitter and for a signal passing through the axon, is introduced to the model by a delay term.

Refining and developing the analysis on behavior of solutions of these models needs some analytical approaches or numerical methods. There are some substantial researches

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This paper was recommended for publication in revised form by Editor in Chief Ahmet Selim Dalkilic

Published by Yıldız Technical University Press, İstanbul, Turkey

on the stability of the model and the effect of the delay term [2,4-6,8,17-20]. From a dynamical systems point of view, the investigation of the roots of the characteristic equation for the neural field model made by the Routh-Hurwitz criterion or D-curves method can be seen in [11-15]. The details for these methods can be found in [7,10,16].

In this study we are interested in the stability of a neural field model for two neuron populations with small delay. First, the model is introduced. In next section, the stability switches in terms of delay term and system parameters are given. The results obtained are given in summary in the last part of this section. The conclusion of this study is given in the last section.

The Model

The neural field equations for p neural populations on the space $\Omega \subset R^d$ modelling the dynamics of mean membran potential [18-20] is given below

$$\left(\frac{d}{dt}+l_i\right)V_i(t,r) = \sum_{j=1}^{p}\int_{\Omega} J_{ij}(r,\bar{r})S\big[\sigma_j\big(V_j\big(t-\tau_{ij}(r,\bar{r}),\bar{r}\big)-h_j\big)\big]\,d\bar{r} + I_i^{ext}(r,t),\ t\geq 0, 1\leq i\leq p \qquad (1)$$

$$V_i(t,r)=\phi_i(t,r),\ t\in[-\tau_{max},0]$$

In this neural field model, the functions $V_i(t,r)$ represent the synaptic inputs for a large group of neurons at position x and time t. The synaptic connectivity is given by the functions $J_{i,j}(r,\bar{r})$. These functions are π periodic even functions.

Hence the linearized system near (0,0) in terms of the functions $U_1(x,t)$ and $U_2(x,t)$ for the synaptic inputs is the following:

$$\begin{aligned}\frac{d}{dt}U_1(x,t)+l_1U_1(x,t-\varepsilon)&=\sigma_2 s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{12}(x,y)U_2\big(y,t-\tau(x,y)\big)dy\\ \frac{d}{dt}U_2(x,t)+l_2U_2(x,t-\varepsilon)&=\sigma_1 s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x,y)U_1\big(y,t-\tau(x,y)\big)dy\end{aligned} \qquad (2)$$

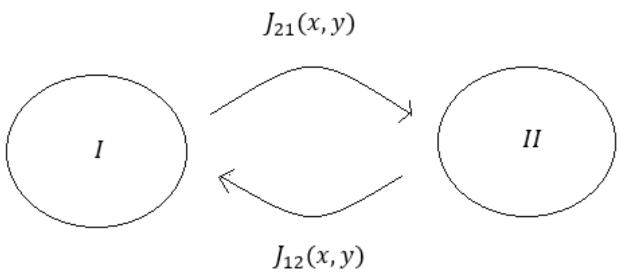

Figure 1. This figure is an illustration of the mathematical model studied. This is an adapted version of the figure given in [3] showing the relation of neurons in population I and II. The main idea of this figure is based on the work by Wilson and Cowan [21]. Wilson and Cowan [21], with permission from Springer.].

Here we consider the effect of the small delay ε in the functions $U_1(x,t)$ and $U_2(x,t)$ of the model written for two neuron populations on $\Omega=(-\frac{\pi}{2},\frac{\pi}{2})$. We assume the propagation delay as $\tau(x,y)=\tau$. The synaptic connectivity functions generally describe how neurons in the J^{th} population at position y influence the neurons in the i^{th} population at position x. As a special case for the model, we consider the case in which synaptic connectivity functions as $J_{11}(x,y) = J_{22}(x,y) = 0$. An illustrative explanation of the model considered here is given below in Figure 1.

This is an adapted version of the figure given in [3] showing the relation of neurons in population I and II. The main idea of this figure is based on the work by Wilson and Cowan [21].

Stability Analysis

If we write the first order Taylor series expansion for the functions $U_1(x,t-\varepsilon)$ and $U_2(x,t-\varepsilon)$ then we get

$$\begin{aligned}\frac{d}{dt}U_1(x,t)+l_1\left(U_1(x,t)-\varepsilon\frac{d}{dt}U_1(x,t)\right)&=\sigma_2 s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{12}(x,y)U_2(y,t-\tau)dy\\ \frac{d}{dt}U_2(x,t)+l_2\left(U_2(x,t)-\varepsilon\frac{d}{dt}U_2(x,t-\varepsilon)\right)&=\sigma_1 s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x,y)U_1(y,t-\tau)dy\end{aligned} \qquad (3)$$

Hence we have

$$\begin{aligned}(1-\varepsilon l_1)\frac{d}{dt}U_1(x,t)+l_1U_1(x,t)&=\sigma_2 s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{12}(x,y)U_2(y,t-\tau)dy\\ (1-\varepsilon l_2)\frac{d}{dt}U_2(x,t)+l_2U_2(x,t)&=\sigma_1 s_1\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x,y)U_1(y,t-\tau)dy\end{aligned} \qquad (4)$$

We are looking for the solutions in the form $U_1(x,t) = u_1(t)e^{ikx} = c_1e^{\lambda t}e^{ikx}$ and $U_2(x,t) = u_2(t)e^{ikx} = c_2e^{\lambda t}e^{ikx}$. Substituting them we get the following

$$(1-\varepsilon l_1)\lambda e^{ikx}u_1(t)+l_1e^{ikx}u_1(t)-K_1e^{-\lambda\tau}u_2(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{12}(x,y)e^{iky}dy=0$$

$$(1-\varepsilon l_2)\lambda e^{ikx}u_2(t)+l_2e^{ikx}u_2(t)-K_2e^{-\lambda\tau}u_1(t)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x,y)e^{iky}dy=0$$

For the simplicity we take $K_1 = \sigma_1 s_1$, $K_2 = \sigma_2 s_1$ and $F_1=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{12}(x,y)e^{iky}dy, F_2=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} J_{21}(x,y)e^{iky}dy.$

For $x = 0$, the system of equations becomes

$$\begin{aligned}(1-\varepsilon l_1)\lambda u_1(t)+l_1u_1(t)-K_1u_2(t)e^{-\lambda\tau}F_1&=0\\ (1-\varepsilon l_2)\lambda u_2(t)+l_2u_2(t)-K_2u_1(t)e^{-\lambda\tau}F_2&=0\end{aligned} \qquad (5)$$

Rearranging the system for $u_1(t)$ and $u_2(t)$ and considering the coefficient determinant, we get the characteristic values λ satisfying the following equation

$$(1-\varepsilon l_1)(1-\varepsilon l_2)\lambda^2+((1-\varepsilon l_1)l_2 + (1-\varepsilon l_2)l_1)\lambda+l_1l_2-K_1K_2e^{-2\lambda\tau}F_1F_2=0 \quad (6)$$

The Case: Absence of Delay Term

For the stability analysis, we can make use of the Routh-Hurwitz criterion [7]. First we analyze the stability considering the absence of the propagation delay term τ.

In this case the equation (6) turns into

$$(1-\varepsilon l_1)(1-\varepsilon l_2)\lambda^2+((1-\varepsilon l_1)l_2 + (1-\varepsilon l_2)l_1)\lambda+l_1l_2-K_1K_2F_1F_2=0 \quad (7)$$

If the coefficients of λ and λ^2 and the constant term are positive, then all the roots have negative real parts. Let $(1-\varepsilon l_1)(1-\varepsilon l_2)>0$. If

$$((1-\varepsilon l_1)l_2+(1-\varepsilon l_2)l_1)>0$$

and

$$l_1l_2-K_1K_2F_1F_2>0$$

then we may conclude that all roots have negative real parts.

According to the model, the conditions $l_1 > 0$ and $l_2 > 0$ are satisfied. Hence the conditions on the stability given above may be summarized to show the relations among parameters of the system in the following theorem.

Theorem 1 : Consider the system (2) and its characteristic equation (7) in case of no delay term exists. Hence, if the conditions $0<l_1<\frac{1}{\varepsilon}$, $0<l_2<\frac{1}{\varepsilon}$ and $K_1K_2F_1F_2<\frac{1}{\varepsilon^2}$ are satisfied then the system is stable.

Proof : Considering the equation (7) and investigating its roots according to the Routh-Hurwitz criteria, if the conditions given are hold then all roots have negative real parts. Hence the system is stable.

The Case: Existence of Delay Term

In this section, as a general case, we investigate the stability considering the propagation delay term. For this aim, we consider the imaginary roots $\lambda = i\sigma$ for the equation. After converting the characteristic equation into a polynomial form, we make an analysis for the roots whether they have positive real parts or not.

Writing $\lambda = i\sigma$ in (6) then we have the following equation

$$-(1-\varepsilon l_1)(1-\varepsilon l_2)\sigma^2+i(1-\varepsilon l_1)\sigma l_2+i(1-\varepsilon l_2)\sigma l_1 + l_1l_2-K_1K_2F_1F_2(\cos(2\sigma\tau)-i sin(2\sigma\tau))=0$$

Separating the real and imaginary parts, we get the following two equations

$$-(1-\varepsilon l_1)(1-\varepsilon l_2)\sigma^2+l_1l_2=K_1K_2F_1F_2\cos(2\sigma\tau)$$
$$(1-\varepsilon l_1)\sigma l_2+(1-\varepsilon l_2)\sigma l_1=-K_1K_2F_1F_2 sin(2\sigma\tau))$$

Now our aim is to apply the Routh-Hurwitz criterion for the stability analysis. Hence we add the squares of both sides of these equations. Then we have

$$(1-\varepsilon l_1)^2(1-\varepsilon l_2)^2\sigma^4+(1-\varepsilon l_1)^2\sigma^2{l_2}^2 + (1-\varepsilon l_2)^2\sigma^2{l_1}^2+(l_1l_2)^2-(K_1K_2F_1F_2)^2=0$$

We will make the analysis using a polynomial including the characteristic values. Hence we make the substitution $\mu = \sigma^2$, then we get the following polynomial equation making the leading coefficient 1,

$$\mu^2+\frac{(1-\varepsilon l_1)^2{l_2}^2+(1-\varepsilon l_2)^2{l_1}^2}{(1-\varepsilon l_1)^2(1-\varepsilon l_2)^2}\mu+\frac{(l_1l_2)^2-(K_1K_2F_1F_2)^2}{(1-\varepsilon l_1)^2(1-\varepsilon l_2)^2}=0 \quad (8)$$

Existence of a single positive root for the equation (8) leads the unstability of the system. Hence we may conclude this case by the following theorem.

Theorem 2 : Consider the system (2) and its characteristic equation (8) in case of delay term exists. If

$$\frac{(l_1l_2)^2-(K_1K_2F_1F_2)^2}{(1-\varepsilon l_1)^2(1-\varepsilon l_2)^2}<0$$

is satisfied then the system is unstable.

Proof : According to the Routh-Hurwitz criteria, since the leading coefficient of (8) is positive, if the condition

$$\frac{(l_1l_2)^2-(K_1K_2F_1F_2)^2}{(1-\varepsilon l_1)^2(1-\varepsilon l_2)^2}<0$$

is satisfied then there is a single positive root. Hence the unstability of the system occurs.

The General Stability and Unstability Cases

In this section we give the conclusions made by two theorems in the preceding sections. The following corollaries include the changes in the stability of the system in terms of the effects of the system parameters.

Corollary 1 : Consider the system (2) with small delay. Preserving the stability conditions given in Theorem 1, the following two cases occur.

1. If $K_1K_2F_1F_2 < 0$ and $l_1 l_2 < |K_1K_2F_1F_2|$ are satisfied then there is a positive root. Hence the system becomes unstable.
2. If $K_1K_2F_1F_2 < 0$ but $l_1 l_2 < |K_1K_2F_1F_2|$ is satisfied or the condition $K_1K_2F_1F_2 > 0$ holds, then the system is still stable.

Proof : Considering the system (2) with delay term, we have the characteristic equation (8). Since the leading coefficient is positive, if the constant term is negative then there exists a positive root. If the conditions $K_1K_2F_1F_2 < 0$ and $l_1 l_2 < |K_1K_2F_1F_2|$ are satisfied then $(l_1 l_2)^2 - (K_1K_2F_1F_2)^2 < 0$ and hence there is a positive root. This implies the unstability of the given system. Otherwise, no positive root exists.

Corollary 2 : Consider the system (2). Under the condition $l_1 l_2 - K_1K_2F_1F_2 < 0$, the system is always unstable.

Proof : For the system (2) with or without delay, the characteristic equation has a positive root under given condition. Hence the system is unstable.

CONCLUSION

In this study, a special case of the neural field model for two populations including a small delay is addressed. Investigation of the characteristic equation and using the Routh-Hurwitz criterion, we identify the connections between the parameters and the newly added delay term for the stability of the system. The powerful and practical Routh-Hurwitz criterion captures some practical results that the small delay term ε does not have an effect on the stability of the model considered when the propagation delay τ exists.

ACKNOWLEDGEMENTS

The authors would like to thank the reviewers for their valuable and helpful comments.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Amari SI. Dynamics of pattern formation in lateral-inhibition type neural fields. Biol Cybern 1977;27:77–87. [CrossRef]

[2] Atay FM, Hutt A. Stability and bifurcations in neural fields with finite propagation speed and general connectivity. SIAM J Math Anal 2006;5:670–698.

[3] Cook BJ, Peterson ADH, Woldman W, Terry JR. Neural field models: A mathematical overview and unifying framework. Math Neurosci Appl 2022;2:1–67. [CrossRef]

[4] Coombes S. Waves, bumps, and patterns in neural field theories. Biol Cybern 2005;93:91−108. [CrossRef]

[5] Coombes S, Venkov NA, Shiau L, Bojak L, Liley DTJ, Laing CR. Modeling electrocortical activity through improved local approximations of integral neural field equations. Phys Rev E 2007;76:051901. [CrossRef]

[6] Faye G, Faugeras O. Some theoretical and numerical results for delayed neural field equations. Phys D Nonlinear Phenom 2010;239:561−578. [CrossRef]

[7] Forde J, Nelson P. Applications of Sturm sequences to bifurcation analysis of delay differential equation models. J Math Anal Appl 2004;300:273-284. [CrossRef]

[8] Huang C, Vandewalle S. An analysis of delay dependent stability for ordinary and partial differential equations with fixed and distributed delays. SIAM J Sci Comput 2004;25:1608−1632. [CrossRef]

[9] Hodgkin AL, Huxley AF. A quantitative description of membrane current and its application to conduction and excitation in nerve. J Physiol 1952;117:500-544. [CrossRef]

[10] Insperger T, Stepan G. Semi-discretization for time-delay systems, Stability and engineering applications. New York: Springer; 2011. [CrossRef]

[11] Özgür B, Demir A. Some stability charts of a neural field model of two neural populations. Commun Math Appl 2016;7:159−166.

[12] Özgür B, Demir A. On the stability of two neuron populations interacting with each other. Rocky Mt J Math 2018;48:2337−2346. [CrossRef]

[13] Özgür B, Demir A, Erman S. A note on the stability of a neural field model. Hacettepe J Math Stat 2018;47:1495−1502.

[14] Özgür B. Stability switches of a neural field model: An algebraic study on the parameters. Sakarya Univ J Sci 2020;24:178−182. [CrossRef]

[15] Özgür B. Investigation of stability changes in a neural field model. Kocaeli J Sci Eng 2021;4:46−50. [CrossRef]

[16] Stepan, G. Retarded dynamical systems: stability and characteristic functions. London: Longman Scientific & Technical, England; 1989.

[17] Van Gils SA, Janssens SG, Kuznetsov Yu. A, Visser S. On local bifurcations in neural field models with transmission delays. J. Math. Biol 2013;66:837−887. [CrossRef]

[18] Veltz R, Faugeras O. Stability of the stationary solutions of neural field equations with propagation delay. J Math Neurosci 2011;1:1. [CrossRef]

[19] Veltz R. Interplay between synaptic delays and propagation delays in neural field equations. SIAM J Appl Dyn 2013;12:1566−1612. [CrossRef]

[20] Veltz R, Faugeras O. A center manifold result for delayed neural fields equations. SIAM J Math Anal 2013;45:1527−1562. [CrossRef]

[21] Wilson H, Cowan J. A Mathematical theory of the functional dynamics of cortical and thalamic nervous tissue. Biol Cybern 1973;13:55−80. [CrossRef]